

Measure change under conditioning

1 Setup and notation

- Let $\mathbf{D} = (D^-, D^+)$ a $\mathbb{N} \times \mathbb{N}$ -valued random variable and let

$$\mu := \mathbb{E}[D^-] = \mathbb{E}[D^+] \quad (1)$$

$$\beta_- := \mathbb{E}[(D^-)^2] \quad (2)$$

$$\gamma_- := \mathbb{E}[(D^-)^3] \quad (3)$$

$$\rho := \mathbb{E}[D^- D^+] \quad (4)$$

- Let $\mathbf{D}_1, \dots, \mathbf{D}_n$ be i.i.d. copies of \mathbf{D} let

$$\Xi_{n-m}^\pm := \sum_{i=m+1}^n D_i^\pm \quad (5)$$

$$\Delta_{n-m} := \Xi_{n-m}^- - \Xi_{n-m}^+. \quad (6)$$

- Let Σ_n be a sized biased ordering of $\mathbf{D}_1, \dots, \mathbf{D}_n$ be in-degree. Ririgously Σ_n is a random permutation on $\{1, \dots, n\}$ such that

$$\mathbb{P}(\Sigma_n = \sigma \mid D_1^- = k_1^-, \dots, D_n^- = k_n^-) := \prod_{i=1}^n \frac{k_{\sigma(i)}^-}{\sum_{j=i}^n k_{\sigma(j)}^-}. \quad (7)$$

- Let $(\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,n}) := (\mathbf{D}_{\Sigma_n(1)}, \dots, \mathbf{D}_{\Sigma_n(n)})$.
- Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be i.i.d. with law

$$\mathbb{P}(Z_i^- = k^-, Z_i^+ = k^+) := \frac{1}{\mu} k^- \mathbb{P}(D^- = k^-, D^+ = k^+). \quad (8)$$

Then

$$\nu_- := \mathbb{E}[Z_i^-] = \frac{\beta_-}{\mu} \quad (9)$$

$$\nu_+ := \mathbb{E}[Z_i^+] = \frac{\rho}{\mu} \quad (10)$$

2 Exact form of the measure change

First we state the form of the measure change between $\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,m}$ and $\mathbf{Z}_1, \dots, \mathbf{Z}_m$. The proof of [1, Proposition 4.2] covers the undirected configuration model but the same proof works for the directed case.

Theorem 1

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \rightarrow \mathbb{R}$,

$$\mathbb{E}[u(\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,m})] = \mathbb{E}[u(\mathbf{Z}_1, \dots, \mathbf{Z}_m) \psi_m^n(\mathbf{Z}_1, \dots, \mathbf{Z}_m)] \quad (11)$$

where

$$\psi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) := \mathbb{E} \left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-} \right]. \quad (12)$$

Now we give the form of the measure change after conditioning on the event

$$\{\Delta_n = 0\} = \left\{ \sum_{i=1}^n \hat{D}_{n,i}^- = \sum_{i=1}^n \hat{D}_{n,i}^+ \right\} = \left\{ \sum_{i=1}^n D_i^- = \sum_{i=1}^n D_i^+ \right\}. \quad (13)$$

Theorem 2

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[u(\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,m}) \mid \Delta_n = 0 \right] = \mathbb{E}[u(\mathbf{Z}_1, \dots, \mathbf{Z}_m) \phi_m^n(\mathbf{Z}_1, \dots, \mathbf{Z}_m)] \quad (14)$$

where

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \frac{\mathbb{E} \left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-} \mathbb{1} \{ \Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-) \} \right]}{\mathbb{P}(\Delta_n = 0)}. \quad (15)$$

Proof Proof: By theorem 1, since

$$\psi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^-} \quad (16)$$

we have

$$\mathbb{E} \left[u(\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,m}) \mathbb{1} \left\{ \sum_{i=1}^n (\hat{D}_{n,i}^- - \hat{D}_{n,i}^+) = 0 \right\} \right] \quad (17)$$

$$= \mathbb{E} \left[u(\mathbf{Z}_1, \dots, \mathbf{Z}_m) \mathbb{1} \left\{ \sum_{i=1}^n (Z_i^- - Z_i^+) = 0 \right\} \prod_{i=1}^n \frac{(n-i+1)\mu}{\sum_{j=i}^n Z_j^-} \right] \quad (18)$$

$$= \mathbb{E} \left[u(\mathbf{Z}_1, \dots, \mathbf{Z}_m) \mathbb{E} \left[\mathbb{1} \left\{ \sum_{i=m+1}^n (Z_i^- - Z_i^+) = \sum_{i=1}^m (Z_i^+ - Z_i^-) \right\} \right. \right. \\ \left. \left. \times \prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m Z_j^- + \sum_{j=m+1}^n Z_j^-} \prod_{i=m+1}^n \frac{(n-i+1)\mu}{\sum_{j=i}^n Z_j^-} \mid \mathbf{Z}_1, \dots, \mathbf{Z}_m \right] \right]. \quad (19)$$

Thus eq. (14) holds with

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \frac{1}{\mathbb{P}(\Delta_n = 0)} \mathbb{E} \left[\mathbb{1} \left\{ \sum_{i=m+1}^n (Z_i^- - Z_i^+) = \sum_{i=1}^m (k_i^+ - k_i^-) \right\} \right. \\ \left. \times \prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m k_j^- + \sum_{j=m+1}^n Z_j^-} \prod_{i=m+1}^n \frac{(n-i+1)\mu}{\sum_{j=i}^n Z_j^-} \right]. \quad (20)$$

Since the Z_i are i.i.d. we can shift indices to get

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \frac{1}{\mathbb{P}(\Delta_n = 0)} \mathbb{E} \left[\mathbb{1} \left\{ \sum_{i=1}^{n-m} (Z_i^- - Z_i^+) = \sum_{i=1}^m (k_i^+ - k_i^-) \right\} \right. \\ \left. \times \prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^{n-m} k_j^- + \sum_{j=1}^{n-m} Z_j^-} \prod_{i=1}^{n-m} \frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m} Z_j^-} \right]. \quad (21)$$

Note that

$$\prod_{i=1}^{n-m} \frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m} Z_j^-} = \psi_{n-m}^{n-m}(\mathbf{Z}_1, \dots, \mathbf{Z}_{n-m}) \quad (22)$$

therefore by theorem 1

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \frac{1}{\mathbb{P}(\Delta_n = 0)} \mathbb{E} \left[\mathbb{1} \left\{ \sum_{i=1}^{n-m} (\hat{D}_{n-m,i}^- - \hat{D}_{n-m,i}^+) = \sum_{i=1}^m (k_i^+ - k_i^-) \right\} \right. \\ \left. \times \prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m k_j^- + \sum_{j=1}^{n-m} \hat{D}_{n-m,j}^-} \right]. \quad (23)$$

Finally since

$$\sum_{i=1}^{n-m} \hat{D}_{n-m,i}^\pm = \sum_{i=1}^{n-m} D_i^\pm \stackrel{d}{=} \sum_{i=m+1}^n D_i^\pm = \Xi_{n-m}^\pm \quad (24)$$

we obtain the desired result of

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) = \frac{\mathbb{E} \left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m k_j^- + \Xi_{n-m}^-} \mathbb{1} \left\{ \Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-) \right\} \right]}{\mathbb{P}(\Delta_n = 0)}. \quad (25)$$

3 Approximation of the measure change

We explore the scaling of the measure change ϕ_m^n as $n \rightarrow \infty$ in the regime $m(n) = \Theta(n^{2/3})$.

Theorem 3

Define

$$s^\pm(i) := \sum_{j=1}^i (k_j^\pm - \nu_\pm). \quad (26)$$

Assume that $m = \Theta(n^{2/3})$ and

$$|s^\pm(i)| \leq n^{\frac{1}{3}+\epsilon} \quad \forall i = 1, \dots, m. \quad (27)$$

Then

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) \geq \exp \left(\frac{1}{\mu n} \sum_{i=0}^m (s^-(i) - s^-(m)) - \frac{\text{var}(Z^-)}{6\mu^2} \frac{m^3}{n^2} \right) + o(1) \quad (28)$$

where the $o(1)$ term is independent of $\mathbf{k}_1, \dots, \mathbf{k}_m$ satisfying our assumptions.

This result only provides a lower bound, but this proves to be sufficient since we can show asymptotically that the lower bound has expectation 1. This shows we haven't lost a significant amount of mass in the limit. The condition on $\mathbf{k}_1, \dots, \mathbf{k}_m$ occurs with high probability when $\mathbf{k}_i = \mathbf{Z}_i$ since the s^- and s^+ become centered random walks.

3.1 Exponential tilting

The key ingredient of the proof is to tilt the measure exponentially so that Δ_{n-m} has mean approximately $\sum_{i=1}^m (k_i^+ - k_i^-)$.

Lemma 1

Define an measure \mathbb{P}_θ , for $\theta \geq 0$, by its Radon–Nikodym derivative

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} := \exp \left(-\Xi_{n-m}^- - (n-m)\alpha(\theta) \right) \quad (29)$$

where

$$\alpha(\theta) := \log \mathbb{E} \left[e^{-\theta D^-} \right]. \quad (30)$$

Take $\theta_n = \frac{1}{\mu} \frac{m}{n}$ and write $\mathbb{P}_n = \mathbb{P}_{\theta_n}$. Then

$$\mathbb{E}_n[] \quad (31)$$

3.2 Local limit theorems

Next we recall a local limit theorem for triangular arrays on lattices. For this section it is convenient to view all the variables as living in the same probability space rather than applying a measure change.

References

- [1] Guillaume Conchon-Kerjan and Christina Goldschmidt. “The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees”. In: *arXiv:2002.04954 [math]* (Feb. 28, 2020). arXiv: 2002.04954. URL: <http://arxiv.org/abs/2002.04954> (visited on 05/13/2020).