Advanced Topics in Stochastic Calculus

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1 Stochastic Integration with Jumps

Stochastic integration is often introduced by working with continuous processes. Such processes have a lot of nice properties, but in the real world a lot of process you want to integrate against have discontinuities. First let's put aside the probability aspect and recall the possible types of discontinuities a function $\mathbb{R} \to \mathbb{R}$ can have.

Definition 1.1

Let $f : \mathbb{R} \to \mathbb{R}$ be discontinuous at $x \in \mathbb{R}$. We use the notation

$$f(x-) := \lim_{t \uparrow x} f(t) \quad \text{and} \quad f(x+) := \lim_{t \downarrow x} f(t) \tag{1.1}$$

when these limits exist. Then we say

1. *f has an* essential discontinuity *at x if at least of one $f(x+)$ or $f(x-)$ does not exist or is infinite.*

2. *f has a* jump discontinuity *at x if $f(x+)$ and $f(x-)$ both exist and are finite but $f(x-) \neq f(x+)$.*

3. *f has a* removable discontinuity *at x if $f(x+)$ and $f(x-)$ both exist and are finite and $f(x-) = f(x+)$.*

We might as well pretend removable discontinuities don't exist, we can make f continuous just by changing the value of f at the discontinuity. Allowing essential discontinuities would mean working with all manner of horrendous functions, so we'll ignore them also. This leaves us with functions having jump discontinuities, which is what we're really interested in anyway.

1.1 Càdlàg Functions

Consider $f : [0, \infty) \to \mathbb{R}$ such that f has only jump discontinuities. Commonly we'll want f to be the equal to the left or right limit at the discontinuity in order to make f left or right continuous.

Definition 1.2

A function $f : [0, \infty) \to \mathbb{R}$ is càdlàg *if it is right-continuous with finite left limits. For such a process the* jump size *can be defined as*

$$\Delta f(t) := f(t) - f(t-). \tag{1.2}$$

Similarly f is càglàd *if it is left-continuous with finite right limits, with*

$$\Delta f(t) := f(t+) - f(t). \tag{1.3}$$

We now ask how far away a càdlàg function is from being a continuous function. This following proposition gives what can be considered a weaker version of uniform continuity for a càdlàg function on $[0, t]$.

Proposition 1.3

Consider $f : [0, \infty) \to \mathbb{R}$. Fix $t \geq 0$ and $\varepsilon > 0$. Then exists a partition $\Pi_\varepsilon^t = \{t_0, \dots, t_{k(\varepsilon,t)}\}$ of $[0, t]$ such that

1. *if f has only jump discontinuities*

$$\sup_{r,s \in (t_i, t_{i+1})} |f(s) - f(r)| \leq \varepsilon \quad \forall i = 0, \dots, t_{k(\varepsilon,t)} \tag{1.4}$$

2. *if f is càdlàg*

$$\sup_{r,s \in [t_i, t_{i+1})} |f(s) - f(r)| \leq \varepsilon \quad \forall i = 0, \dots, t_{k(\varepsilon,t)} \tag{1.5}$$

3. *if f is càglàd*

$$\sup_{r,s \in (t_i, t_{i+1}]} |f(s) - f(r)| \leq \varepsilon \quad \forall i = 0, \dots, t_{k(\varepsilon,t)} \tag{1.6}$$

For a uniformly continuous function we could just specify a maximum mesh size rather than a specific partition. But when we have jumps, we need to make sure the jumps are positioned at the end points of some partition. This is why the supremum in eq. (1.4) uses an open interval. With càdlàg and càglàd functions we have additional continuity assumptions that can be incorparated by making left or right side of the interval closed respectively.

Proof. Suppose that f has only jump discontinuities. For any $x \in (0, \infty)$, by existence of left limits there exists $x^- < x$ such that

$$\forall s \in (x^-, x),\ |f(t-) - f(s)| < \tfrac{1}{2}\varepsilon \implies \forall r, s \in (x^-, x),\ |f(r) - f(s)| < \varepsilon. \tag{1.7}$$

For any $x \in [0, \infty)$, by existence of right limits there exists $x^+ > x$ such that

$$\forall s \in (x, x^+),\ |f(s) - f(t+)| < \tfrac{1}{2}\varepsilon \implies \forall r, s \in (x, x^+),\ |f(r) - f(s)| < \varepsilon. \tag{1.8}$$

Then $\mathcal{U} := \{[0, 0^+)\} \cup \{(x^-, x^+) : t \in (0, t]\}$ is an open cover of $[0, t]$. By compactness this has a finite subcover $\mathcal{U}' := \{[0, 0^+)\} \cup \{((x_i^-, x_i^+) : i = 1, \dots, k\}$. WLOG

$$a = x_0 < x_1 < \cdots < x_k = b. \tag{1.9}$$

Since $\mathcal{U}'$ covers $[a, b]$, we must have $x_i^- < x_{i-1}^+$ for all $i = 1, \ldots, k$. Therefore we can pick $y_i \in (x_i^-, x_{i-1}^+) \cap (x_{i-1}, x_i)$. Then $\{x_0, y_1, x_1, \ldots, y_k, x_k\}$ is a partition satsifying proposition 1.3 on the previous page.

If further f is càdlàg then

$$\sup_{r,s\in(t_i,t_{i+1})} |f(r) - f(s)| = \sup_{r,s\in[t_i,t_{i+1})} |f(r) - f(s)| \tag{1.10}$$

thus the result follows from the result follows. The case where f is càglàd follows similarly. □

Importantly proposition 1.3 on the preceding page shows that càdlàg or càglàd functions are locally bounded.

Corollary 1.4

Suppose $f : [0, \infty) \to \mathbb{R}$ is càdlàg or càglàd. Then for all $t \geq 0$, $\sup_{s\leq t}|f(s)| < \infty$.

Proof. WLOG f is càglàd and for any $\varepsilon > 0$ let $\{t_1, \ldots, t_k\}$ be a mesh as in proposition 1.3 on the previous page. Then we have

$$\sup_{s\leq t}|f(s)| \leq k\varepsilon + \sum_{i=1}^{k} \Delta f(t_i) < \infty. \tag{1.11}$$

□

Another key corollary of proposition 1.3 on the preceding page is that it allows us to bound the number of jumps that f will make.

Corollary 1.5

Suppose $f : [0, \infty) \to \mathbb{R}$ has only jump discontinuities. Then on any finite interval $[0, t]$ and $\forall \varepsilon > 0$, f has finitely many discontinuities of size greater than ε. In particular f has only countably many discontinuities.

Since f has countably many jumps, the sum

$$\sum_{0\leq s\leq t} |\Delta f(s)| \tag{1.12}$$

actually makes sense (the order of summation does not matter since the summands are non-negative). Thus it makes sense to talk about the total length of jumps a càdlàg or càglàd function makes. The following proposition bounds that length.

Proposition 1.6

If $f : [0, \infty) \to \mathbb{R}$ is of finite variation and càdlàg or càglàd. Then

$$\sum_{0\leq s\leq t} |\Delta f(s)| \leq \mathrm{TV}_{[0,t]}(f) \tag{1.13}$$

where $\mathrm{TV}_{[0,t]}(f)$ *is the total variation of* f *on* $[0, r]$.

1.2 Semimartingales and Simples Stochastic Integrals

Now we bring back the probability space. Throughout this section let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be the underlying filtered probability space, unless specified otherwise all processes are functions $\Omega \times [0, \infty) \to \mathbb{R}$. We first make some obvious definitions and set up some notation.

Definition 1.7

X is a càdlàg process *if X is has càdlàg sample paths $t \mapsto X_t(\omega)$. Let $\mathbb{D}$ denote the space of càdlàg processes and b$\mathbb{D}$ denote the space of bounded càdlàg processes.*

Definition 1.8

X is a càglàd process *if X is has càglàd sample paths $t \mapsto X_t(\omega)$. Let $\mathbb{L}$ denote the space of càglàd processes and b$\mathbb{L}$ denote the space of bounded càglàd processes.*

Remark 1. Some sources assume adaptability in the definition of a càglàd and càglàd process. While we will try and explicitly say when a process is adapted, unless it's explicilty said otherwise it's usually safe to assume adaptedness.

Next we define the space of (simple) integrands.

Definition 1.9

A process $H : [0, \infty) \times \Omega \to \mathbb{R}$ is a simple predictable process *if H is of the form*

$$H_t = H_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i(\omega)\mathbb{1}_{(\tau_i,\tau_{i+1}]}(t) \tag{1.14}$$

where $0 = \tau_1 \leq \cdots \leq \tau_{n+1} < \infty$ are finite stopping times and $H_i \in L^\infty(\Omega, \mathcal{F}_{\tau_i}, \mathbb{P})$. Let $\mathbf{S}$ denote the space of all simple stochastic integrands. $\mathbf{S}_u$ refers to the space $\mathbf{S}$ with norm

$$\|H\|_u = \sup\{H_t(\omega) : (t, \omega) \in [0, \infty) \times \Omega\}. \tag{1.15}$$

Remark 2. All process $H \in \mathbf{S}$ are càglàd. An important property of such processes is that they are *previsible.* To understand what this means consider the σ-algebra defined by

$$\mathcal{P} := \sigma(\{A \times (s, t] : s, t \geq 0, s \leq t, A \in \mathcal{F}_s\}). \tag{1.16}$$

This called the *previsible σ-algebra.* H is previsible in that it is measurable with respect to the previsble σ-algebra. In fact it is unnecessary to assume that the right limits exist to show previsibility, we only need left continuity and adaptedness.

To begin with, we'll start by defining the integral of these simple processes. Then we'll extend the definition to a wider array of processes by density and continuity arguments.

Also to begin with, we'll define integrals from $t = 0$ to $t = \infty$. Because of this the integral will be a random variable instead of a random process. In particular for simple integrands the integral belongs to the following space.

Definition 1.10

Let $\mathbf{L}^0$ be the space of bounded random variables with topology defined by convergence in probability.

Everything is now in place to define what the integral is.

Definition 1.11

Given a process X, the (simple) integral *is the linear operator $I_X : \mathbf{S}_u \to \mathbf{L}^0$ where (assuming H takes the form in eq.* (1.14) *on the previous page)*

$$I_X(H) := \int_0^\infty H(t)\,\mathrm{d}X(t) := H_0 X_0 + \sum_{i=0}^{n} H_i(X_{\tau_{i+1}} - X_{\tau_i}). \tag{1.17}$$

The above definition works for any process X, but in order to extend the definition of the integral using density arguements we will need the operator I_X to be continuous. This places restrictions on which càdlàg processes we can integrate against. In particular we will only integrate agains *total semimartingales*, as defined below.

Definition 1.12

An process X is a total semimartingale *if X is adapted and $I_X : \mathbf{S}_u \to \mathbf{L}_0$ is continuous.*

Now we define the integral up to a finite time $t \in [0, \infty)$. This is done by stopping the process we are integrating against. We will use the notation $(X_t^T)_{t \geq 0}$ for the stopped process $(X_{T \wedge t})_{t \geq 0}$.

Definition 1.13

Given a process X, the (simple) integral *is given by $J_X : \mathcal{S} \to \mathbb{D}$ where*

$$\int_0^t H(t)\,\mathrm{d}X(t) := I_{X^t}(H). \tag{1.18}$$

Correspondingly we no longer need I_X to be continuous, just I_{X^t} for all finite times $t \in [0, \infty)$. This gives rise the the concept of *semimartingales*.

Definition 1.14

An process X is a semimartingale *if X is adapted and X^t, the stopped process at time t, is a total semimartingale for all $t \geq 0$.*

Remark 3. The class of semimartingales is a vector space since for processes X, Y and $\mu, \nu \in \mathbb{R}$ we have $I_{\mu X + \lambda Y} = \mu I_X + \lambda I_Y$.

This looks remarkable like the localisation condition given for local martingales, just with deterministic times instead of stopping times. Is there any value in defining a local semimartingale, i.e. a process that is a semimartingale when stopped on a specific sequence of stopping times almost surely converging to infinity? The answer is no, and this is since the definition of semimartingales is already localised. We now prove this formally.

Proposition 1.15

Let X be an adapted process and (T_m) be a sequence of non-negative random variables such that $T_m \to \infty$ almost surely. Suppose that X^{T_m} is a semimartingale for all m. Then X is a semimartingale.

Proof. Suppose that $\|H^{(n)}\|_u \to 0$ where $H^{(n)} \in \mathbf{S}_u$. Note on the event $T_m \leq t$ we have $X^t = (X^{T_m})^t$. Therefore for any $\varepsilon > 0$ and $m \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}(|I_{X^t}(H^{(n)})| \geq \varepsilon) &\leq \mathbb{P}(|I_{(X^{T_m})^t}(H^{(n)})| \geq \varepsilon, T_m \leq t) \\ &\quad + \mathbb{P}(|I_{X^t}(H^{(n)})| \geq \varepsilon, T_m > t) \qquad (1.19)\\ &\leq \mathbb{P}(|I_{(X^{T_m})^t}(H^{(n)}| \geq \varepsilon) + \mathbb{P}(T_m > t). \qquad (1.20)\end{aligned}$$

For any $\delta > 0$ since $T_m \to \infty$ a.s. there exists a m_0 such that $\mathbb{P}(T_{m_0} > t) < \delta$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}(|I_{X^t}(H^{(n)})| \geq \varepsilon) &\leq \limsup_{n\to\infty} \mathbb{P}(|I_{(X^{T_{m_0}})^t}(H^{(n)})| \geq \varepsilon) + \delta \qquad (1.21)\\ &= \delta \qquad (1.22)\end{aligned}$$

since $X^{T_{m_0}}$ is a semimartingale. Since δ is arbitrary, this shows $I_{X^t}(H^{(n)}) \to 0$ in probability, hence X is a semimartingale. □

Now we show J_X is actually continuous.
If X is a semimartingale then $J_X : \mathbf{S}_{\text{ucp}} \to \mathbb{D}_{\text{ucp}}$ is continuous.

Proof. See lecture notes □

So what kind of processes are semimartingales? We'll give some classes of processes that are always semimartingales.

Proposition 1.16

Suppose that X is an càdlàg adapted process.

1. *If X is constant then X is a semimartingale*
2. *If X is of finite variation then X is a semimartingale*
3. *If X is a local martingale then X is a semimartingale*

Proof. Suppose that $H^{(k)} \in \mathbf{S}_u$ and $\|H^{(k)}\|_u \to 0$ as $n \to \infty$.

Part (1): If X is constant then $I_{X^t}(H^{(k)}) = H_0^{(k)} X_0$ which will converge to 0 almost surely as $n \to \infty$ and hence in probability. Thus X is a semimartingale.

Part (2): Suppose X is of finite variation. Since constant processes are semimartingales and semimartingales are vector spaces, we can assume $X_0 = 0$ by subtracting off the initial value. Then $|I_{X^t}(H^{(k)})| \leq \|H^{(k)}\|_u \,\mathrm{TV}_{[0,t]}(X) \to 0$ almost surely and thus in probability.

Part (3): First suppose X is a martingale bounded by $M > 0$. Like for finite variation processes, WLOG $X_0 = 0$. Then

$$\mathbb{E}[I_{X^t}(H^{(k)})^2] = \sum_{i,j} \mathbb{E}[H_i H_j (X^t_{\tau_{i+1}} - X^t_{\tau_i})(X^t_{\tau_{j+1}} - X^t_{\tau_j})]. \tag{1.23}$$

If $i < j$, then $H_i, H_j, (X^t_{\tau_{i+1}} - X^t_{\tau_i})$ and $X^t_{\tau_j}$ are $\mathcal{F}_{\tau_j}$ measurable. Therefore

$$\begin{aligned}
&\mathbb{E}[H_i H_j (X^t_{\tau_{i+1}} - X^t_{\tau_i})(X^t_{\tau_{j+1}} - X^t_{\tau_j})] \\
=&\mathbb{E}[H_i H_j (X^t_{\tau_{i+1}} - X^t_{\tau_i})\mathbb{E}[X^t_{\tau_{j+1}} - X^t_{\tau_j} \mid \mathcal{F}_{\tau_j}]] = 0 \qquad (1.24)
\end{aligned}$$

by the martinagle property of X^t. Thus the cross terms die out and we have

$$\begin{aligned}
\mathbb{E}[I^t_X(H^{(k)})^2] &= \sum_i \mathbb{E}[H_i^2 (X^t_{\tau_{i+1}} - X^t_{\tau_i})^2] && (1.25)\\
&\leq \|H\|_u^2 \sum_i \mathbb{E}[(X^t_{\tau_{i+1}} - X^t_{\tau_i})^2] = \|H\|_u^2 \mathbb{E}[(X^t_{\tau_n})^2] && (1.26)\\
&\leq M^2 \|H\|_u^2 \quad \text{as } n \to \infty. && (1.27)
\end{aligned}$$

Since we have L^2 convergence, we will have convergence in probability. Therefore X is a semimartingale. For a general càdlàg local martingale X, since càdlàg paths are locally bounded we can localise X to a bounded martingale. Thus X is a semimartingale by proposition 1.15 on the preceding page. □

If you've done a past course on stochastic analysis, you'll have seen a different definition of a semi-martingale which we will now give.

Definition 1.17

A process X is a classical semimartingale *or* decomposable *if there exists a local martingale M, a finite variation process N and $X_0 \in \mathcal{F}_0$ such that $M_0 = N_0 = 0$ and*

$$X_t = X_0 + M_t + N_t. \tag{1.28}$$

From proposition 1.16 on the previous page and the fact that set of semimartingales is a vector space, we know that decomposable processes are semimartingales. It turns out the converse is also true!

Proposition 1.18

Suppose X is a càdlàg adapted process. Then X is a semimartingale if and only if

X is decomposable.

The converse direction is much harder to prove, and can be found in Protter's book 'Stochastic Integration and Differential Equations'.

1.3 Integrating

Definition 1.19

A sequence of process $H^{(n)}$ converges uniformly on compact sets in probability *(u.c.p.) to H*

$$|H^n - H|_t^* \to H \quad \textit{where } |X|_t^* := \sup_{s \le t} |X_s|. \tag{1.29}$$

Remark 4. The topology of u.c.p. convergence can be characterised by the metric

$$d(H, G) := \sum_{n=1}^{\infty} 2^{-n} \mathbb{E}[|H - G|_n^* \wedge 1]. \tag{1.30}$$

We write $\mathbb{D}_{\text{ucp}}$ for the space $\mathbb{D}$ equipped with the topology of u.c.p. convergence, and similarly for other space of processes. $\mathbb{D}_{\text{ucp}}$ and $\mathbb{L}_{\text{ucp}}$ are complete metric spaces.

Theorem 1.20

$\mathbf{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$.

Proof. We first show $\mathrm{b}\mathbb{L}_{\text{ucp}}$ is dense in $\mathbb{L}_{\text{ucp}}$. Given any $X \in \mathbb{L}$ let

$$X_t^n := X_{t \wedge \tau_n} \mathbb{1}_{\tau_n > 0} \quad \text{where } \tau_n := \inf\{t \ge 0 : X_t > n\}. \tag{1.31}$$

By left continuity, $\sup_{t \ge 0} |X_t^n| \le n$. The purpose of the indicator is for the case where $X_0 \ge n$. Then

$$\mathbb{P}(|X^n - X|_t^* > 0) = \mathbb{P}(\tau_n < t) \to 0 \quad \text{as } n \to \infty \tag{1.32}$$

since càglàd processes are bounded on bounded interval $\square$

Since S_{ucp} is dense in $\mathbb{D}_{\text{ucp}}$, $\mathbb{L}_{\text{ucp}}$ is a completely metrizable space and $J_X : \mathbf{S}_{\text{ucp}} \to \mathbb{L}_{\text{ucp}}$ is continuous, we can extend it to a continuous operator $J_X : \mathbb{L}_{\text{ucp}} \to \mathbb{D}_{\text{ucp}}$.

Properties

$$\Delta\Big(\int H \, \mathrm{d}X\Big)_t = H_t \Delta X_t \tag{1.33}$$

$$\int H\Big(\int G \, \mathrm{d}X\Big) = \int HG \, \mathrm{d}X \tag{1.34}$$

Theorem 1.21

If X is of finite variation, then the integral agrees with the Lebesgue-Stieljes integration.

Theorem 1.22

Let X be a semimartingale. Let $H \in \mathbb{L} \cup \mathbb{D}$.

$$\Pi_n = \left\{\tau_1^n, \dots, \tau_{k_n+1}^n\right\} \tag{1.35}$$

be a sequence of random partitions tending to the identity:

1. $\limsup_{n\to\infty} \sup_{i=1,\dots,k_n+1} \tau_i^n = \infty$ *almost surely*
2. *The mesh tends to 0 almost surely*

Then

$$\int_0^{\cdot} H_{s-}\, \mathrm{d}X_s = \mathit{lim\ ucp}_{n\to\infty} \sum_{i=1}^{k_n} H_{\tau_i^n}(X_{\cdot}^{\tau_{i+1}^n} - X_{\cdot}^{\tau_i^n}) \tag{1.36}$$

1.4 Quadratic Variation

Definition 1.23

If X and Y are semimartingales then

$$[X]_t := X_t^2 - 2\int_0^t X_{s-}\, \mathrm{d}X_s \tag{1.37}$$

$$[X,Y]_t := X_t Y_t \int_0^t X_{s-}\, \mathrm{d}Y_s - \int_0^t Y_{s-}\, \mathrm{d}X_s \tag{1.38}$$

Remark 5. This definition can be motivated by integration by parts. Note $[\cdot]$ will be different from $\langle\cdot\rangle$.

Proposition 1.24

If X is continuous and FV, then $[X] = X_0^2$

Proof. Lebesgue-Stieljes theory and continuity gives $\int_0^t X_{s-}\, \mathrm{d}X_s = \int_0^t X_s\, \mathrm{d}X_s = \frac{1}{2}(X_t^2 - X_0^2)$ □

Theorem 1.25

If $X \in \mathbb{D}$ is a semimartingale, then $[X] \in \mathbb{D}$ and

1. $[X]_0 = X_0^2$, $\Delta[X] = (\Delta X)^2$
2. *If $\Pi^n = \{\tau_1^n, \dots, \tau_{k_n+1}^n\}$ is a sequence of random partitions tending to identity then*

$$X_0^2 + \sum_{i-1}^{k_n}(X^{\tau_{i+1}^n} - X^{\tau_i^n})^2 \to [X] \tag{1.39}$$

u.c.p. as $n \to \infty$.

3. *$[X]$ is F.V.*
4. *$[X]^\tau = [X^\tau] = [X^\tau, X]$ for all stopping times τ*

Proof. By definition it is the sume of two càdlàg functions.

$$(\Delta X_t)^2 = (X_t - X_{t-})^2 = X_t^2 + X_{t-}^2 - 2X_t X_{t-} \tag{1.40}$$
$$= (X_t^2 - X_{t-}^2) - 2X_{t-}(X_t - X_{t-}) = \Delta X_t^2 - 2X_{t-}\Delta X_t \tag{1.41}$$
$$= \Delta X_t^2 - 2\Delta \left(\int X_{\cdot -} \,\mathrm{d}X \right)_t = \Delta[X]_t \tag{1.42}$$

WLOG $X_0 = 0$

$$(X^{\tau_{i+1}^n} - X^{\tau_i^n})^2 = (X^{\tau_{i+1}^n})^2 - (X_t^{\tau_i^n})^2 - 2X_t^{\tau_i^n}(X^{\tau_{i+1}^n} - X^{\tau_i^n}) \tag{1.43}$$
$$\sum = (X^{\tau_{k_n+1}^n})^2 - 2\sum_{i=1}^{k_n} X_{\tau_i}(X_t^{\tau_{i+1}} - X^{\tau_i^n}) \tag{1.44}$$
$$\to [X] \tag{1.45}$$

u.c.p. □

Proposition 1.26 (*Kunita-Watanabe Inequality*)

$$\int |H||G||d[X,Y]| \leq \left(\int |H|^2 \,\mathrm{d}[X] \right)^{1/2} \left(\int |G|^2 \,\mathrm{d}[X] \right)^{1/2} \tag{1.46}$$
$$\left[\int H \,\mathrm{d}X, \int G \,\mathrm{d}Y \right] = \int HG \,\mathrm{d}[X,Y] \tag{1.47}$$

Proposition 1.27 (*Lebesgue Stieljes*)

Let $t \mapsto A_t$ be right-continuous and GV (on compact). If $f \in C^1(\mathbb{R})$ then

$$f(A_t) - f(A_0) = \int_{(0,t]} f'(A_s') \,\mathrm{d}A_s + \sum_{0<s\leq t} (\Delta A_s - f'(A_s)\Delta A_s) \tag{1.48}$$

Proposition 1.28 (*Pure jump quadratic variation*)

1. *If $A \in \mathcal{D}$ is FV, then $[A]$ is purely discontinuous (pure jump) with*

$$[A]_t = [A]_0 + \sum_{0<s\leq t} (\Delta A_t)^2 \tag{1.49}$$

2. If X, Y are semimartingales and $[Y]^c = 0$, then

$$[X, Y] = [X, Y]_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s \tag{1.50}$$