Project : Computing the Polar Decomposition

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1 Norms and Singular Value Decomposition

1.1 Vector Norms

Before introducing matrix norm, a brief illustration of vector norm is necessary.

Definition 1.1 (Vector Norm)**.** A vector norm on $\mathbb{C}^n$ is a function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ such that it satisfies the following properties

1. $\|x\| \geq 0$ for all $x \in \mathbb{C}^n$,
2. $\|x\| = 0$ if and only if $x = 0$,
3. $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$,
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{C}^n$.

Example 1.2. For $x \in \mathbb{C}^n$,

$$\begin{aligned}
\text{1-norm} : \|x\|_1 &= \sum_{i=1}^{n} |x_i|, \\
\text{2-norm (Euclidean Norm)} : \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^* x}, \\
\infty\text{-norm} : \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i|.
\end{aligned}$$

These are all special cases of the p-norm,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad q \geq 1. \tag{1.1}$$

1.2 Matrix Norms

Definition 1.3 (Matrix Norms)**.** A matrix norm is a function $\|\cdot\| : \mathbb{C}^{m \times n} \to \mathbb{R}$ satisfies the analogues of the four properties in the Definition 1.1.

Example 1.4 (Frobenius norm)**.** For $A \in \mathbb{C}^{m \times n}$, the Frobenius norm is defined as

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \left(\operatorname{trace}(A^* A)\right)^{1/2}.$$

Example 1.5 (Subordinate matrix norm)**.** The matrix norm that is induced by a vector norm is called the subordinate norm. Suppose $\|\cdot\|$ is a vector norm, the corresponding subordinate matrix norm is defined as

$$\|A\| = \max_{\|x\|=1} \|Ax\|, \quad A \in \mathbb{C}^{m \times n}, \quad x \in \mathbb{C}^n.$$

Apply this definition into (1.1), we have the definition for the matrix p-norm

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p.$$

The subordinate matrix norms for 1-, 2- and ∞-norms can be shown to have the following form

$$\begin{aligned}
\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|,\\
\|A\|_2 &= (\rho(A^*A))^{1/2} = \sigma_{\max}(A),\\
\|A\|_\infty &= \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|,
\end{aligned}$$

where $\rho(A)$ represents the spectral radius of the matrix A which defined as the largest eigenvalue in magnitude of A and $\sigma_{\max}(A)$ represents the largest singular value of A.

We say a matrix norm $\|\cdot\|$ is consistent if for all $A, B \in \mathbb{C}^{m\times n}$, the following inequality holds whenever the product AB defines

$$\|AB\| \le \|A\|\|B\|.$$

The Frobenius norm and all subordinate norms are consistent.

1.3 Singular Value Decomposition

Theorem 1.6 (Singular value decomposition)**.** *If $A \in \mathbb{C}^{m\times n}$, $m \ge n$, then there exists two unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that*

$$A = U\Sigma V^*, \quad \Sigma = \operatorname{diag}(\sigma_1,\dots,\sigma_p) \in \mathbb{R}^{m\times n}, \quad p = \min\{m,n\}, \tag{1.2}$$

where $\sigma_1,\dots,\sigma_p$ are all non-negative and arranged in non-ascending order. We denote (1.2) as the singular value decomposition (SVD) of A and $\sigma_1,\dots,\sigma_p$ are the singular values of A.

2 Polar Decomposition and its Properties

Throughout this project, we focused on $A \in \mathbb{C}^{n\times n}$. In complex analysis, it is known that for any $\alpha \in \mathbb{C}$, we can write α in polar form, namely $\alpha = re^{\mathrm{i}\theta}$. The polar decomposition is its matrix analogue.

Theorem 2.1 (Polar Decomposition [1, 2008, Theorem 8.1])**.** *Let $A \in \mathbb{C}^{m\times n}$ with $m \ge n$. There exists a matrix $U \in \mathbb{C}^{m\times n}$ with orthonormal columns and a unique Hermitian positive semidefinite matrix $H \in \mathbb{C}^{n\times n}$ such that $A = UH$. The matrix H is given by $H = (A^*A)^{1/2}$. If the matrix A is full rank, then H is Hermitian positive definite and U is uniquely determined.*

Proof. Suppose $\operatorname{rank}(A) = r$, then let A has a SVD $A = P\Sigma_r V^*$. The polar decomposition of A can be formed in terms of the SVD:

$$A = P\begin{bmatrix} I_r & 0\\ 0 & I_{m-r,n-r}\end{bmatrix} V^*V \begin{bmatrix}\Sigma_r & 0\\ 0 & 0_{n-r,n-r}\end{bmatrix} V^* =: UH.$$

where

$$U = P\begin{bmatrix} I_r & 0\\ 0 & I_{m-r,n-r}\end{bmatrix} V^*, \quad H = V\begin{bmatrix}\Sigma_r & 0\\ 0 & 0_{n-r,n-r}\end{bmatrix} V^*.$$

We can test the columns' orthonormality of U by performing

$$U^*U = V \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r,m-r} \end{bmatrix} P^*P \begin{bmatrix} I_r & 0 \\ 0 & I_{m-r,n-r} \end{bmatrix} V^* = I_{n,n}.$$

The symmetry of H is obvious. Notice that, H and $\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0_{n-r,n-r} \end{bmatrix}$ are unitarily similar, hence they share the same eigenvalues. Equivalently speaking, the eigenvalues of H are the singular values of A. From the Theorem 1.6, the singular values of A are all real and non-negative, therefore H is Hermitian positive semidefinite and it is uniquely determined via

$$(A^*A)^{1/2} = (H^*U^*UH)^{1/2} = \left(H^2\right)^{1/2} = H. \tag{2.1}$$

If A is full rank, namely $r = n$, then all the singular values of A are positive and consequently H's eigenvalues are all positive, therefore it is Hermitian positive definite. Clearly if H is nonsingular, then U is uniquely determined by $U = AH^{-1}$. □

We will refer to U as the unitary polar factor.

Theorem 2.2. *For $A \in \mathbb{C}^{m\times n}$, let $A = UH$ be its polar decomposition. A is normal if and only if U and H are commute.*

Proof. ($\Leftarrow$): If U and H are commute, then

$$\begin{aligned} AA^* &= (UH)(UH)^* \\ &= (HU)(HU)^* = HUU^*H^* = H^2. \\ A^*A &= H^2 = AA^*. \end{aligned}$$

Hence A is normal.

($\Rightarrow$): If A is normal □

References

[1] N. J. Higham, *Functions of Matrices: Theory and Computation*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008. https://doi.org/10.1137/1.9780898717778. (Cited on page 3.)