Measure change under conditioning

1 Setup and notation

- Let $\mathbf{D} = (D^-, D^+)$ a $\mathbb{N} \times \mathbb{N}$-valued random variable and let

$$\mu := \mathbb{E}[D^-] = \mathbb{E}[D^+] \tag{1}$$
$$\beta_- := \mathbb{E}[(D^-)^2] \tag{2}$$
$$\gamma_- := \mathbb{E}[(D^-)^3] \tag{3}$$
$$\rho := \mathbb{E}[D^- D^+] \tag{4}$$

- Let $\mathbf{D}_1, \dots, \mathbf{D}_n$ be i.i.d. copies of $\mathbf{D}$ let

$$\Xi^\pm_{n-m} := \sum_{i=m+1}^{n} D_i^\pm \tag{5}$$
$$\Delta_{n-m} := \Xi^-_{n-m} - \Xi^+_{n-m}. \tag{6}$$

- Let Σ_n be a sized biased ordering of $\mathbf{D}_1, \dots, \mathbf{D}_n$ be in-degree. Rigirously Σ_n is a random permutation on $\{1, \dots, n\}$ such that

$$\mathbb{P}(\Sigma_n = \sigma \mid D_1^- = k_1^-, \dots, D_n^- = k_n^-) := \prod_{i=1}^{n} \frac{k^-_{\sigma(i)}}{\sum_{j=i}^n k^-_{\sigma(j)}}. \tag{7}$$

- Let $(\hat{\mathbf{D}}_{n,1}, \dots, \hat{\mathbf{D}}_{n,n}) := (\mathbf{D}_{\Sigma_n(1)}, \dots, \mathbf{D}_{\Sigma_n(n)})$.
- Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be i.i.d. with law

$$\mathbb{P}(Z_i^- = k^-, Z_i^+ = k^+) := \frac{1}{\mu} k^- \mathbb{P}(D^- = k^-, D^+ = k^+). \tag{8}$$

Then

$$\nu_- := \mathbb{E}[Z_i^-] = \tfrac{\beta_-}{\mu} \tag{9}$$
$$\nu_+ := \mathbb{E}[Z_i^+] = \tfrac{\rho}{\mu} \tag{10}$$

2 Exact form of the measure change

First we state the form of the measure change between $\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m}$ and $\mathbf{Z}_1, \ldots, \mathbf{Z}_m$. The proof of [1, Proposition 4.2] covers the undirected configuration model but the same proof works for the directed case.

Theorem 1

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \to \mathbb{R}$,

$$\mathbb{E}[u(\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m})] = \mathbb{E}[u(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)\psi_m^n(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)] \tag{11}$$

where

$$\psi_m^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) := \mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-}\right]. \tag{12}$$

Now we gives the form of the measure change after conditioning on the event

$$\{\Delta_n = 0\} = \left\{\sum_{i=1}^n \hat{D}_{n,i}^- = \sum_{i=1}^n \hat{D}_{n,i}^+\right\} = \left\{\sum_{i=1}^n D_i^- = \sum_{i=1}^n D_i^+\right\}. \tag{13}$$

Theorem 2

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \to \mathbb{R}$,

$$\mathbb{E}\left[u(\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m}) \mid \Delta_n = 0\right] = \mathbb{E}[u(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)\phi_m^n(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)] \tag{14}$$

where

$$\phi_m^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) = \frac{\mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-} \mathbb{1}\left\{\Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-)\right\}\right]}{\mathbb{P}(\Delta_n = 0)}. \tag{15}$$

Proof. By theorem 1, since

$$\psi_n^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) = \prod_{i=1}^n \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^-} \tag{16}$$

we have

$$\mathbb{E}\left[u(\hat{\mathbf{D}}_{n,1},\dots,\hat{\mathbf{D}}_{n,m})\mathbb{1}\left\{\sum_{i=1}^{n}(\hat{D}^-_{n,i}-\hat{D}^+_{n,i})=0\right\}\right] \tag{17}$$

$$=\mathbb{E}\left[u(\mathbf{Z}_1,\dots,\mathbf{Z}_m)\mathbb{1}\left\{\sum_{i=1}^{n}(Z^-_i-Z^+_i)=0\right\}\prod_{i=1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\right] \tag{18}$$

$$=\mathbb{E}\Bigg[u(\mathbf{Z}_1,\dots,\mathbf{Z}_m)\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=m+1}^{n}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(Z^+_i-Z^-_i)\right\}$$
$$\times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}Z^-_j+\sum_{j=m+1}^{n}Z^-_j}\prod_{i=m+1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\mid \mathbf{Z}_1,\dots,\mathbf{Z}_m\Bigg]\Bigg]. \tag{19}$$

Thus eq. (14) holds with

$$\phi^n_m(\mathbf{k}_1,\dots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=m+1}^{n}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}$$
$$\times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=m+1}^{n}Z^-_j}\prod_{i=m+1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\Bigg]. \tag{20}$$

Since the Z_i are i.i.d. we can shift indices to get

$$\phi^n_m(\mathbf{k}_1,\dots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=1}^{n-m}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}$$
$$\times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=1}^{n-m}Z^-_j}\prod_{i=1}^{n-m}\frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m}Z^-_j}\Bigg]. \tag{21}$$

Note that

$$\prod_{i=1}^{n-m}\frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m}Z^-_j}=\psi^{n-m}_{n-m}(\mathbf{Z}_1,\dots,\mathbf{Z}_{n-m}) \tag{22}$$

therefore by theorem 1

$$\phi^n_m(\mathbf{k}_1,\dots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=1}^{n-m}(\hat{D}^-_{n-m,i}-\hat{D}^+_{n-m,i})=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}$$
$$\times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=1}^{n-m}\hat{D}^-_{n-m,j}}\Bigg]. \tag{23}$$

Finally since

$$\sum_{i=1}^{n-m}\hat{D}^\pm_{n-m,i}=\sum_{i=1}^{n-m}D^\pm_i\overset{d}{=}\sum_{i=m+1}^{n}D^\pm_i=\Xi^\pm_{n-m} \tag{24}$$

we obtain the desired result of

$$\phi_m^n(\mathbf{k}_1,\ldots,\mathbf{k}_m) = \frac{\mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-} \mathbb{1}\left\{\Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-)\right\}\right]}{\mathbb{P}(\Delta_n = 0)}. \tag{25}$$

□

3 Approximation of the measure change

We explore the scaling of the measure change ϕ_m^n as $n \to \infty$ in the regime $m(n) = \Theta(n^{2/3})$.

Theorem 3

Define

$$s^{\pm}(i) := \sum_{j=1}^i (k_i^{\pm} - \nu_{\pm}). \tag{26}$$

1. *Assume* $\mathbf{k}_1,\ldots,\mathbf{k}_m$ *satisfies*

$$\max_{i=1,\ldots,m} |s^{\pm}(i)| \leq m^{\frac{1}{2}+\epsilon}. \tag{27}$$

2. *Assume* $D^- - D^+$ *is not supported on any proper sublattice of* $\mathbb{Z}$, *in that for all* $a, b \in \mathbb{Z}$ *if* $b \geq 2$ *then*

$$\mathbb{P}(D^- - D^+ \in a + b\mathbb{Z}) < 1. \tag{28}$$

Then in the regime $m = \Theta(n^{2/3})$

$$\phi_m^n(\mathbf{k}_1,\ldots,\mathbf{k}_m) \geq \exp\left(\frac{1}{\mu n}\sum_{i=0}^m (s^-(i) - s^-(m)) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2}\right) + o(1) \tag{29}$$

where the $o(1)$ *term is independent of* $\mathbf{k}_1,\ldots,\mathbf{k}_m$ *satisfying our assumptions.*

This result only provides a lower bound, but this proves to be sufficient since we can show asymptotically that the lower bound has expectation 1. This shows we haven't lost a significant amount of mass in the limit. The condition on $\mathbf{k}_1,\ldots,\mathbf{k}_m$ occurs with high probability when $\mathbf{k}_i = \mathbf{Z}_i$ since the s^- and s^+ become centered random walks.

3.1 Exponential tilting

Note that $Z^- - Z^+$ is non-centered so we expect $\sum_{i=1}^m (k_i^+ - k_i^-)$ to be of order m. This is a moderate deviation event for Δ_{n-m}. To deal with this, we tilt the measure exponentially so that Δ_{n-m} has mean approximately $\sum_{i=1}^m (k_i^+ - k_i^-)$.

The next result defines the tilting and then gives asympotics for cumulant generating function of D^- and for the behaviour of Ξ_{n-m}^- and Ξ_{n-m}^+ under this tilting.

Lemma 1

Define an measure $\mathbb{P}_\theta$, for $\theta \geq 0$, by its Radon–Nikodym derivative

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} := \exp\left(-\theta D^- - \alpha(\theta)\right) \quad \textit{where} \quad \alpha(\theta) := \log \mathbb{E}\left[e^{-\theta D^-}\right]. \tag{30}$$

Then as $\theta \downarrow 0$ we have

$$\alpha(\theta) = -\mu\theta + \tfrac{1}{2}\operatorname{var}(D^-)\theta^2 - \tfrac{1}{6}\mathbb{E}\left[(D^- - \mu)^3\right]\theta^3 + o(\theta^3), \tag{31}$$

$$\mathbb{E}_\theta[D^-] = \mu - \operatorname{var}(D^-)\theta + \tfrac{1}{2}\mathbb{E}\left[(D^- - \mu)^3\right]\theta^2 + o(\theta^2), \tag{32}$$

$$\mathbb{E}_\theta[D^+] = \mu - \operatorname{cov}(D^-, D^+)\theta + \tfrac{1}{2}\mathbb{E}\left[(D^- - \mu)^2(D^+ - \mu)\right]\theta^2 + o(\theta^2), \tag{33}$$

$$\operatorname{var}_\theta(D^-) = \operatorname{var}(D^-) - \mathbb{E}\left[(D^- - \mu)^3\right]\theta + o(\theta), \tag{34}$$

$$\operatorname{var}_\theta(D^+) = \operatorname{var}(D^+) - \mathbb{E}\left[(D^- - \mu)(D^+ - \mu)^2\right]\theta + o(\theta), \tag{35}$$

$$\operatorname{cov}_\theta(D^-, D^+) = \operatorname{cov}(D^-, D^+) + \mathbb{E}\left[(D^- - \mu)^2(D^+ - \mu)\right]\theta + o(\theta). \tag{36}$$

Proof. Since $\mathbb{E}\left[|D^-|^3\right] < \infty$ and D^- is non-negative, by the dominated convergence theorem

$$\mathbb{E}\left[(D^-)^3 \exp(-\theta D^-)\right] = \mathbb{E}\left[(D^-)^3\right] + o(1) \tag{37}$$

as $\theta \downarrow 0$. Integrating eq. (37) with respect to θ and applying Fubini's theorem gives

$$\int_0^\theta \mathbb{E}\left[(D^-)^3 e^{-\theta' D^-}\right] \mathrm{d}\theta' = \int_0^\theta \left(\mathbb{E}\left[(D^-)^3\right] + o(1)\right) \mathrm{d}\theta \tag{38}$$

$$\iff \mathbb{E}\left[\int_0^\theta (D^-)^3 e^{-\theta' D^-}\, \mathrm{d}\theta'\right] = \mathbb{E}\left[(D^-)^3\right]\theta + o(\theta) \tag{39}$$

$$\iff \mathbb{E}\left[(D^-)^2\right] - \mathbb{E}\left[(D^-)^2 e^{-\theta D^-}\right] = \mathbb{E}\left[(D^-)^3\right]\theta + o(\theta) \tag{40}$$

$$\iff \mathbb{E}\left[(D^-)^2 e^{-\theta D^-}\right] = \mathbb{E}\left[(D^-)^2\right] - \mathbb{E}\left[(D^-)^3\right]\theta + o(\theta). \tag{41}$$

Repeating this method yields

$$\mathbb{E}\left[D^- e^{-\theta D^-}\right] = \mu - \mathbb{E}\left[(D^-)^2\right]\theta + \tfrac{1}{2}\mathbb{E}\left[(D^-)^3\right]\theta^2 + o(\theta^2), \tag{42}$$

$$\text{and} \quad \mathbb{E}\left[e^{-\theta D^-}\right] = 1 - \mu\theta + \tfrac{1}{2}\mathbb{E}\left[(D^-)^2\right]\theta^2 - \tfrac{1}{6}\mathbb{E}\left[(D^-)^3\right]\theta^3 + o(\theta^3). \tag{43}$$

Similary integrating the equation

$$\mathbb{E}\left[(D^-)^2 D^+ \exp(-\theta D^-)\right] = \mathbb{E}\left[(D^-)^2 D^+\right] + o(1) \tag{44}$$

gives

$$\mathbb{E}\left[D^- D^+ e^{-\theta D^-}\right] = \mathbb{E}\left[D^- D^+\right] - \mathbb{E}\left[(D^-)^2 D^+\right]\theta + o(\theta), \tag{45}$$

$$\text{and} \quad \mathbb{E}\left[D^+ e^{-\theta D^-}\right] = \mu\theta - \mathbb{E}\left[D^- D^+\right]\theta + \tfrac{1}{2}\mathbb{E}\left[(D^-)^2 D^+\right]\theta^2 + o(\theta^2). \tag{46}$$

Integrating the equation

$$\mathbb{E}\left[D^-(D^+)^2\exp(-\theta D^-)\right] = \mathbb{E}\left[D^-(D^+)^2\right] + o(1) \tag{47}$$

gives

$$\mathbb{E}\left[(D^+)^2 e^{-\theta D^-}\right] = \mathbb{E}\left[(D^+)^2\right] - \mathbb{E}\left[D^-(D^+)^2\right]\theta + o(\theta). \tag{48}$$

eq. (43) gives the expansion of the normalising constant of the measure change. Taking the logarithm on both sides of eq. (43) and expanding the right hand side gives the expansion for the cumulant generating function. Combining this with eq. (42) and eq. (46) gives the expansions for $\mathbb{E}_\theta[D^-]$ and $\mathbb{E}_\theta[D^+]$ respectively. Similarly eq. (41), eq. (48) and eq. (45) are used to get the exanpsions for $\mathrm{var}_\theta(D^-)$, $\mathrm{var}_\theta(D^+)$ and $\mathrm{cov}_\theta(D^-, D^+)$ respectively. □

3.2 Local limit theorems

Lemma 2

Let c be given by

$$c := \frac{\mathrm{cov}(D^-, D^- - D^+)}{\mathrm{var}(D^- - D^+)} \tag{49}$$

such that $\mathrm{cov}(D^- - c(D^- - D^+), D^- - D^+) = 0$. *Also let*

$$\Theta_{n-m} := \Xi^-_{n-m} - c\Delta_{n-m}. \tag{50}$$

Then

$$\begin{aligned}&\mathbb{P}\left(\Theta_{n-m} - \mathbb{E}[\Theta_{n-m}] = x, \Delta_{n-m} = y\right)\\ =&\frac{1}{\sqrt{2\pi\sigma_1^2 n}}\exp\left(-\frac{1}{2\sigma_1^2}\frac{x^2}{n}\right)\frac{1}{\sqrt{2\pi\sigma_2^2 n}}\exp\left(-\frac{1}{2\sigma_2^2}\frac{y^2}{n}\right) + o\left(n^{-1}\right)\end{aligned} \tag{51}$$

as $n \to \infty$ where the $o(n^{-1})$ term is uniform in x and y and

$$\sigma_1^2 = \mathrm{var}(D^- - c(D^- - D^+)) \quad \text{and} \quad \sigma_2^2\,\mathrm{var}(D^- - D^+). \tag{52}$$

In addition

$$\mathbb{P}\left(\Delta_{n-m} = y\right) = \frac{1}{\sqrt{2\pi\sigma_2^2 n}}\exp\left(-\frac{1}{2\sigma_2^2}\frac{y^2}{n}\right) + o\left(n^{-1/2}\right) \tag{53}$$

as $n \to \infty$ where the $o(n^{-1/2})$ term is uniform in y.

Lemma 3

$$\begin{aligned}&\mathbb{P}_n\left(\Theta_{n-m} - \mathbb{E}_n[\Theta_{n-m}] = x, \Delta_{n-m} - \mathbb{E}_n[\Delta_{n-m}] = y\right)\\ =&\frac{1}{\sqrt{2\pi\sigma_1^2 n}}\exp\left(-\frac{1}{2\sigma_1^2}\frac{x^2}{n}\right)\frac{1}{\sqrt{2\pi\sigma_2^2 n}}\exp\left(-\frac{1}{2\sigma_2^2}\frac{y^2}{n}\right) + o\left(n^{-1}\right)\end{aligned} \tag{54}$$

as $n \to \infty$ where the $o(n^{-1})$ term is uniform in x and y.

Lemma 4

$$\mathbb{P}_n\left(|\Omega_n| \leq n^{\frac{1}{2}+\epsilon},\ \Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right) \geq \mathbb{P}\left(\Delta_{n-m} = 0\right)(1 + o(1)) \tag{55}$$

Proof. Firstly by lemma 1,

$$\begin{aligned}
\mathbb{E}_n[\Delta_{n-m}] &= (n-m)\mathbb{E}_{\theta_n}[D^- - D^+] &(56)\\
&= (n-m)\left(-\frac{\mathbb{E}[(D^-)^2] - \mathbb{E}[D^-D^+]}{\mu}\frac{m}{n} + O(n^{-2/3})\right) &(57)\\
&= -(\nu_- - \nu_+)m + O(n^{1/3}). &(58)
\end{aligned}$$

Therefore

$$\begin{aligned}
a_n &:= \sum_{i=1}^{m}(k_i^+ - k_i^-) - \mathbb{E}_n[\Delta_{n-m}] &(59)\\
&= \sum_{i=1}^{m}(k_i^+ - k_i^-) - (\nu_+ - \nu_-)m + O(n^{1/3}) &(60)\\
&= s_+(m) - s_-(m) + O(n^{1/3}) = O(n^{1/3+\epsilon}) &(61)
\end{aligned}$$

by assumption

Then on the event

$$\left\{\Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right\} = \{\Delta_{n-m} = \mathbb{E}_n[\Delta_{n-m}] + a_n\} \tag{62}$$

we have

$$\begin{aligned}
\Omega_{n-m} &= \Xi^-_{n-m} - \mu n + \nu_- m &(63)\\
&= \Xi^-_{n-m} - \mathbb{E}_n[\Xi^-_{n-m}] + O(n^{1/3}) &(64)\\
&= \left(\Xi^-_{n-m} - \mathbb{E}_n[\Xi^-_{n-m}]\right) - c\left(\Delta_{n-m} + \mathbb{E}_n[\Delta_{n-m}]\right) + |a_n| + O(n^{1/3}) &(65)\\
&= \Theta_{n-m} - \mathbb{E}_n[\Theta_{n-m}] + a_n + O(n^{1/3}). &(66)
\end{aligned}$$

Therefore for all fixed $L > 0$

$$|\Theta_{n-m} - \mathbb{E}_n[\Theta_{n-m}]| \leq Ln^{1/2} \implies |\Omega_{n-m}| \leq Ln^{1/2} + a_n + O(n^{1/3}) \leq n^{1/2+\epsilon} \tag{67}$$

for all n sufficiently large. Hence

$$\begin{aligned}
&\mathbb{P}_n\left(|\Omega_n| \geq n^{\frac{1}{2}+\epsilon},\ \Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right) \\
\geq &\mathbb{P}_n\left(|\Theta_{n-m} - \mathbb{E}_n[\Theta_{n-m}]| \leq Ln^{1/2},\ \Delta_{n-m} - \mathbb{E}_n[\Delta_{n-m}] = a_n\right) && (68)\\
= &\sum_{|x|\leq Ln^{1/2}} \mathbb{P}_n\left(|\Theta_{n-m} - \mathbb{E}_n[\Theta_{n-m}]| \leq Ln^{1/2},\ \Delta_{n-m} - \mathbb{E}_n[\Delta_{n-m}] = a_n\right) && (69)\\
= &\frac{1}{\sqrt{2\pi\sigma_2^2 n}} \exp\left(-\frac{1}{2\sigma_2^2}\frac{-a_n^2}{n}\right) \sum_{|x|\leq Ln^{1/2}} \frac{1}{\sqrt{n}} f\left(\frac{x}{\sqrt{n}}\right) + o(n^{-1/2}) && (70)
\end{aligned}$$

using eq. (54) of lemma 3 since there are $O(n^{1/2})$ terms in the summation and the $o(n^{-1})$ term in eq. (54) is uniform in x and a_n. Here

$$f(x) := \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right) \tag{71}$$

is the density of a $\mathcal{N}(0, \sigma_1^2)$ distribution. Moreover by the Euler method for approximating integrals

$$\sum_{|x|\leq Ln^{1/2}} \frac{1}{\sqrt{n}} f\left(\frac{x}{\sqrt{n}}\right) = \int_{-L}^{L} f(u)\,\mathrm{d}u + o(1) \tag{72}$$

In addition since by assumption $a_n = O(n^{1/3+\epsilon})$, we have $\exp\left(-\frac{1}{2\sigma_1^2}\frac{a_n^2}{n}\right) = 1 + o(1)$. Therefore

$$\mathbb{P}_n\left(|\Omega_n| \geq n^{\frac{1}{2}+\epsilon},\ \Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right) \geq \frac{1}{\sqrt{2\pi\sigma_2^2 n}} \cdot \int_{-L}^{L} f(u)\,\mathrm{d}u \cdot (1 + o(1))\,. \tag{73}$$

Next note that by eq. (53) in lemma 2,

$$\mathbb{P}\left(\Delta_{n-m} = 0\right) = \frac{1}{\sqrt{2\pi\sigma_2^2 n}}(1 + o(1)) \tag{74}$$

thus

$$\begin{aligned}
&\mathbb{P}_n\left(|\Omega_n| \geq n^{\frac{1}{2}+\epsilon},\ \Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right) && (75)\\
\geq &\mathbb{P}(\Delta_{n-m} = 0) \cdot \int_{-L}^{L} f(u)\,\mathrm{d}u \cdot (1 + o(1))\,. && (76)
\end{aligned}$$

This holds for all $L > 0$ and $\int_{-\infty}^{\infty} f(u)\,\mathrm{d}u = 1$, thus

$$\mathbb{P}_n\left(|\Omega_n| \geq n^{\frac{1}{2}+\epsilon},\ \Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right) \geq \mathbb{P}(\Delta_{n-m} = 0)\,(1 + o(1)) \tag{77}$$

as required. □

3.3 Combining the parts

Proof. Firstly

$$\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-} = \exp(X_n - Y_n) \tag{78}$$

where

$$X_n = \sum_{i=1}^{m} \log\left(1 - \frac{i-1}{n}\right) \quad \text{and} \quad Y_n = \sum_{i=1}^{m} \log\left(\frac{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-}{\mu n}\right). \tag{79}$$

Note

$$\sum_{j=i}^{m} k_j^- = s^-(m) - s^-(i-1) + (m-i+1)\nu_- \tag{80}$$

Thus

$$\begin{aligned} Y_n &= \sum_{i=1}^{m} \log\left(\frac{s^-(m) - s^-(i-1) + (m-i+1)\nu_- + \Omega_n^- + \mu n - \nu_- m}{\mu n}\right) && (81) \\ &= \sum_{i=1}^{m} \log\left(1 + A_n^i + B_n + C_n^i\right) && (82) \end{aligned}$$

where

$$A_n^i = -\frac{1}{\mu}\frac{1}{n}\left[s^-(i-1) - s^-(m)\right], \quad B_n = \frac{1}{\mu}\frac{1}{n}\Omega_n^-, \quad C_n^i = -\frac{\nu_-}{\mu}\frac{1}{n}(i-1). \tag{83}$$

When expanding $\log(1 + A_n^i + B_n + C_n^i)$, the summation contributes order $m = O(n^{2/3})$. Thus we keep terms of order $\Omega(n^{-2/3})$ in the expansion. Write

$$\mathcal{E}_n := \left\{|\Omega_n^-| \leq n^{\frac{1}{2}+\epsilon}\right\} \tag{84}$$

On the event $\mathcal{E}_n$, we can check that the A_n, B_n, C_n^i and $(C_n^i)^2$ terms are the only terms in the expansion which has order $\Omega(n^{-2/3})$, moreover

$$\sum_{i=1}^{m} C_n^i = -\frac{\nu_-}{2\mu}\frac{m^2}{n} + o(1) \quad \text{and} \quad \sum_{i=1}^{m} (C_n^i)^2 = \frac{\nu_-^2}{3\mu^2}\frac{m^3}{n^2} + o(1). \tag{85}$$

Therefore

$$\begin{aligned} Y_n &= \sum_{i=1}^{m} (A_n^i + B_n + C_n^i - \tfrac{1}{2}(C_n^i)^2) + o(1) && (86) \\ &= -\frac{1}{\mu}\frac{1}{n}\sum_{i=0}^{m}\left(s^-(i) - s^-(m)\right) + \frac{1}{\mu}\frac{m}{n}\Omega_n^- - \frac{\nu_-}{2\mu}\frac{m^2}{n} - \frac{\nu_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1). && (87) \end{aligned}$$

where we use that $\sum_{i=1}^{m}(s^-(i-1) - s^-(m)) = \sum_{i=0}^{m}(s^-(i) - s^-(m))$.

Similarly we can expand X_n as

$$X_n = -\frac{1}{2}\frac{m}{n} - \frac{1}{3}\frac{m^3}{n^2} + o(1). \tag{88}$$

Thus

$$\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-} \geq \exp\left(\frac{1}{\mu}\frac{1}{n}\sum_{i=1}^{m}(s^-(i) - s^-(m)) \right.$$
$$\left. - \frac{1}{\mu}\frac{m}{n}\Omega_n^- + \frac{\nu_- - \mu}{2\mu}\frac{m^2}{n} + \frac{\nu_-^2 - \mu^2}{6\mu^2}\frac{m^3}{n^2}\right)\mathbb{1}_{\mathcal{E}_n} \tag{89}$$

In addition measure change can be expanded as

$$\frac{d\mathbb{P}_n}{d\mathbb{P}} = \exp\left(-\frac{1}{\mu}\frac{m}{n}\Omega_n^- + \frac{\nu_- - \mu}{2\mu}\frac{m^2}{n} + \frac{\nu_-^2 - \mu^2}{6\mu^2}\frac{m^3}{n^2} + \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1)\right). \tag{90}$$

Hence

$$\phi_m^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) \tag{91}$$
$$= \frac{1}{\mathbb{P}(\Delta_n = 0)}\mathbb{E}\left[\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-}\mathbb{1}_{A_n}\right] \tag{92}$$
$$\geq \frac{1}{\mathbb{P}(\Delta_n = 0)}\mathbb{E}_n\left[\exp\left(\frac{1}{\mu n}\sum_{i=1}^{m}\big(s^-(i) - s^-(m)\big) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1)\right)\mathbb{1}_{\mathcal{E}_n \cap A_n}\right] \tag{93}$$
$$\geq \exp\left(\frac{1}{\mu n}\sum_{i=1}^{m}\big(s^-(i) - s^-(m)\big) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2}\right)(1 + o(1))\frac{\mathbb{P}_n(\mathcal{E} \cap A_n)}{\mathbb{P}(\Delta_n = 0)}. \tag{94}$$

Finally

$$\frac{\mathbb{P}_n(\mathcal{E} \cap A_n)}{\mathbb{P}(\Delta_n = 0)} \geq 1 + o(1) \tag{95}$$

as $n \to \infty$ by lemma 4 which gives the desired final result. □

References

[1] Guillaume Conchon–Kerjan and Christina Goldschmidt. "The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees". In: *arXiv:2002.04954 [math]* (Feb. 28, 2020). arXiv: `2002.04954`. URL: `http://arxiv.org/abs/2002.04954` (visited on 05/13/2020).