STRONG FEASIBILITY IN INPUT-MOVE-BLOCKING MODEL PREDICTIVE CONTROL

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Abstract: Time-invariant input-move-blocking regimes are used in many practical online model predictive control systems in order to reduce the computational complexity of the associated finite-horizon optimal control problem, and have been shown to be beneficial for offline model predictive control methods also. However, until now there exists no method to ensure strong feasibility. In this paper a least-restrictive method to enforce strong feasibility in time-invariant input-move-blocking model predictive control problems is proposed, where the state of the first prediction step is constrained to a novel type of controlled invariant set, called here a *controlled invariant feasible* set. An algorithm to determine maximal controlled invariant feasible sets is proposed. This algorithm is shown to be semi-decidable for the case of linear, time-invariant plants with time-invariant, polytopic state and control input constraint sets.

Keywords: Predictive control, constraint satisfaction, invariance, sets.

1. INTRODUCTION

Model predictive control (MPC) schemes are attractive for their ability to explicitly accommodate hard constraints on states and control inputs [Borrelli (2003); Camacho and Bordons (1999)]. However, the computational complexity of solving the associated finite-horizon optimal control problem restricts applications of MPC to suitably slow plants and/or use of low complexity optimization problems. For fixed state and input dimension, the main source of complexity is in the choice of prediction horizon length. In general, longer prediction horizons yield superior control performance to shorter ones. To reduce the computational complexity of finite-horizon optimal control problems of a given prediction horizon length it is common to parameterize the predicted control input trajectory in some manner [Qin and Badgewell (1997)]. In input-move-blocking MPC schemes an input sequence of N predicted control moves is approximated by one of $\tilde{N} < N$ degrees of freedom, where predicted control moves are held constant, 'blocked', over multiple prediction steps [Tøndel and Johansen (2002)].

Strong feasibility describes the quality that the closed-loop state trajectory starting from any feasible initial state never reaches a state for which the finite-horizon optimal control problem admits no feasible solution, and has received much research attention in the context of full degree of freedom optimal control problems and discrete-time systems [Mayne *et al.* (2000); Scokaert and Rawlings (1999)]. In this case, strong feasibility is ensured by constraining the terminal state of the finite-horizon optimal control problem to a controlled invariant set [Blanchini (1999); Kerrigan and Maciejowski (2000)]. This method does not work for time-invariant input-move-blocking regimes [see Sec. 2.4; Cagienard *et al.* (2007)]. A move-blocking regime which does guarantee strong feasibility was proposed in Cagienard *et al.* (2007), which uses a time-dependent trajectory parameterization combined with time-dependent constraint sets. However, the complication of a time-dependent move-blocking regime is in some ways undesirable, as time-dependence does not allow the use of some common tools of controller design and performance evaluation, for example explicit determination of control laws achieved by multi-parametric quadratic-programming methods for linear-quadratic MPC [Bemporad *et al.* (2002)], and the subsequent performance analysis of (time-invariant) piecewise-affine systems [Gondhalekar and Imura (2007); Grieder *et al.* (2003)].

The absence of strong feasibility guarantees in time-invariant move-blocking regimes, widely used in industry, somewhat defeats a main motivation for choosing MPC. Without the flexibility offered by a time-dependent solution, ensuring strong feasibility by means of a time-invariant method is more challenging.

In this paper a generalized method for enforcing strong feasibility of MPC problems is proposed, which can be applied to time-invariant input-move-blocking MPC schemes. Instead of constraining the *terminal* state to a controlled invariant set, the state at the *first* prediction step is constrained to lie within a novel type of controlled invariant set, called a *controlled invariant feasible* set (Sec. 3). The property of controlled invariant feasibility ensures recursive feasibility of the finite-horizon optimal control problem, assuming a feasible initial state. An algorithm to compute maximal controlled invariant feasible sets is proposed (Sec. 4). This follows strongly along the lines of that in Vidal *et al.* (2000) for the determination of maximal controlled invariant sets. The algorithm is shown to be semi-decidable for the case of a linear, time-invariant plant with time-invariant, polytopic constraint sets (Sec. 5). Two numerical examples are given in Sec. 6.

2. PRELIMINARIES

2.1 Notation

Real number and integer sets are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively ($\mathbb{R}_0$, $\mathbb{N}_0$: non-negative. $\mathbb{R}_+$, $\mathbb{N}$: strictly positive). I_n and $0_{\{n,m\}}$ denote a square identity matrix of dimension n and rectangular zero matrix of dimension $n \times m$, respectively, while 0 without subscript denotes a zero matrix with dimension deemed obvious by context. The transpose of matrix A is denoted by A^{T}. $A \otimes B$ denotes the Kronecker product of arbitrary matrices A and B. ${}^{\mathrm{c}}\mathbb{X}$ denotes the complement of a set $\mathbb{X}$. The product set of a set $\mathbb{X}$ with itself k times is denoted by $\mathbb{X}^k$. A sequence of elements $x_i \in \mathbb{X}\ \forall i \in \{j, \cdots, k\}$ is denoted by $\{x_i \in \mathbb{X}\}_{i=j}^{k} \in \mathbb{X}^{k-j+1}$. If the elements' parent set is obvious by context the sequence is denoted simply by $\{x_i\}_{i=j}^{k}$. Let $i \in \mathbb{Z}$ denote a system's actual step index, while $k \in \mathbb{N}_0$ denotes the $(i+k)^{\mathrm{th}}$ step as predicted from actual step i. Thus, $\psi_{(i,k)}$ denotes the value of variable ψ at step $i + k$, as predicted from step i. For compact notation $\psi_{(i,0)} \equiv \psi_i$. *Polytope* refers to a closed and bounded intersection of a finite number of half-spaces. Denote by $[\tilde{G}, \tilde{W}] \longleftarrow \mathrm{proj}(G, W, \tilde{n})$ the computation of orthogonal projection $\tilde{\Omega} = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} |\ \tilde{G}\tilde{x} \leq \tilde{W}\}$ of polytope $\Omega = \{x \in \mathbb{R}^n |\ Gx \leq W\}$ onto the first $\tilde{n} < n$ dimensions.

2.2 Standard Discrete-Time MPC

Consider the nonlinear, time-invariant plant

$$x_{i+1} = f(x_i, u_i) \tag{1}$$

with step index $i \in \mathbb{Z}$, system state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$. The function $f(\cdot,\cdot)$ is assumed uniquely defined over $\mathbb{R}^n \times \mathbb{R}^m$. State and control input are required to satisfy the general constraints

$$x_i \in \mathbb{X} \subseteq \mathbb{R}^n \ \wedge\ u_i \in \mathbb{U} \subseteq \mathbb{R}^m \ \ \forall i \in \mathbb{Z}. \tag{2}$$

MPC achieves closed-loop control action of system (1) subject to constraints (2) by applying at each step the first control input of a predicted optimal open-loop control input sequence. At each step i, this optimal open-loop control input sequence is evaluated via the minimization of prediction cost $J(x_i, \mathcal{U}_i)$ over all permissible control input sequences:

$$\mathcal{U}_i^*(x_i) \equiv \arg \min_{\mathcal{U}_i \equiv \{u_{(i,k)} \in \mathbb{U}\}_{k=0}^{N-1}} \{J(x_i, \mathcal{U}_i)\}. \tag{3}$$

The predicted state trajectory evolves according to $x_{(i,k+1)} = f(x_{(i,k)}, u_{(i,k)})\ \ \forall k \in \{0, \cdots, N-1\}$, and must satisfy prediction state constraints

$$x_{(i,k)} \in \mathbb{X}\ \ \forall k \in \{0, \cdots, N\}. \tag{4}$$

The prediction cost function $J : \mathbb{X} \times \mathbb{U}^N \mapsto \mathbb{R}_0$ is given in the general form

$$J(x_i, \mathcal{U}_i) = F(x_{(i,N)}) + \sum_{k=0}^{N-1} L(x_{(i,k)}, u_{(i,k)}), \tag{5}$$

with terminal cost function $F : \mathbb{X} \to \mathbb{R}_0$ and stage cost function $L : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_0$.

The optimal control law $\kappa : \mathbb{X} \mapsto \{\mathbb{U}, \emptyset\}$ is given by

$$u_i^* = \kappa(x_i) = u_{(i,0)}^*(x_i)\ ,$$

and the closed-loop system evolves according to

$$x_{i+1} = f(x_i, u_i^*)\ .$$

2.3 Input-Move-Blocking MPC

Rewriting control input sequence $\mathcal{U}_i$ in vector form $U_i \equiv [u_{(i,0)}^{\mathrm{T}}, \cdots, u_{(i,N-1)}^{\mathrm{T}}]^{\mathrm{T}}$, in input-move-blocking the predicted control input sequence $U_i \in \mathbb{U}^N$ of N control moves is parameterized by a control input sequence $\tilde{U}_i \in \mathbb{U}^{\tilde{N}}$ of $\tilde{N} < N$ control moves and a blocking matrix $M \in \{0,1\}^{N \times \tilde{N}}$ according to [Cagienard *et al.* (2007); Tøndel and Johansen (2002)]:

$$U_i = (M \otimes I_m)\, \tilde{U}_i \equiv \tilde{M}\tilde{U}_i\ . \tag{6}$$

Blocking matrix M requires the following properties:

- $M_{1,1} = 1$. First control input of U equals the first

parameterized control input of $\tilde{U}$.
• $M_{N,\tilde{N}} = 1$. Final control input of U equals the final parameterized control input of $\tilde{U}$.
• $\sum_{j=1}^{\tilde{N}} M_{i,j} = 1 \forall i \in \{1, \cdots, N\}$. One non-zero element per row.
• $\sum_{i=1}^{N} M_{i,j} \geq 1 \forall j \in \{1, \cdots, \tilde{N}\}$. No zero columns.
• $\forall (i,j) \in \{2, \cdots, N\} \times \{1, \cdots, \tilde{N}\}$ $M_{i,j} = 1$ iff $[M_{i-1,j} = 1] \vee [M_{i-1,j-1} = 1]$. Matrix in staircase form with non-zero elements on or below diagonal.

For example, a four-move control input sequence could be parameterized by a two-move one, with moves $(2, 3, 4)$ blocked, as follows:

$$\begin{bmatrix} u_{(i,0)} \\ u_{(i,1)} \\ u_{(i,2)} \\ u_{(i,3)} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \otimes I_m \right) \begin{bmatrix} \tilde{u}_{(i,0)} \\ \tilde{u}_{(i,1)} \end{bmatrix}.$$

The input-move-blocked MPC problem is given by

$$\tilde{U}_i^*(x_i) = \arg \min_{\tilde{U}_i \in \mathbb{U}^{\tilde{N}}} \left\{ J(x_i, \tilde{M}\tilde{U}_i) \right\} \qquad (7)$$

subject to (4), and the input-move-blocked optimal control law $\tilde{\kappa} : \mathbb{X} \mapsto \{\mathbb{U}, \emptyset\}$ is then given by

$$u_i^* = \tilde{\kappa}(x_i) = F\tilde{M}\tilde{U}_i^*(x_i)$$

with $F \equiv \left[I_m \ , \ 0_{\{m,(N-1)m\}} \right]$.

2.4 *Strong Feasibility*

Definition 1. Feasible. An MPC problem is called *feasible* at state x if and only if there exists a permissible predicted open-loop control input sequence such that the predicted state trajectory satisfies all prediction state constraints. If an MPC problem is feasible at state x, then x is called a *feasible state*. For an infeasible state x we write $\kappa(x) = \emptyset$ (alternatively $\tilde{\kappa}(x) = \emptyset$).

Definition 2. Recursively feasible. An MPC problem is called *recursively feasible* from a feasible initial state x_0 if and only if the MPC problem is feasible, recursively, at every state along the closed-loop trajectory $x_i \ \forall i \in \mathbb{N}$.

Definition 3. Strongly feasible. [Kerrigan (2000)] [1] An MPC problem is called *strongly feasible* if and only if the MPC problem is recursively feasible *from every* feasible initial state *for all* feasible control input sequences.

Definition 4. Controlled invariant. [Blanchini (1999)] A non-empty set $\mathbb{C} \subseteq \mathbb{X}$ is a *controlled invariant* set of system (1) within $\mathbb{X}$ if and only if $\forall x \in \mathbb{C}\ \exists u \in \mathbb{U}$ s.t. $f(x, u) \in \mathbb{C}$.

[1] Note that this is different to [Kerrigan and Maciejowski (2000)].

A sufficient method for ensuring strong feasibility of full degree of freedom MPC problem (3) is the choice of terminal state constraint $x_{(i,N)} \in \mathbb{C}$ [Mayne *et al.* (2000)]. In this case, the optimal sequence from the previous step, shifted by one step, is always a feasible (but generally not optimal) solution to the MPC problem at the current step, up to but not including the final prediction step: $\{u_{(i,k)}\}_{k=0}^{N-2} = \{u^*_{(i-1,k)}\}_{k=1}^{N-1}$. Existence of final control move $u_{(i,N-1)}$ is ensured by the new terminal constraint and Definition 4. This is however not true for blocked MPC problem (7), because due to move-blocking regime (6) the shifted input sequence from the previous step is not guaranteed to be a feasible input sequence at the current step.

3. CONTROLLED INVARIANT FEASIBILITY

Definition 5. Controlled invariant feasible. A non-empty set $\Omega \subseteq \mathbb{R}^n$ is a *controlled invariant feasible* set of system (1) under input-move-blocking regime (6) with prediction state constraints (4) iff $\forall x \in \Omega$ $\exists \tilde{U} \in \mathbb{U}^{\tilde{N}}$ s.t. (4) $\wedge$ $f(x, F\tilde{M}\tilde{U}) \in \Omega$.

Each element of a controlled invariant feasible set is feasible, as for each element there exists a feasible parameterized open-loop input sequence $\tilde{U} \in \mathbb{U}^{\tilde{N}}$ satisfying prediction state constraints (4). Furthermore the set is controlled invariant, in the sense that for each element there exists a feasible parameterized open-loop input sequence such that the closed-loop state trajectory, achieved by applying only the first control move of the open-loop input sequence, starting from any initial state within the controlled invariant feasible set, remains within this set.

Definition 6. Maximal controlled invariant feasible set. The *maximal controlled invariant feasible* set $\mathbb{F} \subseteq \mathbb{X}$ within $\mathbb{X}$ of system (1) under blocking regime (6) with prediction state constraints (4) is defined as:

$$\mathbb{F} \equiv \Big\{ x_0 \in \mathbb{X} \mid \exists \{\tilde{U}_i \in \mathbb{U}^{\tilde{N}}\}_{i=0}^{\infty} \text{ s.t. } \forall i \in \mathbb{N}_0 \ , \quad (4) \ \wedge \ x_{i+1} = f\big(x_i, F\tilde{M}\tilde{U}_i\big) \in \mathbb{X} \Big\}.$$

Theorem 1. Assume the non-empty set $\mathbb{F}$ exists. Then, MPC problem (7) is recursively feasible from a feasible initial state x_0 if and only if the optimal control input sequence $\tilde{U}_i^*(x_i)$ is such that $x_{(i,1)} \in \mathbb{F} \ \forall i \in \mathbb{N}_0$.

Proof: *(If)* If at feasible state x_i MPC problem (7) yields a solution $\tilde{U}_i^*(x_i)$ such that $x_{(i,1)} \in \mathbb{F}$, then $x_{i+1} \in \mathbb{F}$, which guarantees that $\tilde{\kappa}(x_{i+1}) \neq \emptyset$, by Definition 5. By induction the argument holds recursively. *(Only if)* By Definition 6, $\mathbb{F}$ contains every member of $\mathbb{X}$ which not only admits a feasible solution to (7), but ensures the existence of a closed-loop control input sequence which remains feasible. If at feasible state x_i MPC problem (7) yields a solution

$\tilde{U}_i^*(x_i)$ such that $x_{(i,1)} \in {}^{c}\mathbb{F}$, then $\tilde{\kappa}(x_j) = \emptyset$ for some $j > i$ (but not necessarily $j = i + 1$). □

Motivated by Theorem 1, we no longer consider prediction state constraints (4), instead solve MPC problem (7) subject to prediction state constraints

$$x_{(i,1)} \in \mathbb{F} \;\; \wedge \;\; x_{(i,k)} \in \mathbb{X} \;\; \forall k \in \{0, 2, \cdots, N\} \quad (8)$$

to yield the *least restrictive* control law based on MPC problem (7) and prediction state constraints (4), but strongly feasible: $u_i^* = \hat{\kappa}(x_i) = F\tilde{M}\tilde{U}_i^*(x_i)$, $\hat{\kappa} : \mathbb{F} \mapsto \mathbb{U}$. Least restrictive means that the MPC problem is recursively feasible from the largest subset of the state space possible for any controller based on MPC problem (7) subject to prediction state constraints (4).

Note that maximal controlled invariant feasible set $\mathbb{F}$ does not depend on the prediction cost function $J(x_i, \mathcal{U}_i)$ of (5). It depends only on the plant dynamics (1), control input and state constraint sets (2) and input-move-blocking regime (6). This allows the tuning of cost function (5) to affect stability and cost performance, without affecting the domain of feasibility of the controller.

4. COMPUTATION OF MAXIMAL CONTROLLED INVARIANT FEASIBLE SETS

Define the following operator on a set $\Omega \subseteq \mathbb{R}^n$:

$$Pre(\Omega) \equiv \Big\{ x \in \Omega \mid \exists \tilde{U} \in \mathbb{U}^{\tilde{N}} \text{ s.t. (4)} \wedge f(x, F\tilde{M}\tilde{U}) \in \Omega \Big\}. \quad (9)$$

The $Pre(\cdot)$ operator retains from an initial set Ω all states which admit a feasible solution to MPC problem (7) subject to prediction state constraints (4) and can be controlled to remain within Ω for one closed-loop step. Put another way, all states which are infeasible and/or cannot be retained within Ω by one closed-loop step are removed. The $Pre(\cdot)$ operator is contractive, i.e. $Pre(\Omega) \subseteq \Omega$. A set Ω is a controlled invariant feasible set if it is a fixed-point of the $Pre(\cdot)$ operator, i.e. $Pre(\Omega) = \Omega$. We thus formulate the following algorithm for the determination of maximal controlled invariant feasible set $\mathbb{F}$ [Vidal *et al.* (2000)].

Algorithm 1.

Initialize: $\mathbb{F}_0 \leftarrow \mathbb{X}$, $\mathbb{F}_1 \leftarrow Pre(\mathbb{X})$, $j \leftarrow 1$
While $(\mathbb{F}_j \neq \mathbb{F}_{j-1})$
    $j \leftarrow j + 1$
    $\mathbb{F}_j \leftarrow Pre(\mathbb{F}_{j-1})$
End While
$\mathbb{F} \leftarrow \mathbb{F}_j$

To implement Algorithm 1 one requires the ability to

(a) encode sets of states and test for set equality,
(b) compute the $Pre(\cdot)$ operator, and
(c) guarantee that a fixed-point is reached after a finite number of iterations.

Using terminology introduced in Vidal *et al.* (2000), if conditions (a) and (b) are satisfied we say the problem is *semi-decidable*. If all three conditions are met the problem is termed *decidable*.

5. A SEMI-DECIDABLE CLASS OF SYSTEM

One class of system which is semi-decidable is that of a linear, time-invariant plant $x_{i+1} = Ax_i + Bu_i$, with time-invariant, polytopic control input and state constraint sets; $\mathbb{U}$, $\mathbb{X}$. The paper will focus on this class of system hereafter. Prediction state constraints (4) and constraints on control input sequence U can then be expressed by [Bemporad *et al.* (2002)]:

$$G_{\mathrm{U}} U_i \leq W_{\mathrm{U}} + E_{\mathrm{U}} x_i. \quad (10)$$

5.1 Computation of $Pre(\cdot)$

Assume that the parameter passed to the $Pre(\cdot)$ operator is a polytope $\Omega = \{x \in \mathbb{R}^n \mid G_\Omega x \leq W_\Omega\}$. As will be shown below, this ensures $Pre(\Omega)$ to be a polytope also. The assumption is true for the first call of $Pre(\cdot)$ of Algorithm 1, because $\mathbb{X}$ is a polytope, ensuring the assumption to be valid recursively. The conditions

$$x \in \Omega \;\wedge\; \tilde{U} \in \mathbb{U}^{\tilde{N}} \text{ s.t. (4)} \;\wedge\; Ax + BF\tilde{M}\tilde{U} \in \Omega$$

of (9) can now be neatly reformulated as

$$\begin{bmatrix} G_\Omega & 0 \\ -E_{\mathrm{U}} & G_{\mathrm{U}}\tilde{M} \\ G_\Omega A & G_\Omega BF\tilde{M} \end{bmatrix} \begin{bmatrix} x \\ \tilde{U} \end{bmatrix} \leq \begin{bmatrix} W_\Omega \\ W_{\mathrm{U}} \\ W_\Omega \end{bmatrix} \implies G \begin{bmatrix} x \\ \tilde{U} \end{bmatrix} \leq W,$$

where the pair (G, W) defines a polytope in $\mathbb{R}^{n+\tilde{N}m}$. The existential quantifier $\exists$ is eliminated by orthogonal projection onto $\mathbb{R}^n$: $\big[\tilde{G}, \tilde{W}\big] \longleftarrow \mathrm{proj}(G, W, n)$. Subsequently $Pre(\Omega) \equiv \big\{x \in \mathbb{R}^n | \tilde{G}x \leq \tilde{W}\big\}$.

With all sets given by polytopes they are simply encoded, and the $Pre(\cdot)$ operator is easily computed. Testing equality of polytopes is straight-forward. Conditions for *semi-decidability* are thus satisfied.

5.2 Existence, Boundedness and Convexity of $\mathbb{F}$

The existence of a non-empty controlled invariant feasible set can be ensured, under mild assumptions [2]. The following assumptions will be made hereafter; (A, B) is stabilizable, $0 \in \mathbb{U}$ and $0 \in \mathbb{X}$.

[2] These assumptions are reasonable for most practical systems.

Lemma 1. A non-empty controlled invariant feasible set $\Omega \subseteq \mathbb{X}$, $\Omega \neq \emptyset$ exists.

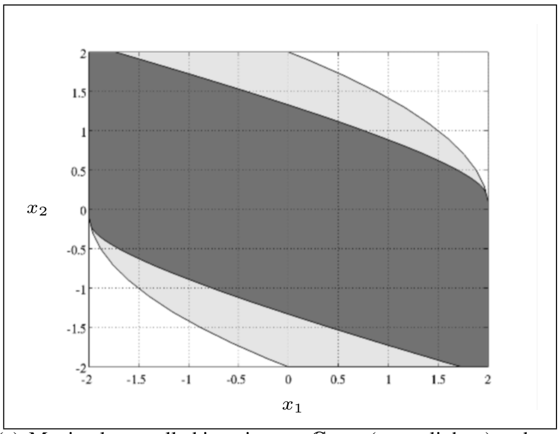
Proof: The origin is always a controlled invariant feasible set, as at the origin the parameterized predicted control input sequence $\tilde{U} = 0_{\{\tilde{N}m,1\}}$ is always permissible and feasible, for any blocking regime. $\square$

Lemma 2. $\mathbb{F}$ exists.

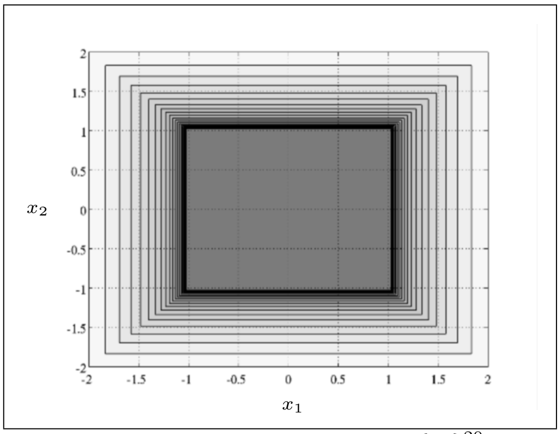
Proof: By closedness assumption on $\mathbb{X}$ and Lemma 1 the maximum over all controlled invariant feasible sets exists. $\square$

Multi-parametric quadratic-programming solvers [Bemporad *et al.* (2002)], used in MPC of linear systems with quadratic performance index, require constraints of the form of (10). It is therefore convenient that the $Pre(\cdot)$ operator returns a polytope, because if Algorithm 1 terminates, then strong feasibility constraints (8) can be written in the form of (10).

However, $\mathbb{F}$ is not always a polytope, i.e. bounded, closed, convex and parameterized by a finite number of linear inequalities. By Lemma 3 however, $\mathbb{F}$ is ensured to be bounded, closed and convex.



(a) Maximal controlled invariant set $\mathbb{C}_{\max}$ (outer, lighter) and maximal controlled invariant feasible set $\mathbb{F}$ (inner, darker).



(b) The sequence of sets of Algorithm 1: $\{\mathbb{F}_j\}_{j=0}^{20}$.

Fig. 1. Decidable (a) and undecidable (b) example.

Lemma 3. $\mathbb{F}$ is bounded, closed and convex.

Proof: *Boundedness:* By boundedness assumption on $\mathbb{X}$, $\mathbb{F}$ is bounded because $\mathbb{F} \subseteq \mathbb{X}$ by definition. *Closure and convexity:* Consider the set $\mathbb{F}_j \equiv \{x_0 \in \mathbb{X} \mid \exists\{\tilde{U}_i \in \mathbb{U}^{\tilde{N}}\}_{i=0}^{j-1} \text{ s.t. } \forall i \in \{0, \cdots, j-1\}$, (4) $\wedge\ x_{i+1} = f(x_i, F\tilde{M}\tilde{U}_i) \in \mathbb{X}\}$ for some $j \in \mathbb{N}$. Constraints on $\mathbb{F}_j$ can be written in polytopic form $G_j[x_0^{\mathrm{T}}, \tilde{U}_0^{\mathrm{T}}, \cdots, \tilde{U}_{j-1}^{\mathrm{T}}]^{\mathrm{T}} \leq W_j$. The pair (G_j, W_j) describes a polytope in $\mathbb{R}^{n+j\tilde{N}m}$. Closure and convexity are retained by the projection $[\tilde{G}_j, \tilde{W}_j] \leftarrow \text{proj}(G_j, W_j, n)$. Therefore $\mathbb{F}_j = \{x \in \mathbb{R}^n | \tilde{G}_j x \leq \tilde{W}_j\}$ is closed and convex. These properties are maintained in the limit as $j \to \infty$. $\square$

5.3 Undecidablity

Algorithm 1 is not guaranteed to terminate after a finite number of steps. This problem has been met with algorithms for the computation of controlled invariant sets also [Dórea and Hennet (1999); Vidal *et al.* (2000)]. While some (in general very restrictive) sufficient conditions can be imposed to make it decidable, in general "there appears to be no finite algorithmic procedure for showing that (it) is not finitely determined" [Gilbert and Tan (1991)]. Unfortunately a thorough discussion is beyond the scope of this paper.

6. NUMERICAL EXAMPLE

6.1 Decidable Example

The double integrator with sample-period $\tau = 0.2$s, $x_{i+1} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u_i$, input constraint set $\mathbb{U} \equiv \{u \in \mathbb{R} | \|u\|_\infty \leq 1\}$, state constraint set $\mathbb{X} \equiv \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 2\}$, prediction horizon $N = 40$ and parameterized prediction horizon $\tilde{N} = 2$, and blocking matrix $M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$ is considered.

Figure 1(a) shows maximal controlled invariant set $\mathbb{C}_{max}$ within $\mathbb{X}$ as the lighter, outside area, and maximal controlled invariant feasible set $\mathbb{F}$ within $\mathbb{X}$ of the blocked MPC problem as the darker, inside area. These sets correspond to the feasible domains of the MPC problem without and with input-move-blocking, respectively. The difference in size is clearly visible.

6.2 Undecidable Example

The system $x_{i+1} = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix} x_i + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} u_i$ with constraint sets $\mathbb{U} \equiv \{u \in \mathbb{R}^2 | \|u\|_\infty \leq 1\}$, $\mathbb{X} \equiv \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 2\}$ and prediction horizon length $N = 1$ was chosen. Note that for a one step prediction horizon, maximal controlled invariant feasible set $\mathbb{F}$ and maximal controlled invariant set $\mathbb{C}_{\max}$ are the same: $\mathbb{F} = \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 1\}$. This example was chosen to illustrate the undecidability property, rather than the controlled invariant feasible property. The two modes of this systems are identical but decoupled. This was chosen only to aid graphical presentation.

Consider only the mode of dimension 1. The sets $\{x \in \mathbb{R}^2 | x_1 = -1\}$ ($\{x \in \mathbb{R}^2 | x_1 = +1\}$) are stationary under maximum (minimum) control input $\{u \in \mathbb{R}^2 | u_1 = +1\}$ ($\{u \in \mathbb{R}^2 | u_1 = -1\}$). The same holds for the mode of dimension 2. The $Pre(\cdot)$ operator is therefore not able to remove members of such sets from the parameter sets passed to them. Therefore Algorithm 1 is known to be undecidable. Figure 1(b) shows the sequence of sets $\{\mathbb{F}_j\}_{j=0}^{20}$ of Algorithm 1. Theses sets contract with each iteration but never converge to $\mathbb{F}$.

7. CONCLUSION

In this paper a generalized method for enforcing strong feasibility of MPC problems was proposed, which is applicable to input-move-blocking MPC schemes (or for example the use of irregular prediction horizon time-discretization [Gondhalekar and Imura (2007); Goodwin *et al.* (2006)]), where present methods of applying terminal constraints fail. The state at the first prediction step is constrained to lie within a controlled invariant feasible set. The property of controlled invariant feasibility was introduced, and an algorithm to compute maximal controlled invariant feasible sets proposed. This algorithm was shown to be easily implementable and semi-decidable for linear, time-invariant plants with time-invariant, polytopic constraint sets. Necessary and sufficient conditions for the algorithm to be decidable remain elusive.

The main contribution of this paper is the notion of controlled invariant feasibility and Theorem 1 (Sec. 3), which are valid for MPC of general nonlinear plants as well as linear ones, and for general constraint sets also. However, due to the considerable difficulty in computing the $Pre(\cdot)$ operator (Sec. 5.1) for anything but linear systems with polytopic constraint sets, this is the only class of system considered by the authors sofar. However, the success of the proposed method for linear systems is motivation to develop this method further for MPC of nonlinear systems, where input-move-blocking and irregular prediction horizon time-discretization play a prominent role.

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