Measure change under conditioning

1 Setup and notation

- Let $\mathbf{D} = (D^-, D^+)$ a $\mathbb{N} \times \mathbb{N}$-valued random variable and let

$$\mu := \mathbb{E}[D^-] = \mathbb{E}[D^+] \tag{1}$$
$$\beta_- := \mathbb{E}[(D^-)^2] \tag{2}$$
$$\gamma_- := \mathbb{E}[(D^-)^3] \tag{3}$$
$$\rho := \mathbb{E}[D^- D^+] \tag{4}$$

- Let $\mathbf{D}_1, \ldots, \mathbf{D}_n$ be i.i.d. copies of $\mathbf{D}$ let

$$\Xi^{\pm}_{n-m} := \sum_{i=m+1}^{n} D_i^{\pm} \tag{5}$$
$$\Delta_{n-m} := \Xi^-_{n-m} - \Xi^+_{n-m}. \tag{6}$$

- Let Σ_n be a sized biased ordering of $\mathbf{D}_1, \ldots, \mathbf{D}_n$ be in-degree. Rigirously Σ_n is a random permutation on $\{1, \ldots, n\}$ such that

$$\mathbb{P}(\Sigma_n = \sigma \mid D_1^- = k_1^-, \ldots, D_n^- = k_n^-) := \prod_{i=1}^{n} \frac{k^-_{\sigma(i)}}{\sum_{j=i}^{n} k^-_{\sigma(j)}}. \tag{7}$$

- Let $(\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,n}) := (\mathbf{D}_{\Sigma_n(1)}, \ldots, \mathbf{D}_{\Sigma_n(n)})$.
- Let $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ be i.i.d. with law

$$\mathbb{P}(Z_i^- = k^-, Z_i^+ = k^+) := \frac{1}{\mu} k^- \mathbb{P}(D^- = k^-, D^+ = k^+). \tag{8}$$

  Then

$$\nu_- := \mathbb{E}[Z_i^-] = \tfrac{\beta_-}{\mu} \tag{9}$$
$$\nu_+ := \mathbb{E}[Z_i^+] = \tfrac{\rho}{\mu} \tag{10}$$

2 Exact form of the measure change

First we state the form of the measure change between $\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m}$ and $\mathbf{Z}_1, \ldots, \mathbf{Z}_m$. The proof of [1, Proposition 4.2] covers the undirected configuration model but the same proof works for the directed case.

Theorem 1

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \to \mathbb{R}$,

$$\mathbb{E}[u(\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m})] = \mathbb{E}[u(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)\psi_m^n(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)] \tag{11}$$

where

$$\psi_m^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) := \mathbb{E}\left[\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{j=i}^{n} k_j^- + \Xi_{n-m}^-}\right]. \tag{12}$$

Now we gives the form of the measure change after conditioning on the event

$$\{\Delta_n = 0\} = \left\{\sum_{i=1}^{n} \hat{D}_{n,i}^- = \sum_{i=1}^{n} \hat{D}_{n,i}^+\right\} = \left\{\sum_{i=1}^{n} D_i^- = \sum_{i=1}^{n} D_i^+\right\}. \tag{13}$$

Theorem 2

For any test function $u : (\mathbb{N} \times \mathbb{N})^m \to \mathbb{R}$,

$$\mathbb{E}\left[u(\hat{\mathbf{D}}_{n,1}, \ldots, \hat{\mathbf{D}}_{n,m}) \mid \Delta_n = 0\right] = \mathbb{E}[u(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)\phi_m^n(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)] \tag{14}$$

where

$$\phi_m^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) = \frac{\mathbb{E}\left[\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{j=i}^{n} k_j^- + \Xi_{n-m}^-} \mathbb{1}\left\{\Delta_{n-m} = \sum_{i=1}^{m}(k_i^+ - k_i^-)\right\}\right]}{\mathbb{P}(\Delta_n = 0)}. \tag{15}$$

Proof. By theorem 1, since

$$\psi_n^n(\mathbf{k}_1, \ldots, \mathbf{k}_m) = \prod_{i=1}^{n} \frac{(n-i+1)\mu}{\sum_{j=i}^{n} k_j^-} \tag{16}$$

we have

$$\mathbb{E}\left[u(\hat{\mathbf{D}}_{n,1},\ldots,\hat{\mathbf{D}}_{n,m})\mathbb{1}\left\{\sum_{i=1}^{n}(\hat{D}^-_{n,i}-\hat{D}^+_{n,i})=0\right\}\right] \tag{17}$$

$$=\mathbb{E}\left[u(\mathbf{Z}_1,\ldots,\mathbf{Z}_m)\mathbb{1}\left\{\sum_{i=1}^{n}(Z^-_i-Z^+_i)=0\right\}\prod_{i=1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\right] \tag{18}$$

$$\begin{aligned}=\mathbb{E}\Bigg[u(\mathbf{Z}_1,\ldots,\mathbf{Z}_m)\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=m+1}^{n}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(Z^+_i-Z^-_i)\right\}\\ \times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}Z^-_j+\sum_{j=m+1}^{n}Z^-_j}\prod_{i=m+1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\mid \mathbf{Z}_1,\ldots,\mathbf{Z}_m\Bigg]\Bigg].\end{aligned} \tag{19}$$

Thus eq. (14) holds with

$$\begin{aligned}\phi^n_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=m+1}^{n}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}\\ \times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=m+1}^{n}Z^-_j}\prod_{i=m+1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n}Z^-_j}\Bigg].\end{aligned} \tag{20}$$

Since the Z_i are i.i.d. we can shift indices to get

$$\begin{aligned}\phi^n_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=1}^{n-m}(Z^-_i-Z^+_i)=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}\\ \times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=1}^{n-m}Z^-_j}\prod_{i=1}^{n-m}\frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m}Z^-_j}\Bigg].\end{aligned} \tag{21}$$

Note that

$$\prod_{i=1}^{n-m}\frac{(n-m-i+1)\mu}{\sum_{j=i}^{n-m}Z^-_j}=\psi^{n-m}_{n-m}(\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-m}) \tag{22}$$

therefore by theorem 1

$$\begin{aligned}\phi^n_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\frac{1}{\mathbb{P}(\Delta_n=0)}\mathbb{E}\Bigg[\mathbb{1}\left\{\sum_{i=1}^{n-m}(\hat{D}^-_{n-m,i}-\hat{D}^+_{n-m,i})=\sum_{i=1}^{m}(k^+_i-k^-_i)\right\}\\ \times\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k^-_j+\sum_{j=1}^{n-m}\hat{D}^-_{n-m,j}}\Bigg].\end{aligned} \tag{23}$$

Finally since

$$\sum_{i=1}^{n-m}\hat{D}^{\pm}_{n-m,i}=\sum_{i=1}^{n-m}D^{\pm}_i\overset{d}{=}\sum_{i=m+1}^{n}D^{\pm}_i=\Xi^{\pm}_{n-m} \tag{24}$$

we obtain the desired result of

$$\phi_m^n(\mathbf{k}_1,\dots,\mathbf{k}_m) = \frac{\mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^n k_j^- + \Xi_{n-m}^-} \mathbb{1}\left\{\Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-)\right\}\right]}{\mathbb{P}(\Delta_n = 0)}. \tag{25}$$

□

3 Approximation of the measure change

We explore the scaling of the measure change ϕ_m^n as $n \to \infty$ in the regime $m(n) = \Theta(n^{2/3})$.

Theorem 3

Define

$$s^{\pm}(i) := \sum_{j=1}^i (k_i^{\pm} - \nu_{\pm}). \tag{26}$$

1. *Assume* $\mathbf{k}_1,\dots,\mathbf{k}_m$ *satisfies*

$$\max_{i=1,\dots,m} |s^{\pm}(i)| \le m^{\frac{1}{2}+\epsilon}. \tag{27}$$

2. *Assume* $D^- - D^+$ *is not supported on any proper sublattice of* $\mathbb{Z}$*, in that for all* $a, b \in \mathbb{Z}$ *if* $b \ge 2$ *then*

$$\mathbb{P}(D^- - D^+ \in a + b\mathbb{Z}) < 1. \tag{28}$$

Then in the regime $m = \Theta(n^{2/3})$

$$\phi_m^n(\mathbf{k}_1,\dots,\mathbf{k}_m) \ge \exp\left(\frac{1}{\mu n}\sum_{i=0}^m (s^-(i) - s^-(m)) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2}\right) + o(1) \tag{29}$$

where the $o(1)$ *term is independent of* $\mathbf{k}_1,\dots,\mathbf{k}_m$ *satisfying our assumptions.*

This result only provides a lower bound, but this proves to be sufficient since we can show asymptotically that the lower bound has expectation 1. This shows we haven't lost a significant amount of mass in the limit. The condition on $\mathbf{k}_1,\dots,\mathbf{k}_m$ occurs with high probability when $\mathbf{k}_i = \mathbf{Z}_i$ since the s^- and s^+ become centered random walks.

3.1 Exponential tilting

Note that $Z^- - Z^+$ is non-centered so we expect $\sum_{i=1}^m (k_i^+ - k_i^-)$ to be of order m. This is a moderate deviation event for Δ_{n-m}. To deal with this, we tilt the measure exponentially so that Δ_{n-m} has mean approximately $\sum_{i=1}^m (k_i^+ - k_i^-)$.

The next result defines the tilting and then gives asympotics for cumulant generating function of D^- and for the behaviour of Ξ_{n-m}^- and Ξ_{n-m}^+ under this tilting.

Lemma 1

Define an measure $\mathbb{P}_\theta$, for $\theta \geq 0$, by its Radon–Nikodym derivative

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} := \exp\left(-\Xi^-_{n-m} - (n-m)\alpha(\theta)\right) \quad \textit{where} \quad \alpha(\theta) := \log \mathbb{E}\left[e^{-\theta D^-}\right]. \tag{30}$$

Take $\theta_n = \frac{1}{\mu}\frac{m}{n}$. Then

$$(n-m)\alpha(\theta_n) = -m + \frac{\nu_- - \mu}{2\mu}\frac{m^2}{n} - \frac{\sigma_-^2 + \nu_-^2 - \mu^2}{6\mu^2}\frac{m^3}{n^2} + o(1), \tag{31}$$

$$\mathbb{E}_{\theta_n}[D^\pm] = \mu - (\nu_\pm - \mu)\frac{m}{n} + O(n^{-2/3}), \tag{32}$$

$$\mathrm{var}_{\theta_n}(D^\pm) = \mathrm{var}(D^\pm) + o(1), \tag{33}$$

$$\textit{and } \mathrm{cov}_{\theta_n}(D^-, D^+) = \mathrm{cov}(D^-, D^+) + o(1). \tag{34}$$

3.2 Local limit theorems

Next we recall a local limit theorem for triangular arrays on lattices. For this section it is convenient to view all the variables as living in the same probabilty space rather than applying a measure change.

Lemma 2

Writing $\tilde{\sigma}^2 = \mathrm{var}(D^- - D^+)$,

$$\mathbb{P}_n\left(\Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-)\right) \sim \mathbb{P}(\Delta_n = 0) \sim \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 n}} \tag{35}$$

as $n \to \infty$.

When expanding the measure change, instead of expanding in terms of Ξ^-_{n-m} it is convenient to expand in terms of Ω_n^- which is almost $\mathbb{P}_n$-centered. The following lemma provides probabilistic bounds on Ω_n^-.

Lemma 3

Define

$$\Omega_n^- := \Xi^-_{n-m} - \mu n + \nu_- m. \tag{36}$$

Then

$$\lim_{n\to\infty} \mathbb{P}_n\left(|\Omega_n^-| \leq n^{\frac{1}{2}+\epsilon} \,\middle|\, \Delta_{n-m} = \sum_{i=1}^m (k_i^+ - k_i^-)\right) = 1. \tag{37}$$

3.3 Combining the parts

Proof. Firstly

$$\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-} = \exp(X_n - Y_n) \tag{38}$$

where

$$X_n = \sum_{i=1}^{m} \log\left(1 - \frac{i-1}{n}\right) \quad \text{and} \quad Y_n = \sum_{i=1}^{m} \log\left(\frac{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-}{\mu n}\right). \tag{39}$$

Note

$$\sum_{j=i}^{m} k_j^- = s^-(m) - s^-(i-1) + (m-i+1)\nu_- \tag{40}$$

Thus

$$Y_n = \sum_{i=1}^{m} \log\left(\frac{s^-(m) - s^-(i-1) + (m-i+1)\nu_- + \Omega_n^- + \mu n - \nu_- m}{\mu n}\right) \tag{41}$$

$$= \sum_{i=1}^{m} \log\left(1 + A_n^i + B_n + C_n^i\right) \tag{42}$$

where

$$A_n^i = -\frac{1}{\mu}\frac{1}{n}\left[s^-(i-1) - s^-(m)\right], \quad B_n = \frac{1}{\mu}\frac{1}{n}\Omega_n^-, \quad C_n^i = -\frac{\nu_-}{\mu}\frac{1}{n}(i-1). \tag{43}$$

When expanding $\log(1 + A_n^i + B_n + C_n^i)$, the summation contributes order $m = O(n^{2/3})$. Thus we keep terms of order $\Omega(n^{-2/3})$ in the expansion. Write

$$\mathcal{E}_n := \left\{|\Omega_n^-| \le n^{\frac{1}{2}+\epsilon}\right\} \tag{44}$$

On the event $\mathcal{E}_n$, we can check that the A_n, B_n, C_n^i and $(C_n^i)^2$ terms are the only terms in the expansion which has order $\Omega(n^{-2/3})$, moreover

$$\sum_{i=1}^{m} C_n^i = -\frac{\nu_-}{2\mu}\frac{m^2}{n} + o(1) \quad \text{and} \quad \sum_{i=1}^{m} (C_n^i)^2 = \frac{\nu_-^2}{3\mu^2}\frac{m^3}{n^2} + o(1). \tag{45}$$

Therefore

$$Y_n = \sum_{i=1}^{m} (A_n^i + B_n + C_n^i - \tfrac{1}{2}(C_n^i)^2) + o(1) \tag{46}$$

$$= -\frac{1}{\mu}\frac{1}{n}\sum_{i=0}^{m}\left(s^-(i) - s^-(m)\right) + \frac{1}{\mu}\frac{m}{n}\Omega_n^- - \frac{\nu_-}{2\mu}\frac{m^2}{n} - \frac{\nu_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1). \tag{47}$$

where we use that $\sum_{i=1}^{m} (s^-(i-1) - s^-(m)) = \sum_{i=0}^{m} (s^-(i) - s^-(m))$.

Similarly we can expand X_n as

$$X_n = -\frac{1}{2}\frac{m}{n} - \frac{1}{3}\frac{m^3}{n^2} + o(1). \tag{48}$$

Thus

$$\begin{aligned}\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-} \geq \exp\Bigg(\frac{1}{\mu}\frac{1}{n}\sum_{i=1}^{m}(s^-(i) - s^-(m)) \\ - \frac{1}{\mu}\frac{m}{n}\Omega_n^- + \frac{\nu_- - \mu}{2\mu}\frac{m^2}{n} + \frac{\nu_-^2 - \mu^2}{6\mu^2}\frac{m^3}{n^2}\Bigg)\mathbb{1}_{\mathcal{E}_n}\end{aligned} \tag{49}$$

In addition measure change can be expanded as

$$\frac{d\mathbb{P}_n}{d\mathbb{P}} = \exp\left(-\frac{1}{\mu}\frac{m}{n}\Omega_n^- + \frac{\nu_- - \mu}{2\mu}\frac{m^2}{n} + \frac{\nu_-^2 - \mu^2}{6\mu^2}\frac{m^3}{n^2} + \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1)\right). \tag{50}$$

Hence

$$\phi_m^n(\mathbf{k}_1, \dots, \mathbf{k}_m) \tag{51}$$

$$= \frac{1}{\mathbb{P}(\Delta_n = 0)}\mathbb{E}\left[\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{k=i}^{m} k_i^- + \Xi_{n-m}^-}\mathbb{1}_{A_n}\right] \tag{52}$$

$$\geq \frac{1}{\mathbb{P}(\Delta_n = 0)}\mathbb{E}_n\left[\exp\left(\frac{1}{\mu n}\sum_{i=1}^{m}\big(s^-(i) - s^-(m)\big) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2} + o(1)\right)\mathbb{1}_{\mathcal{E}_n \cap A_n}\right] \tag{53}$$

$$\geq \exp\left(\frac{1}{\mu n}\sum_{i=1}^{m}\big(s^-(i) - s^-(m)\big) - \frac{\sigma_-^2}{6\mu^2}\frac{m^3}{n^2}\right)(1 + o(1))\frac{\mathbb{P}_n(\mathcal{E} \cap A_n)}{\mathbb{P}(\Delta_n = 0)}. \tag{54}$$

Finally

$$\frac{\mathbb{P}_n(\mathcal{E} \cap A_n)}{\mathbb{P}(\Delta_n = 0)} = \mathbb{P}_n(\mathcal{E} \mid A_n)\frac{\mathbb{P}_n(A_n)}{\mathbb{P}(\Delta_n = 0)} \to 1 \tag{55}$$

as $n \to \infty$ by INSERT REFERENCE and INSERT REFERENCE which gives the desired final result. □

References

[1] Guillaume Conchon–Kerjan and Christina Goldschmidt. "The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees". In: *arXiv:2002.04954 [math]* (Feb. 28, 2020). arXiv: `2002.04954`. URL: `http://arxiv.org/abs/2002.04954` (visited on 05/13/2020).