Discussion 6A :)

Anish's 16A Discussion
Office Hours: Tuesday 3-4
@Cory 144MA Friday 4-5

Transition Matrix:

$\vec{x}[k+1] = A\vec{x}[k]$ $\{\vec{x}[k]$ is the state of the system at timestep $k\}$

$\hookrightarrow$ State Transition Matrix $A \in \mathbb{R}^{n \times n}$, $\vec{x}[k] \in \mathbb{R}^n$

Statement: $\vec{x}[k] = A^k \vec{x}[0]$

Proof: $\vec{x}[1] = A\vec{x}[0]$

$\vec{x}[2] = A\vec{x}[1] = A(A\vec{x}[0]) = A^2\vec{x}[0]$

$\vdots$

$\vec{x}[k] = A^k\vec{x}[0]$ ✓

Assume A has n distinct eigenvalues

$\Rightarrow A$ has n eigenvectors (all linearly independent)

$\Rightarrow \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \mathbb{R}^n$

(any vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$)

$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$

$\vec{x}[1] = A\vec{x}[0] = A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n)$

$= \alpha_1(A\vec{v}_1) + \alpha_2(A\vec{v}_2) + \cdots + \alpha_n(A\vec{v}_n)$

$= \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \cdots + \alpha_n\lambda_n\vec{v}_n$ $\{$Employing $A\vec{v} = \lambda\vec{v}\}$

Statement: If $A\vec{v} = \lambda\vec{v}$, then $A^k\vec{v} = \lambda^k\vec{v}$

Proof: $A\vec{v} = \lambda\vec{v}$

$A(A\vec{v}) = A(\lambda\vec{v})$

$A^2\vec{v} = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$ ✓

$\vdots$

$A^k\vec{v} = \lambda^k\vec{v}$

Therefore $\longrightarrow x[k] = A^k \vec{x}[0]$

$$= A^k(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n)$$

$$= \alpha_1 A^k \vec{v}_1 + \alpha_2 A^k \vec{v}_2 + \cdots + \alpha_n A^k \vec{v}_n$$

$$= \alpha_1 \underline{\lambda_1{}^k} \vec{v}_1 + \alpha_2 \underline{\lambda_2{}^k} \vec{v}_2 + \cdots + \alpha_n \underline{\lambda_n{}^k} \vec{v}_n$$

These will help determine the stability of the system (i.e. whether or not it will converge)

$\longrightarrow$ apply this to question 1

EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 6A

1. Steady and Unsteady States

You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$.

(a) The eigenvalues of $\mathbf{M}$ are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \frac{1}{2}$. Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors corresponding to the eigenvalues. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	–
0	$\neq 0$	$\neq 0$	No	–
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	–
$\neq 0$	$\neq 0$	$\neq 0$	No	–

$$M^n\vec{x} = M^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)$$
$$= \alpha M^n\vec{v}_1 + \beta M^n\vec{v}_2 + \gamma M^n\vec{v}_3$$
$$= \alpha\lambda_1^n\vec{v}_1 + \beta\lambda_2^n\vec{v}_2 + \gamma\lambda_3^n\vec{v}_3$$
$$= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\tfrac{1}{2}\right)^n\gamma\vec{v}_3$$
$$\lim_{n\to\infty} M^n\vec{x} = \lim_{n\to\infty}\left(\cancel{1^n}^{\,1}\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \cancel{\left(\tfrac{1}{2}\right)^n}^{\,0}\gamma\vec{v}_3\right)$$
$$= \lim_{n\to\infty}\left(\alpha\vec{v}_1 + 2^n\beta\vec{v}_2\right)$$

If $\beta \neq 0$, system explodes!

If $\alpha \neq 0$, then the system has a nonzero steady-state component

(b) (**Practice**) Find the eigenspaces associated with the eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$
ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$
iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Review

(i) $\lambda_1 = 1$

$\text{null}(M - \lambda I)$

$= \text{null}\left(\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{cases} x_2 = \text{free} = t \\ x_3 = 0 \\ x_1 - x_2 = 0 \\ x_1 = t \end{cases}$$

$$\underbrace{\vec{v}_1 \in \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}}_{\text{eigenspace}}$$

Same process for (ii) and (iii)

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B, and C. The pumps system between the reservoirs is depicted in Figure 1.

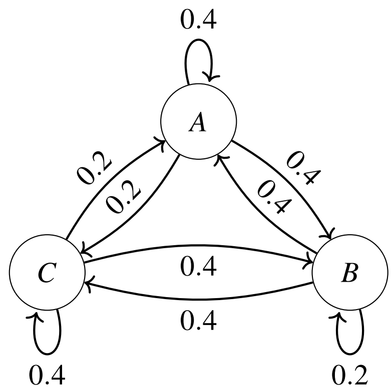

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

$$T = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{1}{5}$, $\lambda_3 = -\frac{1}{5}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

$\lambda = 1$ is the eigenvalue corresponding to the steady state eigenvector

$$T\vec{x} = \lambda_1 \vec{x} = 1 \cdot x$$

$$\vec{x} \in \text{null}(T - \lambda_1 I)$$

$$x \in \text{null}\left(\begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\left[\begin{array}{ccc|c} -0.6 & 0.4 & 0.2 & 0 \\ 0.4 & -0.8 & 0.4 & 0 \\ 0.2 & 0.4 & -0.6 & 0 \end{array}\right] \xrightarrow[\text{by 10 to make life easier}]{\text{multiply everything}} \left[\begin{array}{ccc|c} -6 & 4 & 2 & 0 \\ 4 & -8 & 4 & 0 \\ 2 & 4 & -6 & 0 \end{array}\right]$$

$$\xrightarrow[\text{(look at soln for steps)}]{\text{Gaussian Elimination}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \left.\begin{array}{l} x_3 = \text{free} = t \\ x_2 - t = 0 \longrightarrow x_2 = t \\ x_1 - t = 0 \longrightarrow x_1 = t \end{array}\right\} \vec{x} \in \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

(c) What does the magnitude of the other two eigenvalues λ_2 and λ_3 say about the steady state behavior of their associated eigenvectors?

The magnitude of the other two eigenvalues, $|\lambda_2|$ and $|\lambda_3|$ are both less than 1 $\Rightarrow$ in steady state, their associated eigenvectors trend toward $\vec{0}$

(d) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 150, B_0 = 250, C_0 = 200$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

(1) From part (b), we know $\vec{x}_{ss}$, the steady-state solution, takes the form $\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for any α

(2) Notice that T's columns sum to 1 => conservative system
Initial volume $A_0 + B_0 + C_0$ remains constant at each iteration (timestep)

(3) So far, the sum with $\alpha = 1$ of $\vec{x}_{ss}$ is $1+1+1 = 3$ kL, while the initial state begins with

$$A_0 + B_0 + C_0 = 150 + 250 + 200 = 600 \text{ kL}$$

By inspection, we see then that $\alpha = 200$ is the proper rescaling of $\vec{x}_{ss}$ to sustain the total water of 600 kL:

$$\vec{x}_{ss} = 200 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 200 \\ 200 \\ 200 \end{bmatrix}$$

Sums to 600 kL!

☆ There is an even distribution of water among all 3 pumps at steady-state tinyurl.com/16Anish

Password: steady