Project : Computing the Polar Decomposition

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1 Norms and Singular Value Decomposition

1.1 Vector Norms

Before introducing matrix norm, a brief illustration of vector norm is necessary.

Definition 1.1 (Vector Norm)**.** A vector norm on $\mathbb{C}^n$ is a function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ such that it satisfies the following properties

1. $\|x\| \geq 0$ for all $x \in \mathbb{C}^n$.
2. $\|x\| = 0$ if and only if $x = 0$.
3. $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{C}^n$.

Example 1.2. For $x \in \mathbb{C}^n$,

$$\text{1-norm}: \ \|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

$$\text{2-norm (Euclidean Norm)}: \ \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \sqrt{x^* x}.$$

2 Polar Decomposition and its Properties

Throughout this project, we focused on $A \in \mathbb{C}^{n \times n}$. It is well known that for any complex number $\alpha \in \mathbb{C}$, we can rewrite it into the polar form $\alpha = r\mathrm{e}^{\mathrm{i}\theta}$. The polar decomposition is its matrix analogue.

Theorem 2.1 (Polar Decomposition)**.** *For* $A \in \mathbb{C}^{n \times n}$,