



Online learning of a panoply of quantum objects

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Tomography

- Subsets of positive semidefinite operators.

$$D(\mathcal{X}) := \{\rho \in \text{Pos}(\mathcal{X}) : \text{Tr}(\rho) = 1\} ,$$
$$C(\mathcal{X}, \mathcal{Y}) := \{J \in \text{Pos}(\mathcal{Y} \otimes \mathcal{X}) : \text{Tr}_{\mathcal{Y}}(J) = \mathbb{1}_{\mathcal{X}}\} .$$

- Exponential number of parameters
 - n qubits \rightarrow dimension $D = 2^n$ operator.
- **Physical** objects

- How many **physical** copies of an object to learn its description?
 1. Precision (w.r.t. some norm)
 2. Destructive nature of measurements
- Quantum states: $O(D^2)$, but also $\Omega(D^2)$.
- Are the amplitudes *really* there?

Shadow Tomography

Changing the task

- **Shadow** tomography [Aar18].
 - *Input*: many same quantum objects ρ
 - *Input*: M quantum co-objects E_i
 - *Output*: b_i such that $b_i \approx \langle E_i, \rho \rangle$
- Knowing $b_i \rightarrow$ reasonably good info about ρ .
- Quantum money, advice, copy-protected software, etc.
- Simple approaches (quantum states):
 - Full tomography: $O(D^2)$
 - Just do the measurements: $O(M)$

- **Online learning**: iteratively refine hypothesis, given challenging measurements
 - *Gentle search*: find challenging measurements
- Quantum states:

$$\tilde{O}(\log(D) \cdot \epsilon^{-2}) \times \tilde{O}(\log^4(M) \cdot \epsilon^{-2}) = \tilde{O}(\log(D) \cdot \log^4(M) \cdot \epsilon^{-4}) .$$

- Upper bound on the number of *mistakes*.

Online Learning

- First defined in **machine learning** literature.
- Metric of success borrowed from **game theory**.
- Framework closely tied to **statistical learning theory** and **convex optimization**.

Convex optimization: single interaction

1. Receive convex set \mathcal{K} and convex function f .
2. Choose $\mathbf{x} \in \mathcal{K}$ and suffer loss $f(\mathbf{x})$.

Online convex optimization: iterative process

1. Choose $\mathbf{x}_t \in \mathcal{K}$ and suffer loss $f_t(\mathbf{x}_t)$.
2. Receive convex loss function f_t .
3. Repeat.

- Choose \mathbf{x}_t before f_t is known.
- Losses f_t can be different in general.
- Losses f_t can be selected adversarially.

Loss functions f_t are

- Bounded
- **Convex**

Decision set \mathcal{K} is

- Bounded/structured
- **Convex**

Unbounded f_t

$$f_1(x) = \begin{cases} 0 & x \neq \mathbf{x}_t, \\ \infty & \text{otherwise.} \end{cases}$$

Unbounded \mathcal{K}

Assign loss 1 to all \mathbf{x}_t , while setting aside some strategies with 0 loss. Since \mathcal{K} is infinite, the latter will never run out.

- Adversary can still force the learner to incur constant loss at every round.
- **Metric:** total accumulated loss *versus best fixed action*.

$$\mathcal{R}_T^{\mathcal{A}} = \sup_{f_1, \dots, f_T \subseteq \mathcal{F}} \left\{ \sum_{t=1}^T f_t(\mathbf{x}_t^{\mathcal{A}}) - \min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x}) \right\} .$$

- \mathcal{A} and \sup are typically clear from context and omitted.
- Interested in an **upper bound** for the **worst-case** regret.
- **Goal:** sublinear regret bound.

- Prediction from expert advice (experts problem)
- Online spam filtering
- Online shortest paths
- Recommendation systems
- Portfolio selection
- etc...

Algorithm 1 Regularized Follow-the-Leader (RFTL/FTRL)

Input: $T, \eta > 0$, convex regularization function R , a convex and compact set $\mathcal{K} \subseteq \text{Herm } \mathcal{X}$.

1: Set initial hypothesis $\omega_1 \leftarrow \arg \min_{\varphi \in \mathcal{K}} \{R(\varphi)\}$.

2: **for** $t \leftarrow 1$ **to** T **do**

3: Predict ω_t and incur cost $f_t(\omega_t)$, where $f_t : \mathcal{K} \rightarrow \mathbb{R}$ is convex.

4: Let ∇_t be a subgradient of f_t at ω_t .

5: Update decision according to the RFTL rule

$$\omega_{t+1} \leftarrow \arg \min_{\varphi \in \mathcal{K}} \left\{ \eta \sum_{s=1}^t \langle \nabla_s, \varphi \rangle + R(\varphi) \right\} .$$

6: **end for**

Theorem ([ACH⁺18] Theorems 2 and 3)

Suppose that

- *Loss functions are B -Lipschitz*

RFTL can be applied to n -qubit quantum state tomography for a regret bound of

$$\mathcal{R}_T \leq 2B\sqrt{(2\log 2)nT}$$

Results

Algorithm 2 RFTL for Online Learning of Quantum Objects

Input: $T, \eta > 0$, a convex and compact set $\mathcal{K} \subseteq \text{Pos}(\mathcal{X})$, and a bounded set $\mathcal{E} \subseteq \text{Herm}(\mathcal{X})$.

- 1: Set initial hypothesis $\omega_1 \leftarrow \arg \max_{\varphi \in \mathcal{K}} \{S(\varphi)\}$.
- 2: **for** $t \leftarrow 1$ **to** T **do**
- 3: Predict ω_t and incur cost $f_t(\omega_t) := \ell_t(\langle E_t, \omega_t \rangle)$ with $\ell_t : \mathbb{R} \rightarrow \mathbb{R}$ and $E_t \in \mathcal{E}$.
- 4: Let $\ell'_t(x)$ be a sub-derivative of ℓ_t with respect to x and define

$$\nabla_t \leftarrow \ell'_t(\langle E_t, \omega_t \rangle) E_t .$$

- 5: Update decision according to the RFTL rule

$$\omega_{t+1} \leftarrow \arg \min_{\varphi \in \mathcal{K}} \left\{ \eta \sum_{s=1}^t \langle \nabla_s, \varphi \rangle - S(\varphi) \right\} .$$

- 6: **end for**

Theorem

Suppose that

- *Trace of objects is bounded by A*
- *Loss functions are B -Lipschitz*
- *Operator norm of co-objects is bounded by C*
- *The diameter of \mathcal{K} w.r.t. S is $D^2 := \max_{\varphi, \varphi' \in \mathcal{K}} \{S(\varphi') - S(\varphi)\}$.*
- *There exists $\alpha > 0$ such that $\alpha \mathbb{1} \in \mathcal{K}$*

RFTL for Online Learning of Quantum States achieves a regret bound of

$$\mathcal{R}_T \leq 4BCD\sqrt{AT}.$$

Example corollaries

Objects	Co-objects	Regret bound
States	Effects	$4B\sqrt{\ln(\dim(\mathcal{X}))}\sqrt{T}$
Effects	States	$4B\dim(\mathcal{X})\sqrt{e^{-1} + \log(\dim(\mathcal{X}))}\sqrt{T}$
Gram matrices	Unit ball	$4Bn\sqrt{\log(n)}\sqrt{T}$
Channels	Interactive msmts.	$4B\dim(\mathcal{X})\sqrt{\log(\dim(\mathcal{X})) + \log(\dim(\mathcal{Y}))}\sqrt{T}$
Interactive msmts.	Channels	$4B\dim(\mathcal{X}\mathcal{Y})\sqrt{\log(\dim(\mathcal{X})) + \log(\dim(\mathcal{Y}))}\sqrt{T}$
Strategies	Co-strategies	—"
Co-strategies	Strategies	—"—

- For quantum strategies and co-strategies,

$$\dim(\mathcal{X}) = \prod_{i=1}^n \dim(\mathcal{X}_i) ,$$

$$\dim(\mathcal{Y}) = \prod_{i=1}^n \dim(\mathcal{Y}_i) .$$

Proof Sketch

Challenges

1. Variable traces
2. Derivatives of complex operators
3. Choice of regularizer

Regret bound

Lemma ([Haz16] Lemma 5.3)

RFTL guarantees

$$\mathcal{R}_T \leq \sum_{t=1}^T \langle \nabla_t, \omega_t - \omega_{t+1} \rangle + \frac{1}{\eta} D^2 .$$

Lemma

The function

$$\Phi_E(X) := \langle E, X \rangle + \text{Tr}(X \log(X))$$

is Fréchet-differentiable with

$$\nabla \Phi_E(X) = E + \mathbb{1} + \log(X) .$$




Theorem

For any positive semidefinite P and Q ,

$$\frac{1}{4} \|P - Q\|_{\text{Tr}}^2 \leq \max\{\text{Tr}(P), \text{Tr}(Q)\} [D(P\|Q) - \text{Tr}(P - Q)] .$$

- Also a tightening of Klein's inequality

- Sublinear regret bound for learning general quantum **objects**
- Additional matrix analysis results
- Generalized Pinsker's inequality
- Future directions:
 - Relax assumptions
 - Connect to shadow tomography
 - Other algorithms?

-  Scott Aaronson.
Shadow tomography of quantum states.
Technical Report arXiv:1711.01053, arXiv, November 2018.
arXiv:1711.01053 [quant-ph] type: article.
-  Scott Aaronson, Xinyi Chen, Elad Hazan, Satyen Kale, and Ashwin Nayak.
Online learning of quantum states.
In *Advances in Neural Information Processing Systems*, volume 31. Curran Associates, Inc., 2018.
-  Elad Hazan.
Introduction to online convex optimization.
Foundations and Trends® in Optimization, 2(3-4):157–325, 2016.

Thank you!

Questions?