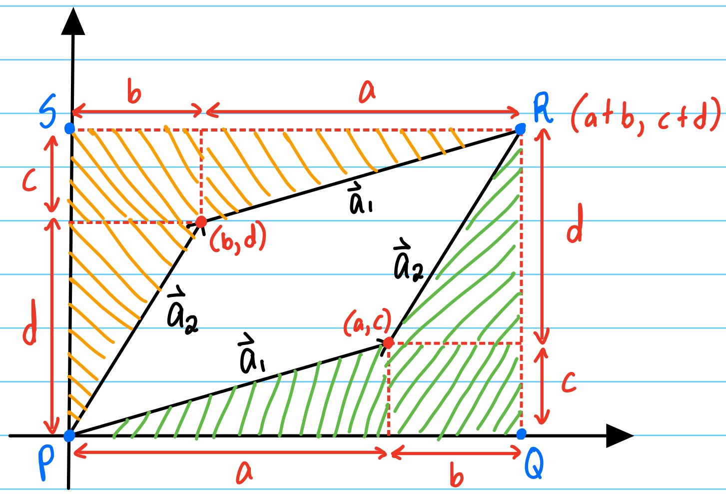
Discussion 5B :)

Anish's 16A Discussion
Office Hours: Tuesday 3-4 @Cory 144MA Friday 4-5

Determinant:

- The determinant is the n-dimensional volume formed by the column vectors
- For $n=2$, area of the parallelogram formed by 2 vectors $= \det(A)$

Proof: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \vec{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ $\quad \vec{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$

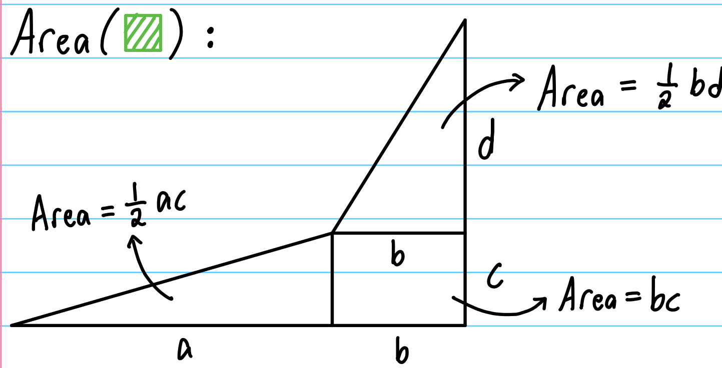


Area of parallelogram = Area(PQRS) − Area(▨) − Area(▨)

By symmetry, we can see that Area(▨) = Area(▨)

$\Rightarrow$ Area of parallelogram = Area(PQRS) − 2 · Area(▨)

Area(▨):



Area(▨) $= \frac{1}{2}ac + \frac{1}{2}bd + bc$

$$\text{Area}(PQRS) = (a+b)(c+d) - 2\left(\tfrac{1}{2}ac + \tfrac{1}{2}bd + bc\right)$$
$$= \cancel{ac} + bc + ad + \cancel{bd} - \cancel{ac} - \cancel{bd} - 2bc$$
$$= ad - bc = \det(A)$$

Proof Complete!

EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 5B

1. **Mechanical Determinants**

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Pattern Match:

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = (2)(3) - (0)(0) = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Pattern Match (again):

$$\det\left(\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}\right) = 2 \cdot \det\left(\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\right)$$

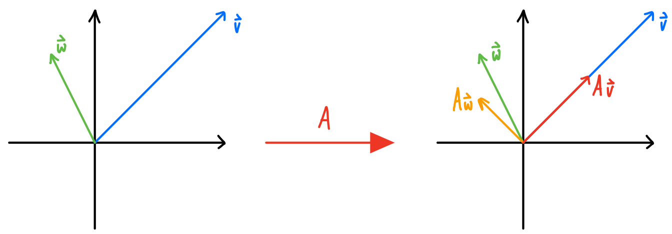
$$= 2(0+4) + 3(10+1) + 1(8-0)$$

$$= 8 + 33 + 8$$

$$= 49$$

Eigenvalues/Eigenvectors:

Geometrical Interpretation:



transformation matrix

eigenvalue

$$A\vec{v} = \lambda\vec{v}$$

eigenvector

$\{\lambda = 0, \ \vec{v} = \vec{0}\}$ is a trivial pair

Under linear transformations like these, vectors typically get knocked off their span; however, some remain on their original span

→ These vectors, who are only scaled by the transformation and thus remain on their original span are called "eigenvectors"; in the example above, $\vec{v}$ is an eigenvector of the transformation A whereas $\vec{w}$ is not

→ Each eigenvector has an associated eigenvalue, which is simply the scaling factor of the eigenvector

Cool Fact: The eigenvector of a 3D rotation is the axis of rotation! Why is this the case?

Solving for Eigenvalues of a matrix A: $\det(A - \lambda I) = 0$

Solving for corresponding Eigenvectors: $\vec{v} \in \text{null}(A - \lambda I)$

Remember: eigenvalues and eigenvectors come in pairs

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and their associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Do you observe anything about the eigenvalues and eigenvectors?

Eigenvalues: $\det(M - \lambda I)$

$$= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(9-\lambda) - (0)(0) = 0$$

$$(1-\lambda)(9-\lambda) = 0$$

$$\lambda = 1, 9$$

Eigenvectors: $\text{null}(M - \lambda I)$

For $\lambda = 1$:

$$\text{null}(M - \lambda I) = \text{null}\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 8 & 0 \end{array}\right] \xrightarrow{\frac{1}{8}R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$x_1 = t$

$0x_1 + x_2 = 0$

$x_2 = 0t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\vec{v}_{\lambda_1} \in \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}$$

eigenspace

eigenvector corresponding to $\lambda = 1$

For $\lambda = 9$:

$$\text{null}(M - \lambda I) = \text{null}\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} -8 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{8}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + 0x_2 = 0$

$x_1 = 0t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\vec{v}_{\lambda_2} \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$$

eigenspace

eigenvector corresponding to $\lambda = 9$

★ The eigenvalues of a diagonal matrix can be obtained by observation

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix **M** and their associated eigenvectors.

(b) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Eigenvalues: $\det(M - \lambda I)$

$$= \det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right)$$

$$= (-\lambda)(-3-\lambda) - (1)(-2)$$

$$= 3\lambda + \lambda^2 + 2$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+2)(\lambda+1) \longrightarrow \lambda = -2, -1$$

Eigenvectors: $\text{null}(M - \lambda I)$

For $\lambda = -2$:

$$\text{null}(M - \lambda I) = \text{null}\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_1 \to R_1]{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + \frac{1}{2}x_2 = 0$

$x_1 = -\frac{1}{2}t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\underbrace{\vec{V}_{\lambda_1} \in \text{span}\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}}$$

eigenspace

eigenvector corresponding to $\lambda = -2$

For $\lambda = -1$:

$$\text{null}(M - \lambda I) = \text{null}\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \text{null}\left(\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + x_2 = 0$

$x_1 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\underbrace{\vec{V}_{\lambda_2} \in \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$$

eigenspace

eigenvector corresponding to $\lambda = -1$

3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

$A\vec{v} = \lambda\vec{v}$

$I\vec{v} = \lambda\vec{v}$, $I^{n\times n}$ and $\vec{v}$ is $n \times 1$

$I\vec{v} = 1 \cdot \vec{v}$ → eigenvalue $\lambda = 1$

$I\vec{v} = \vec{v}$

$\vec{v} \in \mathbb{R}^n$ → all vectors are eigenvectors with corresponding eigenvalues of $\lambda = 1$

Intuition: multiplying by I leaves vectors untouched

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} = D$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{b}_i$ produces $d_i\vec{b}_i$, that is, $D\vec{b}_i = d_i\vec{b}_i$. Thus, the e-values of D are the diagonal entries d_i, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{b}_i$.

$\lambda_1 = d_1$ $\quad \lambda_2 = d_2$ $\quad \ldots\ldots$ $\quad \lambda_n = d_n$

span: $\left\{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}$ $\quad \left\{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right\}$ $\quad \left\{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right\}$

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

Yes:

1) Rotation by $0°$ or any integer multiple $360°$, which yields a rotation matrix $R = I$
   - Yields $\lambda = 1$

2) Rotation by $180°$ or any angle of $180° + n \cdot 360°$ for any integer n, which yields a rotation matrix $R = -I$
   - Yields $\lambda = -1$ (flips vectors)

(d) **(PRACTICE)** Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Practice @ Home

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Yes: $\lambda = 1, -1$ by inspection since T is diagonal

Look at discussion solutions for more intuition behind reflection matricies and their eigenvalues

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

If $\lambda = 0$ is an eigenvalue of M, then we have some subspace that will be mapped to $\vec{0}$ by M:

$$(M - \lambda I)\vec{x} = \vec{0}$$

$$(M - 0 \cdot I)\vec{x} = \vec{0}$$

$$M\vec{x} = \vec{0},$$

which is null(M)

$\Rightarrow$ no unique solution to $M\vec{x} = \vec{b}$ since $M\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Practice @ Home

tinyurl.com/16Anish

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