Notes on papers

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1 Paper [7, 2022]

The authors try to implement a Jacobi algorithm which utilizes the benefit of fastness of low precision. It first computes the approximate eigendecomposition in low precision, then orthogonalize it using MGS approach and apply to original matrix as a preconditioner, and finally compute the eigensystem of preconditioned matrix.

The paper also provides the bound on distance of low precision matrix of eigenvector and its orthogonal QR factor, a bound on $\mathrm{off}(\text{Preconditioned Matrix})$, and a sufficient condition for the Jacobi algorithm to have quadratic convergence that based on both $\mathrm{Egap}(A)$ and $\mathrm{Egap}(Q^TAQ)$, where

$$\mathrm{Egap}(A) = \min_{\lambda_i(A)\neq\lambda_j(A)} \left|\lambda_i(A) - \lambda_j(A)\right|$$

The key element is the preconditioning method. Notice that the quadratic convergence is readily presented by [6], and by preconditioning, the preconditioned matrix is automatically satisfies the condition presented in [6].

One improvement could be construct a simpler proof of the theorems, especially for the bound of $\|Z - Q\|$ where Z is constructed by low precision eigensolver and Q is its orthogonal QR factor.

1.1 Questions

⋆ I am confused about the following inequality arises from [7, 2022,p. 9, Eq. 4.9]:

$$\left|r_{ij}\right| \le \|R^{-1} - R^T\|. \qquad \text{How is this inequality arises?}$$

⋆ Moreover, do we have the following inequality: suppose $\prod_{i=1}^n A_i$ is well-defined, then for any $j \in \{1, \dots, n\}$,

$$\left\|\prod_{i=1}^n A_i\right\|_F \le \|A_j\|_F \prod_{i=1, i\neq j}^n \|A_i\|_2.$$

In fact this inequality should be true as long as the right-hand side have one Frobenius norm component (by $\|A\|_2 \le \|A\|_F$).

My workaround. Based on $\|AB\|_F \le \|A\|_2\|B\|_F$, we can generalized this to the above equation, therefore, I just want to make sure this is indeed correct.

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2 Paper [5]

The authors trying to accomplish improving the one-sided Jacobi SVD algorithm from the LAPACK.[1] The authors claim that they can achieve 2× speedup. To do so, they do the following three steps:

- Preconditioning, mainly using QR factorization. Given $A \in \mathbb{C}^{n\times n}$, first obtain $AP = Q_1R_1$ which is a rank-revealing QR factorization. Then if R_1 is diagonal dominant, then we are good to go. Otherwise, a LQ decomposition will be performed on R_1, $R_1 = L_1Q_2$. Set $X = L_1$, we have $Q_1^*APQ_2^* = X$.
- Compute the SVD in low precision. Before doing so, the authors check whether the low precision SVD is necessary by using the condition number of R_1, i.e. If $\kappa(R_1)$ is smaller than some tolerance, (usually $u_{\text{high}}^{1/4}$) then we can skip the low precision SVD preconditioning.
- Transform the solution back to the working precision using QR factorization and apply the usual one-sided Jacobi to iteratively refine the solution.

Moreover, they pick out some special cases that the mixed precision Jacobi may even slower. These cases are mainly related to apply unnecessary preconditioning methods.

They also gives a backward analysis on their algorithm. Since the algorithm is based on the one in [3], therefore the stability is ensured automatically.

The key elements of this approach are the following:

- Carefully implement low precision SVD as preconditioner.
- Carefully QR preconditioning as discussed in the first bullet point above.
- The Jacobi SVD algorithm can take the advantage of almost orthogonal columns which leads to locally quadratic convergence.

3 Conclusion

These two papers are all following similar idea compare to my MSc Project [8]. We all use the low precision eigen-(singular value decomposition) solver to obtain a preconditioner. The analysis of sufficient condition for quadratic convergence is well developed and the numerical experiments in [5, 7, 8] are all shown that the mixed precision algorithm is much more better than the original fixed precision one. However, I think I need to trace back to the paper [3, 4, 2, 1] in order to fully understand this precondition method so that I am able to utilizes such method in future projects.

Some further work. Purpose a new way of orthogonalizing the low precision orthogonal matrix to high (double) precision. Instead of QR, in [8], we implement a polar decomposition based orthogonalization process, and this method could be better since computing the polar decomposition using the Newton-Schulz iteration fully utilizes the near orthogonal property of the low precision orthogonal matrix. Moreover, we need to prove that the result we get using the polar decomposition method is not worse than the QR decomposition method both in computational complexity and the error.

[1]The computation routine `xGEJSV`, which has been available since Version 3.2, implements the preconditioned Jacobi SVD algorithm introduced by Drmač and Veselič in [3, 4, 2008]. The algorithm used by these authors utilizes careful QR factorizations as preconditioning method.

References

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