SYLVESTER EQUATION SOLVER

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Contents

1 Introduction

Let $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ be given matrices and define the linear transformation: $\phi(X) : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ by

$$\phi(X) = AX + XB.$$

The linear transformation is nonsingular if and only if A and $-B$ have no eigenvalues in common. This has been formalized by Bhatia and Rosenthal in [3] and we will discussed in the Section 1.1.

Linear equations of the form

$$\phi(X) = AX + XB = C \tag{1}$$

is called the Sylvester equation, which first studied by James Joseph Sylvester in 1884. In his paper, he considered the homogeneous case $AX - XB = 0$.

1.1 Solvablility of the Sylvester Equation

Let us first discuss a simple case when $B = -A$. The Sylvester equation (1) becomes

$$AX - XA = C, \tag{2}$$

and taking the trace of both sides gives

$$\operatorname{trace}(C) = \operatorname{trace}(AX - XA) = \operatorname{trace}(AX) - \operatorname{trace}(XA) = 0.$$

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Hence for the simplified Sylvester equation (2), a solution can exist only when C has zero trace. For example, $AX - XA = I$ has no solution.

For the matrix case, Sylvester proved it in his original paper. Moreover, Rosenblum [6] provided a operator case later.

Theorem 1.1 (Sylvester-Rosenblum Theorem)**.** *If A and B are operators such that $\sigma(A) \cup \sigma(-B) = \emptyset$, then the equation $AX + XB = Y$ has a unique solution X for every operator Y.*

Proof. The proof can be seen from the later section when we discuss the Bartel-Stewart algorithm. □

2 The Bartels-Stewart Algorithm

In 1972, Bartels and Stewart proposed an algorithm that computes the solution to the Sylvester equation [1, 1972]. This method has enjoyed considerable success [2, 1976]. The crux of the Bartels-Stewart algorithm is to use the QR algorithm to compute the real Schur decomposition

$$A = URU^T, \quad B = VSV^T, \tag{3}$$

where R, S are upper quasi-triangular and U and V are orthogonal. Then, we premultiplying the Sylvester equation $AX + XB = C$ by U^T and postmultiplying by V, we have

$$U^TCV = U^TAXV + U^TXBV = U^TAUU^TXV + U^TXVV^TBV. \tag{4}$$

Let $F := U^TCV$ and $Y = U^TXV$, the Sylvester equation becomes

$$F = RY + YS, \qquad R, S \text{ are quasi-upper triangular.}$$

Consider $Y = [y_1|\cdots|y_n]$ and $F = [f_1|\cdots|f_n]$, the (4) can be decompose to n upper triangular linear system, where the jth columns on both sides leads to

$$(R + s_{jj}I)y_j = f_j - \sum_{k=1}^{j-1} s_{kj}y_k, \qquad j = 1\colon n. \tag{5}$$

This is a upper triangular linear system which can be solved by backward substitution efficiently. From (5), y_j is uniquely determined if and only if $R + s_{jj}I$ is not singular, which means $\Lambda(R) \cup \Lambda(-S) = \emptyset$ and this proves Theorem 1.1. Finally, if Y can be uniquely determined, then X can also be uniquely determined via $X = UYV^T$.

3 The Hessenberg-Schur Algorithm

In this section, we provide another algorithm described by Golub, Nash and Van Loan [4] which, instead of, computing the Schur decomposition, it uses an upper Hessenberg matrix instead. Namely, it modified (3) into

$$\begin{aligned} H =& U^TAU, \qquad H \text{ is upper Hessenberg,} \\ S =& V^TBV, \qquad S \text{ is upper triangular.} \end{aligned}$$

Recall that a matrix $H = (h_{ij})$ is upper Hessenberg if $h_{ij} = 0$ for all $i > j + 1$. The orthogonal reduction of A to upper Hessenberg form can be accomplished by Householder matrices in $10m^3/3$ flops [5, 2013, Alg. 7.4.2], and this is a significant reduction in computational expenses. For general matrix A, a Schur decomposition will typically requires $10n^3$ flops according to [4, 1979]. Then, following the exactly same procedure, we can transform the Sylvester equation (1) into

$$HY + YS = F, \quad (H + s_{jj}I)y_j = f_j - \sum_{k=1}^{j-1} s_{kj}y_k, \qquad j = 1\colon n.$$

The above system can be solved using Gaussian elimination with partial pivoting, and it only requires $O(m^2)$ flops after the right-hand side has been computed.

References

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