Appendix A. Proof of Theorem 1

We assumed that the numerator of V is positive. In pratice, as the energy hub is usually dispatched every 15 minutes, this assumption is naturally met.

Assuming $E_i^l \leq E_{i,t} \leq E_i^m$, $W_i^l \leq W_{i,t} \leq W_i^m$, after solving online optimization problem (19), we will show $E_i^l \leq E_{i,t+1} \leq E_i^m$, $W_i^l \leq W_{i,t+1} \leq W_i^m$.

The objective function in (19) is

$$F_t = \sum_{i\in\mathcal{N}} [Q_{i,t}^e p_{i,t} + Q_{i,t}^h h_{i,t} + V(f_{i,t}^E + f_{i,t}^G + f_{i,t}^M)]\tau$$

We prove that BSU boundary constraint (3e) is preserved. The constraint (3f) can be proven in the same way.

Case 1: $E_i^l \leq E_{i,t} < E_i^l + \tau P_i^m/\eta_i^{ed}$. According to power balance equation (5), we substitute $p_{i,t}^b$ into objective function F_t. The partial derivative of F_t with respect to $p_{i,t}^c$ is

$$\begin{aligned}\frac{\partial F_t}{\partial p_{i,t}^c} =& \tau(Q_{i,t}^e \eta_i^{ec} + V\lambda_t^b) \\ \leq& \tau((E_{i,t} - \theta_i^e)\eta_i^{ec} + V\lambda^{b,max}) \\ =& \tau(E_{i,t} - E_i^m - \tau P_i^m/\eta_i^{ed})\eta_i^{ec} < 0\end{aligned}$$

On this account, F_t will decrease if $p_{i,t}^c > 0$. Thus, to minimize F_t, the optimal solution is $p_{i,t}^c = P_i^m$ and $p_{i,t}^d = 0$, so $E_{i,t+1} = E_{i,t} + \tau P_i^m \eta_i^{ec}$. Clearly, $E_{i,t+1} > E_i^l$, and $E_{i,t+1} < E_i^m$ follows from the fact that the numerator of V_i^e is positive.

Case 2: $E_i^l + \tau P_i^m/\eta_i^{ed} \leq E_{i,t} \leq E_H$, where $E_H = V(\lambda^{b,max}/\eta_i^{ec} - \eta_i^{ed}\lambda^{b,min}) + E_i^l + \tau P_i^m/\eta_i^{ed}$.

The definition of V indicates

$$\begin{aligned}& V\cdot(\lambda^{b,max}/\eta_i^{ec} - \eta_i^{ed}\lambda^{b,min}) \\ & \leq V_i^e \cdot (\lambda^{b,max}/\eta_i^{ec} - \eta_i^{ed}\lambda^{b,min}) \\ & = E_i^m - E_i^l - \tau P_i^m \eta_i^{ec} - \tau P_i^m/\eta_i^{ed}\end{aligned}$$

which is

$$\begin{aligned}V\cdot(\lambda^{b,max}/\eta_i^{ec} - \eta_i^{ed}\lambda^{b,min}) + E_i^l + \tau P_i^m/\eta_i^{ed} \\ \leq E_i^m - \tau P_i^m \eta_i^{ec}\end{aligned}$$

When $E_{i,t}$ belongs to the indicated interval, either charging or discharging the BSU with the maximum power, we still have $E_i^l \leq E_{i,t+1} \leq E_i^m$.

Case 3: $E_H < E_{i,t} \leq E_i^m$. In this case, we only need to consider $E_{i,t} \in [E_i^m - \tau P_i^m \eta_i^{ec}, E_i^m]$. Otherwise, we must have $E_{i,t} < E_i^m$. Calculate the partial derivative:

$$\frac{\partial F_t}{\partial p_{i,t}^d} = -\tau\left(\frac{Q_{i,t}^e}{\eta_i^{ed}} + V\lambda_t^b\right) < -\tau\left(\frac{E_H - \theta_i^e}{\eta_i^{ed}} + V\lambda_t^b\right)$$

Substituting E_H into the righthand side, we obtain

$$\begin{aligned}\frac{\partial F_t}{\partial p_{i,t}^d} <& -\frac{\tau V}{\eta_i^{ed}}\left(\left(\frac{\lambda^{b,max}}{\eta_i^{ec}} - \eta_i^{ed}\lambda^{b,min}\right) - \frac{\lambda^{b,max}}{\eta_{i,c}^e} + \eta_i^{ed}\lambda_t^b\right) \\ =& \tau V\left(\lambda^{b,min} - \lambda^b\right) \leq 0\end{aligned}$$

In view of this, the objective function will decrease with respect to $p_{i,t}^d$. To minimize F_t, $p_{i,t}^d = P_i^m$ and $p_{i,t}^c = 0$ hold, as a result, $E_{i,t+1} = E_{i,t} - \tau P_i^m/\eta_i^{ed} < E_i^m$, and $E_{i,t+1} > E_i^l$ follows from

$$E_H - \tau P_i^m/\eta_i^{ed} = V(\lambda^{b,max}/\eta_i^{ec} - \eta_i^{ed}\lambda^{b,min}) + E_i^l$$

and the numerator of V_i^e is positive.

In all the above three cases, $E_{i,t+1} \in [E_i^l, E_i^m]$. For the same reason, $W_{i,t+1} \in [W_i^l, W_i^m]$. Theorem 1 follows from induction. ■

Appendix B. Proof of Theorem 2

For any period t, let $\widehat{\boldsymbol{x_t}}$ and $\overline{\boldsymbol{x_t}}$ denote the optimal solution of online problem (19) and hindsight problem (9), and the corresponding costs are $\widehat{f_t}$ and $\overline{f_t}$, respectively. Random variables $\boldsymbol{\xi_t}$ are assumed to be independent identically distributed (i.i.d.). The conclusion remains if $\boldsymbol{\xi_t}$ is Markovian [33].

Take expectation on the drift-plus-cost function:

$$\begin{aligned}
&\mathbb{E}[\Delta(\boldsymbol{\Theta_t})] + V\mathbb{E}[\widehat{f_t}]\tau \\
&\quad \le A + \sum_{i\in\mathcal{N}} \mathbb{E}[(Q_{i,t}^e \widehat{p_{i,t}} + Q_{i,t}^h \widehat{h_{i,t}})]\tau + V\mathbb{E}[\widehat{f_t}]\tau \\
&\quad \le A + \sum_{i\in\mathcal{N}} \mathbb{E}[(Q_{i,t}^e \overline{p_{i,t}} + Q_{i,t}^h \overline{h_{i,t}})]\tau + V\mathbb{E}[\overline{f_t}]\tau
\end{aligned}$$

where the first inequality is obtained from (17); the second inequality follows from the fact that $\widehat{\boldsymbol{x_t}}$ is the optimal solution of problem (19).

Note that $\boldsymbol{\xi_t}$ is i.i.d., so is $\overline{\boldsymbol{x_t}}$. According to the strong law of large numbers

$$\Pr(\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \overline{\boldsymbol{x_t}} = \mathbb{E}[\overline{\boldsymbol{x_t}}]) = 1$$

so we have

$$\begin{aligned}
&\mathbb{E}[\Delta(\boldsymbol{\Theta_t})] + V\mathbb{E}[\widehat{f_t}]\tau \\
&\le A + V\mathbb{E}[\overline{f_t}]\tau + \sum_{i\in\mathcal{N}} \mathbb{E}[\lim_{T\to\infty} \frac{\tau}{T}\sum_{t=1}^{T}(Q_{i,t}^e \overline{p_{i,t}} + Q_{i,t}^h \overline{h_{i,t}})]
\end{aligned}$$

According to equation (13),

$$\sum_{i\in\mathcal{N}} \mathbb{E}[\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}(Q_{i,t}^e \overline{p_{i,t}} + Q_{i,t}^h \overline{h_{i,t}})] = 0$$

Therefore,

$$\mathbb{E}[\Delta(\boldsymbol{\Theta_t})] + V\mathbb{E}[\widehat{f_t}]\tau \le A + V\mathbb{E}[\overline{f_t}]\tau$$

By summing both sides over $t \in \{1, \cdots, T\}$, we have

$$\sum_{t=1}^{T} V\mathbb{E}[\widehat{f_t}]\tau \le AT + \sum_{t=1}^{T} V\mathbb{E}[\overline{f_t}]\tau - \mathbb{E}[L(\boldsymbol{\Theta_{T+1}})] + \mathbb{E}[L(\boldsymbol{\Theta_1})]$$

Since $L(\boldsymbol{\Theta_{T+1}})$ and $L(\boldsymbol{\Theta_1})$ take finite values, dividing both sides of this inequality by VT and letting $T \to \infty$, we have

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\widehat{f_t}]\tau \le \frac{A}{V} + \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\overline{f_t}]\tau$$

implying that

$$\widehat{C} - \overline{C} \le A/V$$

which is the desired result in Theorem 2. ■