Discussion 5A !!

Anish's 16A Discussion

Office Hours: Tuesday 3-4

@Cory 144MA Friday 4-5

Subspaces:

A subspace U of V is a subset of the vectors in V that satisfy:

1) Contains $\vec{0}$ (zero-vector)
2) Closed under vector addition $\longrightarrow$ If $\vec{u}_1, \vec{u}_2 \in U \Rightarrow \vec{u}_1 + \vec{u}_2 \in U$
3) Closed under scalar multiplication $\longrightarrow$ If $\vec{u}_1 \in U$, $\alpha \in \mathbb{R} \Rightarrow \alpha\vec{u}_1 \in U$

Combined $\vec{u}_1, \vec{u}_2 \in U$
$\Rightarrow \alpha\vec{u}_1 + \beta\vec{u}_2 \in U$,
where $\alpha, \beta \in \mathbb{R}$

1. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

Yes!

a subspace of $\mathbb{R}^3$? Why or why not?

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

$$\vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0} \text{ when } c = d = 0 \quad \checkmark \text{ contains } \vec{0}$$

$$\vec{v}_1 = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + d_1\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \vec{v}_2 = c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + d_2\begin{bmatrix}1\\0\\1\end{bmatrix} \quad (\vec{v}_1, \vec{v}_2 \in V)$$

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2)\begin{bmatrix}1\\1\\1\end{bmatrix} + (d_1 + d_2)\begin{bmatrix}1\\0\\1\end{bmatrix} = c_3\begin{bmatrix}1\\1\\1\end{bmatrix} + d_3\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \begin{aligned} c_3 &:= c_1 + c_2 \\ d_3 &:= d_1 + d_2 \end{aligned}$$

$\Rightarrow (\vec{v}_1 + \vec{v}_2) \in V$ and $(c_3, d_3 \in \mathbb{R})$ ✓ Closed under vector addition

$$\alpha\vec{v}_1 = \alpha c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha d_1\begin{bmatrix}1\\0\\1\end{bmatrix} = c_4\begin{bmatrix}1\\1\\1\end{bmatrix} + d_4\begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \begin{aligned} c_4 &:= \alpha c_1 \\ d_4 &:= \alpha d_1 \end{aligned}$$

$\alpha \in \mathbb{R}$

$\Rightarrow (\alpha\vec{v}_1 \in V)$ and $(c_4, d_4 \in \mathbb{R})$ ✓ Closed under scalar multiplication

EECS 16A Designing Information Devices and Systems I
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$$\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (\vec{v}_1, \vec{v}_2 \in V)$$

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} c_3 := c_1 + c_2 \\ d_3 := d_1 + d_2 \end{matrix}$$

$$\Rightarrow (\vec{v}_1 + \vec{v}_2) \in V \text{ and } (c_3, d_3 \in \mathbb{R}) \quad \checkmark \text{ Closed under vector addition}$$

$$\alpha \vec{v}_1 = \alpha c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = c_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} c_4 := \alpha c_1 \\ d_4 := \alpha d_1 \end{matrix}$$

$\alpha \in \mathbb{R}$

$$\Rightarrow (\alpha \vec{v}_1 \in V) \text{ and } (c_4, d_4 \in \mathbb{R}) \quad \checkmark \text{ Closed under scalar multiplication}$$

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that when multiplied with the matrix result in the zero vector.

For the following matrices, answer the following questions:

Column Spaces and Null Spaces:

$$A = \begin{bmatrix} | & | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \\ | & | & | & & | \end{bmatrix} \quad A \in \mathbb{R}^{m \times n} \quad \Rightarrow \quad \text{col}(A) = \text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n\}$$

Example: $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \text{col}(A) = \text{span}\left\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right\} = \mathbb{R}^2$

Dimension: $\dim(\text{col}(A))$ = minimum # of vectors needed to span $\text{col}(A)$
= dimension of the space $\text{col}(A)$

Intuition: how many different directions can I move in using these vectors

$\text{Null}(A) = \{\vec{v} \mid A\vec{v} = \vec{0},\ \vec{v} \in \mathbb{R}^n\}$ (All vectors that get mapped to zero-vector)

Example: $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 4 \end{bmatrix}$ We solve for $A\vec{v} = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 1 & 3 & 0 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} \Rightarrow \vec{v} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

free variables

$r \to$ free, $s \to$ free

$q + s = 0 \Rightarrow q = -s$

$p + 2q + 3s = 0 \Rightarrow p + 2(-s) + 3(s) = 0 \Rightarrow p = -s$

$$\text{Null}(A) = \left\{ \vec{v} \;\middle|\; \vec{v} = r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad s, r \in \mathbb{R} \right\} = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\dim(\text{null}(A)) = 2$

$\dim(\text{null}(A))$ = minimum # of vectors needed to span null(A)
= dimension of the space null(A)

Intuition: How many vectors in different directions get reduced to $\vec{0}$ by A

Note:
- The nullspace of a matrix always contains $\vec{0}$ as a trivial solution
- A non-trivial nullspace means that a matrix has linearly dependent columns, which also means that $A\vec{v} = \vec{0}$ has infinitely many solutions

i. What is the column space of **A**? What is its dimension?

ii. What is the null space of **A**? What is its dimension?

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

iv. Do the columns of **A** span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (i) $\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ $\dim(\text{col}(A)) = 1$

(ii) $\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$ already row reduced!

$x_2 = \text{free} = t$

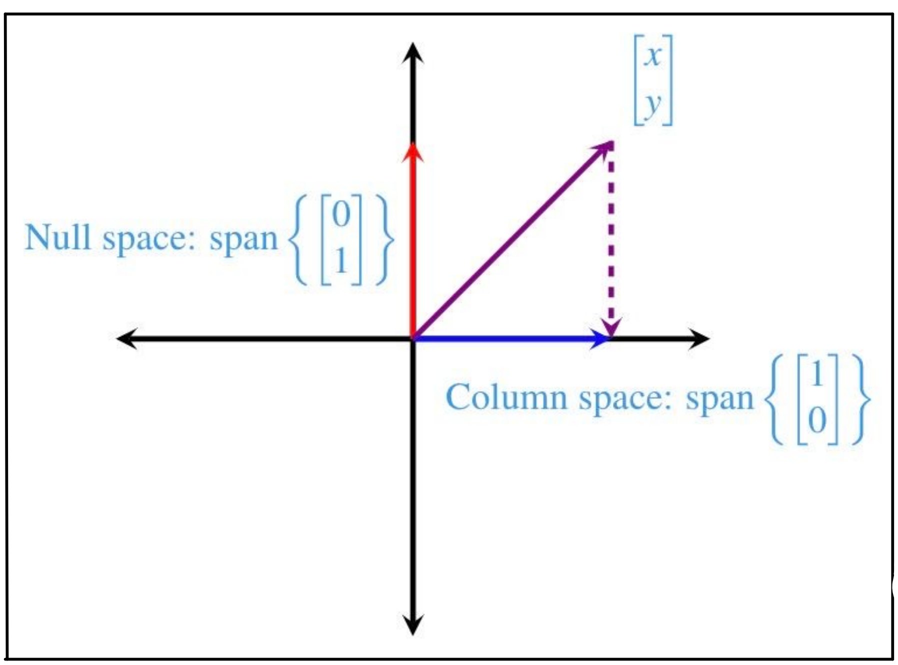
$x_1 + 0x_2 = 0$ $\quad \vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_1 = 0t$

$\text{null}(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ $\dim(\text{null}(A)) = 1$

(iii) Yes, already row reduced to begin with.

(iv) columns of A don't span $\mathbb{R}^2$ and therefore do not form a basis for $\mathbb{R}^2$ because A is 2×2 but has less than 2 linearly independent columns.



Visual Interpretation!

- This matrix projects all vectors onto the x-axis
- Column space is the x-axis any $\vec{v} \mid \vec{v} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$
- Null space is the y-axis any $\vec{v} \mid \vec{v} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (i) $col(A) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $dim(col(A)) = 1$

(ii)
$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array}$$

$x_1 = \text{free} = t$

$0x_1 + x_2 = 0$

$x_2 = 0t$

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$null(A) = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$dim(null(A)) = 1$

(iii) The two column spaces are not the same!

$span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ versus $span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

(iv) A does not span $\mathbb{R}^2 \Rightarrow$ A does not form a basis for $\mathbb{R}^2$ for the same reason as above!

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

No free variables!

(i) $\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$ $\dim(\text{col}(A)) = 2$

(ii) $\text{null}(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ only the trivial nullspace! $\dim(\text{null}(A)) = 0$

(iii) Yes, same column space ($\mathbb{R}^2 \to \mathbb{R}^2$)

(iv) Yes, this is a basis for $\mathbb{R}^2$

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Practice @ Home

$\dim(\text{col}(A)) = 2$

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

By inspection, $\text{col}(A) = \mathbb{R}^2$; however this is not a basis for $\mathbb{R}^2$ since there are lin. dep. columns

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 0 & 2 & 5 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & -7/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right] \end{array}$$

$x_3 = \text{free} = s$
$x_4 = \text{free} = t$
$x_1 = -\frac{1}{2}s + \frac{7}{2}t$
$x_2 = -\frac{5}{2}s - \frac{1}{2}t$

$$\text{null}(A) = s\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\dim(\text{null}(A)) = 2$

(f) What do you notice about the relationship between the dimension of the column space, the dimension of the null space, and their sum in all of these matrices?

$\dim(\text{col}(A)) =$ # of linearly independent columns in A
$\dim(\text{null}(A)) =$ # of linearly dependent columns in A

$\dim(\text{col}(A)) + \dim(\text{null}(A))$
$=$ total # of columns in A

$\Downarrow$

Rank-nullity Theorem:

$m - \dim(\text{col}(A)) = \dim(\text{null}(A))$, where m is the total number of columns in A

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