Butterfly-Core Community Search over Labeled Graphs

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ABSTRACT

Community search aims at finding densely connected subgraphs for query vertices in a graph. While this task has been studied widely in the literature, most of the existing works only focus on finding homogeneous communities rather than heterogeneous communities with different labels. In this paper, we motivate a new problem of cross-group community search, namely Butterfly-Core Community (BCC), over a labeled graph, where each vertex has a label indicating its properties and an edge between two vertices indicates their cross relationship. Specifically, for two query vertices with different labels, we aim to find a densely connected cross community that contains two query vertices and consists of butterfly networks, where each wing of the butterflies is induced by a k-core search based on one query vertex and two wings are connected by these butterflies. Indeed, the BCC structure admits the structure cohesiveness and minimum diameter, and thus can effectively capture the heterogeneous and concise collaborative team. Moreover, we theoretically prove this problem is NP-hard and analyze its non-approximability. To efficiently tackle the problem, we develop a heuristic algorithm, which first finds a BCC containing the query vertices, then iteratively removes the farthest vertices to the query vertices from the graph. The algorithm can achieve a 2-approximation to the optimal solution. To further improve the efficiency, we design a butterfly-core index and develop a suite of efficient algorithms for butterfly-core identification and maintenance as vertices are eliminated. Extensive experiments on eight real-world networks and four novel case studies validate the effectiveness and efficiency of our algorithms.

1 INTRODUCTION

Graphs are extensively used to represent real-life network data, e.g., social networks, academic collaboration networks, expertise networks, professional networks, and so on. Indeed, most of these
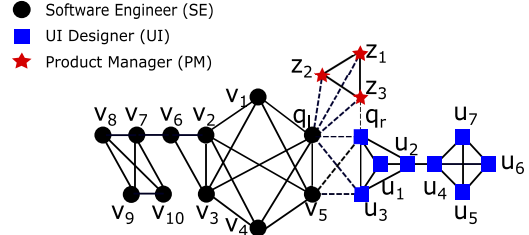
Figure 1: An example of labeled graph G in IT professional networks with three labels denote in different shapes and colors: SE, UI and PM. The collaborations between two employees of the same role (across over different roles) denote by the solid edges (dashed edges).

networks can be regarded as labeled graphs, where vertices are usually associated with attributes as labels (e.g., roles in IT professional networks). A unique topological structure of labeled graph is the cross-group community, which refers to the subgraph formed by two knit-groups with close collaborations but different labels. For example, a cross-role business collaboration naturally forms a cross-group community in the IT professional networks.

In the literature, numerous community models have been proposed for community search based on various kinds of dense subgraphs, e.g. quasi-clique [10], k-core [11, 26, 36], k-truss [20], and densest subgraph. For example, in the classical k-core based community model, a subgraph of k-core requires that each vertex has at least k neighbors within k-core [3, 35]. The cohesive structure of k-core ensures that group members are densely connected with at least k members. However, most of the existing studies only focus on finding homogeneous communities [13, 18], which treat the semantics of all vertices and edges without differences.

Motivating example. Figure 1 shows an example of IT professional networks G, where each vertex represents an employee and an edge represents the collaboration relationship between two employees. The vertices have three shapes and colors, which represent three different roles: "Software Engineer (SE)", "UI Designer (UI)", and "Product Manager (PM)". The edges have two types of solid and dashed lines. A solid edge represents a collaboration between two employees of the same role. A dashed edge represents a collaboration across over different roles, e.g., the dashed edge (q_l, q_r) represents the collaboration between two employees of SE and UI roles respectively. Our motivation is how to effectively find communities formed by these cross-group collaborations given two employees with different roles. Interestingly, considering a search for cross-group communities containing two query vertices $Q = \{q_l, q_r\}$, we find that conventional community search models cannot discover satisfactory results:

- **Structural community search**. This kind algorithms find communities containing all query vertices over a simple graph,
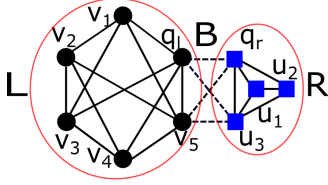
Figure 2: An example of butterfly-core community H on G in Figure 1. B is a bow tie formed by all dashed edges across two labeled groups.

which ignores the vertex labels and treats G as a homogeneous graph. W.l.o.g. we select k-core [35] as an example, the maximum core value of q_l, q_r are 4 and 3 respectively, one limitation of this model is the smaller vertex coreness dominates k value to contain all query vertices. Each vertex on G has a degree of at least 3, thus the whole graph of G is returned as the answer. However, the model suffers from several disadvantages: (1) it fails to capture different community densities of two teams. (2) it treats the semantics of all edges equally, which not only ignores the semantics of different edges but also mixes different teams. (3) the vertices span a long distance to others, e.g., the distance between v_8 and u_7 is 8. Many vertices are irrelevant to the query vertices, such as the vertices $\{v_6, v_7, v_8, v_9, v_{10}\}$ and $\{u_4, u_5, u_6, u_7\}$. (4) the vertices with irrelevant labels to the query vertices, e.g., z_1's label is PM different with SE and UI. Other community search models add graph size constraints such as the minimum size of k-core [23] or the minimum diameter [20]. However, such improved models find the answer of $\{q_l, q_r, v_5, u_3\}$, which suffers from missing many group members with no cross-group edges.

- **Attributed community search**. The studies of attributed community [13, 18] focus on identifying the communities that have cohesive structure and share homogeneous labels. For instance, using query vertices $Q = \{q_l, q_r\}$, keywords $W_q = \{SE, UI\}$ as input, [13] returns a k-core subgraph and maximizes the number of keywords all the vertices share. Since the vertices only contain one label (keyword) on the labeled graph, the cross-group community share no common attributes, then the keyword cohesiveness is always 0, it will return empty result.

- **Ours**. The expected answer of our proposed *Butterfly-Core Community (BCC) search*, aims to find cross-group communities using two query vertices, as shown in Figure 2. The cross-group community has three key parts. The first L is the induced subgraph formed by the vertices with the label SE, which is a 4-core. The second R is the 3-core of vertices with the label UI. The third part B is the bipartite graph (subgraph induced by vertices $\{q_l, q_r, v_5, u_3\}$ with only dashed edges) across over two groups of SE and UI vertices containing a butterfly, i.e., a complete 2×2 biclique.

Motivated by the above example, in this paper, we study a novel problem of cross-group community search in the labeled graph, namely BCC Search. Specifically, given a labeled graph G, two query vertices with different labels $Q = (q_l, q_r) \in G$ and integers $\{k_1, k_2, b\}$, we define the (k_1, k_2, b)-BCC search problem as to find out a densely connected cross-group community that is related to query vertices Q.

In light of the above, we are interested in developing efficient algorithms for the BCC search problem. However, the efficient extraction of BCCs raises significant challenges. We theoretically prove that the BCC search problem is NP-hard and cannot be approximated in polynomial time within a factor $(2 - \varepsilon)$ of an optimal answer with the smallest diameter for any small $\varepsilon \in (0, 1)$ unless $P = NP$. Therefore, we develop a greedy algorithmic framework, which first finds a BCC containing Q and then iteratively maintains BCC by removing the farthest vertices to Q from the graph. The method can achieve a 2-approximation to the optimal BCC answer, obtaining no greater than twice the smallest diameter. To further improve efficiency, we construct the offline butterfly-core index and develop efficient algorithms for butterfly-core identification and maintenance. In addition, we further develop a fast algorithm L^2P-BCC, which integrates several optimization strategies including the bulk deletion of removing multiple vertices each time, the fast query distance computation, leader pair strategy, and the local exploration to generate a small candidate graph. We further discuss how to extend the BCC model to handle queries with multiple vertex labels. To summarize, we make the following contributions.

- We study and formulate a novel problem of BCC search over labeled graphs. We propose a (k_1, k_2, b)-BCC model to find a cross-group community containing two query vertices q_l and q_r with different labels. Moreover, we give the BCC-problem analysis and illustrate useful applications. (Section 3).
- We show the BCC problem is NP-hard and cannot be approximated in polynomial time within a factor $(2 - \varepsilon)$ of the optimal diameter for any small $\varepsilon \in (0, 1)$ unless $P = NP$ (Section 4).
- We develop a greedy algorithm for finding BCC containing two query vertices, which achieves a 2-approximation to an optimal BCC answer with the smallest diameter. The algorithm iteratively deletes the farthest vertices from a BCC, which achieves a small diameter (Section 5).
- We develop several improved strategies for fast BCC search. First, we design an efficient bulk deletion strategy to remove multiple vertices at each iteration; Second, we optimize the shortest path computations of two query vertices; Third, we make a leader pair algorithm for butterfly count maintenance; Finally, we propose an index based local search method (Section 6).
- We extend the BCC model to handle cross-group communities with multiple vertex labels and leverage our L^2P-BCC techniques to develop an efficient search solution (Section 7).
- We conduct extensive experiments on eight real-world datasets with ground-truth communities. Four interesting case studies of cross-group communities are discovered by our BCC model on real-world *global flight networks*, *international trade networks*, *complex fiction networks*, and *academic collaboration networks*. The results show that our proposed algorithms can efficiently and effectively discover BCC, which significantly outperform other approaches (Section 8).

We discuss related work in Section 2, and conclude the paper with a summary in Section 9.

2 RELATED WORK

Attributed community discovery. The studies of attributed community discovery involve two problems of attributed community detection and attributed community search. Attributed community detection is to find all communities in an attributed graph where vertices have attributed labels [5, 17, 30, 47]. Thus, attributed community detection is not the same as our problem in terms of community properties and input data. A survey of clustering on attributed graphs can be found in [5]. In addition, given a set of query vertices and query attributes, attributed community search finds the query-dependent communities in which vertices share homogeneous query attributes [13, 18, 29]. Most recently, Zhang et al. [45] proposed an attributed community search model using only query keywords but no query vertices. Other related works to ours are community detection in heterogeneous networks [38] where vertices have various vertex labels. However, heterogeneous communities are defined based on meta patterns, which are different from our communities across over two labeled groups. Compared with all the above studies, our butterfly-core community search is a novel problem over labeled graphs, which has not been studied before.

Community search. Community search finds the query-dependent communities in a graph [6, 14, 19]. Community search models can be categorized based on different dense subgraphs including k-core [2, 11, 26, 27, 36, 40], k-truss [20], quasi-clique [10], and densest subgraph [43]. Sozio and Gionis defined the problem of community search and proposed a k-core based model with the distance and size constraints [36]. All these community models work on simple structural graphs, which ignore the vertex labels. Recently, several complex community models have been studied for various graph data, such as directed graphs [15, 24, 28], weighted graphs [37, 46], spatial-social networks [1, 7, 8, 22] and so on. Most attributed community search studies aim at finding the communities that have a dense structure and similar attributes [13, 18, 29, 45]. This is different from our studies over labeled graphs, which trends to find two groups with different labels. Our community model requires the dense structure to appear not only in the inter-groups but also between two intra-groups. Most recently, there are two studies that investigate community search on heterogeneous information networks [16, 21], where vertices belong to multiple labeled types. Fang et al. [16] leveraged meta-path patterns to find communities where all vertices have the same label of a query vertex and close relationships following the given meta-paths. Jian et al. [21] proposed the relational constraint to require connections between labeled vertices in a community. They developed heuristic solutions for detecting and searching relational communities due to the hard-to-approximate problem. Both studies are different from our BCC search model that takes two query vertices with different labels and finds a leader-pair based community integrating two cross-over groups. Our problem is NP-hard but can be approximately tackled in polynomial time with an approximate ratio of 2.

Butterfly counting. In the bipartite graph analytics [34, 42], butterfly is a cohesive structure of 2×2 biclique. Butterfly counting is to calculate the number of butterflies in a bipartite graph [25, 32, 33, 41, 44]. Sanei et al. [32] proposed exact butterfly counting and approximation solutions using randomized strategies. Wang et al. [41] further optimized the butterfly counting by assigning high

Table 1: Frequently used notations.

Notation	Description
$G = (V, E, \ell)$	a labeled graph G with a vertex label function ℓ
q_l, q_r	query vertices q_l, q_r
L, B, R	left k_1-core, bipartite graph, right k_2-core
V_L, V_R	the vertex sets of left k_1-core and right k_2-core
$dist_H(u, v)$	the length of the shortest path between u and v in H
$diam(H)$	the diameter of a graph H
$\chi(v)$	the butterfly degree of vertex v
$\ell(v)$	the label associated with vertex v
$N(v)$	the set of neighbor vertices of vertex v in graph G
$N(u) \cap N(v)$	the common neighbors of u and v
N_v^2	the vertices within v's 2-hop neighborhood (excluding v)

degree vertex with high priority to visit wedges. However, these studies focus on the *global butterfly counting* to compute the butterfly number over an entire graph. While our BCC search algorithms aim at finding a few vertices with large butterfly degree as the leaders of cross-group communities. Moreover, our algorithms can dynamically update such leader vertices to admit butterfly degree when the graph structure changes. Overall, our proposed butterfly search solutions are efficient to find leader vertices and update butterfly degrees *locally*, which can avoid the *global* butterfly counting multiple times.

3 PROBLEM FORMULATION

In this section, we introduce the definition and our problem.

3.1 Labeled Graph

Let $G = (V, E, \ell)$ be a labeled graph, where V is a set of vertices, $E \subseteq V \times V$ is a set of undirected edges, and $\ell : V \rightarrow \mathcal{A}$ is a vertex label function mapping from vertices V to labels $\mathcal{A}$. For each vertex $v \in V$, v is associated with a label $\ell(v) \in \mathcal{A}$. The edges have two types, i.e., given two vertices $\{u, v\} \in V$, if with same label $\ell(u) = \ell(v)$, (u, v) is a homogeneous edge; otherwise, if $\ell(u) \neq \ell(v)$, (u, v) is a heterogeneous (cross) edge. For example, consider G in Figure 1, G has three labels: SE, UI and PM. The vertex v_1 has a label of SE. The edge (v_1, v_2) is a homogeneous edge for (SE-SE). The edge (v_5, u_3) is a heterogeneous edge for (SE-UI).

Given a subgraph $H \subseteq G$, the degree of a vertex v in H is denoted as $deg_H(v) = |N_H(v)|$, where $N_H(v)$ is the set of v's neighbors in H. For two vertices u, v, we denote $dist_H(u, v)$ as a length of the shortest path between u and v in H, where $dist_H(u, v) = \infty$ if u and v are disconnected. The diameter of H is defined as the maximum length of the shortest path in H, i.e., $diam(H) = \max_{u,v \in V(H)} dist_H(u, v)$ [20].

3.2 K-Core and Butterfly

We give two definitions of k-core [35] and butterfly [32, 41].

DEFINITION 1 (k-CORE). *Given a subgraph $H \subseteq G$ and an integer k, H is a k-core if each vertex v has at least k neighbors within H, i.e., $deg_H(v) \geq k$.*

The coreness $\delta(v)$ of a vertex $v \in V$ is defined as the largest number k such that there exists a connected k-core containing v. In Figure 2, L is a 4-core as each vertex has at least 4 neighbors within L. Next, we define the butterfly [32, 41] in a bipartite graph.

DEFINITION 2 (BUTTERFLY). *Given a bipartite graph $B = (V_L, V_R, E)$ where $E \subseteq V_L \times V_R$, a butterfly H is a 2×2 biclique*

of G induced by four vertices $v_{l_1}, v_{l_2} \in V_L$, $v_{r_1}, v_{r_2} \in V_R$ such that all four edges (v_{l_1}, v_{r_1}), (v_{l_1}, v_{r_2}), (v_{l_2}, v_{r_1}) and (v_{l_2}, v_{r_2}) exist in H.

DEFINITION 3 (BUTTERFLY DEGREE). *Given a bipartite graph $B = (V_L, V_R, E)$, the butterfly degree of vertex v is the number of butterfly subgraphs containing v in G, denoted by $\chi(v)$.*

EXAMPLE 1. *In Figure 2, the subgraph B is a butterfly since it is a 2×2 biclique formed by four vertices $\{q_l, v_5, q_r, u_3\}$. There exists a unique butterfly B containing the vertex q_r. Thus, the butterfly degree of q_r is $\chi(q_r) = 1$.*

3.3 Butterfly-Core Community Model

We next discuss a few choices to model the cross-group relationships between two groups V_L and V_R with different labels in the community H, and analyze their pros and cons. To quantify the strength of cross-group connections, we use the number of butterflies between two groups, denoted as b.

- First, we consider that $\chi(v) \geq b$ for each vertex $v \in H$. It requires that each vertex's butterfly count is at least b, i.e., $\chi(v) \geq b$. This constraint is too strict, which may miss some vertices without heterogeneous edges. Take Figure 1 as an example, some vertices act like leaders or liaisons who are in charge of communications across the groups, i.e., $\{q_l, q_r, v_5, u_3\}$, while some vertices mostly link within their own group with less interactions across the groups such as $\{v_1, v_2, v_3, v_4\}$. If we model in this way, an input $Q = \{v_1, q_r\}$ requires that v_1 exists in at least one butterfly, which is impossible.
- Second, we alternatively consider $\sum_{v \in V(H)} \chi(v) \geq b$, which requires the total butterfly count in H is at least $\lceil b/4 \rceil$. However, it is hard for us to determine the parameter b as we cannot estimate a qualified number of butterflies in community H, which is a global criterion varying significantly over different kinds of graphs.
- Finally, we consider a constraint between two groups of vertices V_L and V_R that $\exists v_l \in V_L$ and $\exists v_r \in V_R$ to make $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$ hold. It is motivated by real applications. Generally, one collaboration community has at least one leader or liaison in each group, so we require there exists at least one vertex in each group whose butterfly count is at least b. In this setting, no matter the input query vertices are leaders biased (e.g., $Q = \{q_l, q_r\}$) or juniors biased (e.g., $Q = \{v_1, u_1\}$), the underlying community is identical.

In view of these considerations, we define the butterfly-core community as follows.

DEFINITION 4 (BUTTERFLY-CORE COMMUNITY). *Given a labeled graph G, a (k_1, k_2, b)-butterfly-core community (BCC) $H \subseteq G$ satisfies the following conditions:*

1. Two labels: there exist two labels $A_l, A_r \in \mathcal{A}$, $V_L = \{v \in H : \ell(v) = A_l\}$ and $V_R = \{v \in H : \ell(v) = A_r\}$ such that $V_L \cap V_R = \emptyset$ and $V_L \cup V_R = V_H$;

2. Left core: the induced subgraph of H by V_L is k_1-core;

3. Right core: the induced subgraph of H by V_R is k_2-core;

4. Cross-group interactions: $\exists v_l \in V_L$ and $\exists v_r \in V_R$ such that butterfly degree $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$ hold.

In terms of vertex labels, condition (1) requires that the BCC contains exactly two labels for all vertices. In terms of homogeneous groups, conditions (2) and (3) ensure that each homogeneous group satisfies the cohesive structure of k-core, in which community members are internally densely connected. In terms of cross-group interactions, condition (4) targets two representative vertices of two homogeneous groups, which have a required number of butterflies with densely cross-group interactions. Moreover, we call the two vertices v_l and v_r with $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$ as a *leader pair*.

EXAMPLE 2. *Figure 2 shows a $(4, 3, 1)$-BCC. The subgraphs L and R are respectively the left 4-core group and the right 3-core group, respectively. The subgraph B is a butterfly across over two groups L and R, and $\chi(q_l) = \chi(q_r) = 1$.*

3.4 Problem Formulation

We formulate the BCC-Problem studied in this paper.

PROBLEM 1. *(BCC-Problem) Given $G(V, E, \ell)$, two query vertices $Q = \{q_l, q_r\} \subseteq V$ and three integers $\{k_1, k_2, b\}$, the BCC-Problem finds a BCC $H \subseteq G$, such that:*

1. Participation & Connectivity: H is a connected subgraph containing Q;

2. Cohesiveness: H is a (k_1, k_2, b)-BCC.

3. Smallest diameter: H has the smallest diameter, i.e., $\nexists H' \subseteq G$, such that $diam(H') < diam(H)$, and H' satisfies the above conditions 1 and 2.

The BCC-Problem prefers a tight BCC with the smallest diameter such that group members have a small communication cost, to remove query unrelated vertices. In addition, we further study the BCC-problem for multiple query vertices, generalizing the BCC model from 2 group labels to m group labels where $m \geq 2$ in Section 7.

EXAMPLE 3. *Consider a labeled graph G in Figure 1. Assume that the inputs $Q = \{q_l, q_r\}$, $k_1 = 4$, $k_2 = 3$, and $b = 1$. The answer is the $(4, 3, 1)$-butterfly-core community containing Q as shown in Figure 2.*

3.5 Why Butterfly-Core Community Model?

Why butterfly. A butterfly is a complete bipartite subgraph of 2×2 vertices, which serves as the fundamental motif in bipartite graphs. For two groups V_L and V_R with different labels, we model the collaborative interactions between two groups V_L and V_R using the butterfly model [4, 12, 31]. More butterflies indicate 1) stronger connections between two groups and 2) similar properties sharing within the same group members, which are validated in many application scenarios. For instance, in the users-items bipartite graph G, a user $x_1 \in V_L$ buy an item $y_1 \in V_R$, then we have an edge (x_1, y_1) in G. Thus, two users x_1 and x_2 buy the same two items y_1 and y_2, which forms a butterfly in G. Two users purchase the same items, indicating the more similar purchasing preferences of them and more butterflies in the community. Similar cases happen in the common members of *the board of directors* between *two different companies* and also the common members of *the steering committee* in *two conference organizations*. Moreover, in the email communication networks, threads of emails are delivered between *two partner groups* and also cc's to superiors on both sides. The superiors of

two partner groups receive the most emails and play the *leader pair* positions of our BCC model. Overall, the butterfly plays an essential role as the basis higher-order motif in bipartite graphs, which can be regarded as an implicit connection measure between two same labeled vertices.

Why BCC model. The BCC model inherits several good structural properties and efficient computations. First, the community structure enjoys high computational efficiency. The k-core is a natural and cohesive subgraph model of communities in real applications, requiring that every person has at least k neighbors in social groups, which can be computed faster than k-clique. In addition, butterfly listing takes a polynomial time complexity and enjoys an efficient enumeration, which could be optimized by assigning the wedge visiting priority based on vertex degrees [31, 41]. Second, two labeled groups in BCC model admit practical cases of different group densities in real-world applications. Our BCC model crosses over two labeled groups, which may have different group sizes and densities. Thus, two different k-core parameters, i.e., k_1 and k_2 are greatly helpful to capture different community structures of two groups. One simple way for parameter setting is to automatically set k_1 and k_2 with the coreness of two queries q_l and q_r respectively. Third, automatic identification of leader pair in the BCC discovery. The constraint of cross-group interactions is motivated by real-world scenarios that leaders or liaisons in each group always take most interactions with the other group.

3.6 Applications

In the following, we illustrate three representative applications of butterfly-core community search.

- **Interdisciplinary collaboration search**. Given two principal investigators from different departments in the universities, who intend to form a team to apply for an interdisciplinary research grant. The team is better formed by two cohesive groups with good inner-group communications. Moreover, the principal investigators or liaisons should also have cross-group communications.
- **Professional team discovery**. In high-tech companies, there are usually many cross-department projects between two teams with different sizes of employees. Moreover, the technical leader and product manager of each team always take charge of the cross-group communications and information sharing, which naturally form a butterfly, i.e., 2×2 biclique.
- **Various real-world cross-group mining tasks**. BCC search can be applied on various real-world labeled graphs, e.g., *global flight networks*, *international trade networks*, *complex fiction networks*, and *academic collaboration networks*, as reported in four interesting case studies in Section 8.

4 HARDNESS AND APPROXIMATION

In this section, we analyze the hardness and non-approximability of the BCC-Problem.

Hardness. We define a decision version of the BCC-Problem.

PROBLEM 2. *(BCC-Decision Problem) Given a labeled graph $G(V, E, \ell)$, two query vertices $Q = \{q_l, q_r\} \subseteq V$ with different labels,*

and parameters $\{k_1, k_2, b\}$, test whether G has a connected butterfly-core subgraph containing Q with a diameter at most d.

THEOREM 1. *The BCC-Problem is NP-hard.*

PROOF. We reduce the well-known NP-hard problem of Maximum Clique (decision version) to BCC-Problem. Given a graph $G(V, E)$ and a number k, the maximum clique decision problem is to check whether G contains a clique of size k. From this, construct an instance $G'(V', E', \ell)$ of BCC-Problem as follows. $G' = (V' : V + V_c, E' : E + E_c + E_b, \ell)$. For each vertex $v \in V$ we assign the label of l_1, i.e., $\ell(v) = l_1, \forall v \in V$. $G_c(V_c, E_c)$ is a copy of $G(V, E)$ associated with labels l_2, i.e., $\ell(v) = l_2, \forall v \in V_c$. E_b is the edge set that connects any two vertices v_i and v_j where $v_i \in G$ and $v_j \in G_c$, i.e., $E_b = V \times V_c$. Set parameters $k_1 = k_2 = k - 1, b = 1$ (actually b could choose any value fits $b \leq (k - 1) \times \binom{k}{2}$), $d = 1$ and the query vertices $Q = \{q_l, q_r\}$ where $q_l \in G$ and $q_r \in G_c$, i.e., $\ell(q_l) \neq \ell(q_r)$. We show that the instance of the maximum clique decision problem is a YES-instance iff the corresponding instance of BCC-Problem is a YES-instance.

$(\Rightarrow)$: Clearly, any clique with at least k vertices is a connected $(k - 1)$-core, since G_c is a copy of G then there will be a connected $(k - 1)$-core in G_c. Because any edge between the vertices in G and G_c are connected then for each vertex $v \in V$, it forms 1 butterfly with any vertex exclude itself in V and any two vertices in V_c; the same proof to the vertex $v \in V_c$. Then H is a $(k - 1, k - 1, 1)$-BCC with a diameter 1.

$(\Leftarrow)$: Given a solution H for BCC-Problem, we split H into two parts H_l whose vertices label is $\ell(q_l)$ and H_r whose vertices label is $\ell(q_r)$. Since H is a $(k - 1, k - 1, 1)$-BCC, H_l and H_r must contain at least k vertices, $diam(H) = d = 1$ implies H_l and H_r are both cliques which implies G has a clique since G_c is a copy of G. $\square$

Given the NP-hardness of the BCC-Problem, it is interesting whether it can be approximately tackled. We analyze the approximation and non-approximability as follows.

Approximation and non-approximability. For $\alpha \geq 1$, we say that an algorithm achieves an α-approximation to BCC-Problem if it outputs a connected (k_1, k_2, b)-BCC $H \subseteq G$ such that $Q \subseteq H$ and $diam(H) \leq \alpha \cdot diam(H^*)$, where H^* is the optimal BCC. That is, H^* is a connected (k_1, k_2, b)-BCC s.t. $Q \subseteq H^*$, and $diam(H^*)$ is the minimum among all such BCCs containing Q.

THEOREM 2. *Unless $P = NP$, for any small $\epsilon \in (0, 1)$ and given parameters $\{k_1, k_2, b\}$, BCC-Problem cannot be approximated in polynomial time within a factor $(2 - \epsilon)$ of the optimal.*

PROOF. We prove it by contradiction. Assume that there exists a $(2 - \epsilon)$-approximation algorithm for the BCC-Problem in polynomial time complexity, no matter how small the $\epsilon \in (0, 1)$ is. This algorithm can distinguish between the YES and NO instances of the maximum clique decision problem. That is, if an approximate answer of the reduction problem has a diameter of 1, it corresponds to the Yes-instance of maximum clique decision problem; otherwise, the answer with a diameter value of no less than 2 corresponds to the No-instance of the maximum clique decision problem. This is impossible unless P=NP. $\square$

Algorithm 1 BCC Online Search (G, Q)

Require: $G = (V, E, \ell)$, $Q = \{q_l, q_r\}$, three integers $\{k_1, k_2, b\}$.
Ensure: A connected (k_1, k_2, b)-BCC O with a small diameter.
1: Find a maximal connected (k_1, k_2, b)-BCC containing Q as G_0; //see Algorithm 2
2: $l \leftarrow 0$;
3: **while** $connect_{G_l}(Q) = \textbf{true do}$
4:     Compute $dist_{G_l}(q, u)$, $\forall q \in Q$ and $\forall u \in G_l$;
5:     $u^* \leftarrow argmax_{u \in G_l} dist_{G_l}(u, Q)$;
6:     $dist_{G_l}(G_l, Q) \leftarrow dist_{G_l}(u^*, Q)$;
7:     Delete u^* and its incident edges from G_l;
8:     Maintain G_l as a (k_1, k_2, b)-BCC; //see Algorithm 4
9:     $G_{l+1} \leftarrow G_l$; $l \leftarrow l + 1$;
10: $O \leftarrow \arg \min_{G' \in \{G_0, ..., G_{l-1}\}} dist_{G'}(G', Q)$;

Algorithm 2 Find $G_0(G, Q)$

Require: $G = (V, E, \ell)$, $Q = \{q_l, q_r\}$, three integers $\{k_1, k_2, b\}$.
Ensure: A connected $\{k_1, k_2, b\}$-BCC G_0 containing Q.
1: $V_L \leftarrow \{v \in V \mid \ell(v) = \ell(q_l)\}$; $V_R \leftarrow \{v \in V \mid \ell(v) = \ell(q_r)\}$;
2: Let L be a k_1-core induced subgraph of G by V_L;
3: Let R be a k_2-core induced subgraph of G by V_R;
4: $B = \{V_B, E_B\}$, where $V_B = V_L \cup V_R$ and $E_B = \{V_L \times V_R\} \cap E$;
5: Butterfly Counting(B); // See Algorithm 3
6: $max_l \leftarrow \max_{u \in V_L} \chi(u)$;
7: $max_r \leftarrow \max_{u \in V_R} \chi(u)$;
8: **if** $max_l < b$ or $max_r < b$ **then**
9:     return $\emptyset$;
10: $G_0 \leftarrow L \cup B \cup R$;

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5 BCC ONLINE SEARCH ALGORITHMS

In this section, we present a greedy algorithm for the BCC-problem, which online searches a BCC. Then, we show that the greedy algorithm can achieve a 2-approximation to optimal answers. Finally, we discuss an efficient implementation of the algorithm and analyze the time and space complexity.

5.1 BCC Online Search Algorithm

We begin with a definition of query distance as follows.

DEFINITION 5 (QUERY DISTANCE). *Given a graph G, a query set Q, and a set of vertices X, the query distance of X is the maximum length of the shortest path from $v \in X$ to a query vertex $q \in Q$, i.e., $dist_G(X, Q) = max_{v \in X, q \in Q} dist_G(v, q)$.*

For simplicity, we use $dist_G(H, Q)$ and $dist_G(v, Q)$ to represent the query distance for the vertex set V_H in $H \subseteq G$ and a vertex $v \in V$. Motivated by [20], we develop a greedy algorithm to find a BCC with the smallest diameter. Here is an overview of the algorithm. First, it finds a maximal connected (k_1, k_2, b)-BCC containing $Q = \{q_l, q_r\}$, denoted as G_0. As the diameter of G_0 may be large, it then iteratively removes from G_0 the vertices far away to Q, meanwhile it maintains the remaining graph as a (k_1, k_2, b)-BCC.

Algorithm. Algorithm 1 outlines a greedy algorithmic framework for finding a BCC. The algorithm first finds G_0 that is a maximal connected (k_1, k_2, b)-BCC containing Q (line 1). Then, we set $l = 0$. For all $u \in G_l$ and $q \in Q$, we compute the shortest distance between q and u, and obtain the vertex query distance $dist_{G_l}(u, Q)$ (line 4). Among all vertices, we pick up a vertex u^* with the maximum distance $dist_{G_l}(u^*, Q)$, which equals $dist_{G_l}(G_l, Q)$ (lines 5-6). Next, we remove the vertex u^* and its incident edges from G_l and also delete vertices/edges to maintain G_l as a (k_1, k_2, b)-BCC (lines 7-8). Then, we repeat the above steps until G_l is disqualified to be a BCC containing Q (lines 3-9). Finally, the algorithm terminates and returns a BCC O, where O is one of the graphs $G' \in \{G_0, ..., G_{l-1}\}$ with the smallest query distance $dist_{G'}(G', Q)$ (line 10). Note that each intermediate graph $G' \in \{G_0, ..., G_{l-1}\}$ is a (k_1, k_2, b)-BCC.

5.2 Butterfly-Core Discovery and Maintenance

We present two important procedures for BCC online search algorithm: finding G_0 (line 1 in Algorithm 1) and butterfly-core maintenance (line 8 in Algorithm 1).

5.2.1 Finding G_0. As an essentially important step, finding G_0 is to identify a maximal connected (k_1, k_2, b)-BCC containing Q in graph G. The challenge lies in finding a butterfly-core structure, which needs to shrink the graph by vertex removals. However, deleting vertices may trigger off the change of vertex coreness and butterfly degree for vertices in the remaining graph. To address it, our algorithm runs the k-core decomposition algorithm twice and then runs the butterfly counting method once. The *general idea* is to first identify a candidate subgraph formed by two groups of vertices sharing the same labels with q_l and q_r. Then, it shrinks the graph by applying core decomposition algorithm, which deletes disqualified vertices to identify k_1-core and k_2-core, denoted by L and R respectively. Then, it counts the butterfly degree for all vertices and checks whether there exists two vertices $v_l \in V_L$ and $v_r \in V_R$ such that $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$ hold.

Algorithm 2 presents the details of finding G_0. For query vertices Q, first we pick out all vertices with the same labels with query vertices (line 1). Each vertex set in V_L and V_R constructs the subgraph and we run the k-core algorithm respectively, find the connected component graph L and R containing query vertices q_l and q_r (lines 2-3). Next we construct a bipartite graph B to find cross-group butterfly structures in the community. V_B consists of the vertex set V_L and V_R, E_B are cross-group edges (line 4). Then, we compute the number of butterflies for each vertex in B using Algorithm 3 (line 5), which is presented in detail in the next paragraph. Algorithm 3 returns the butterfly degree of all the vertices, maintaining two values max_l and max_r to record the maximum butterfly degree on each side. Then we check if there exists at least one vertex whose butterfly degree is no less than b in each side, i.e., $max_l \geq b$ and $max_r \geq b$, otherwise return $\emptyset$ (lines 8-9). Finally, we merge three subgraph parts to form G_0 (line 10).

Next, we describe the details of the butterfly counting algorithm. $N_v^2 = \{w \mid (w, u) \in E \wedge w \neq v, \forall u \in N(v)\}$ is the set of vertices that are exactly in distance 2 from v, i.e., neighbors of the neighbors of v (excluding v itself). To calculate the butterfly degree of each vertex, we take a vertex $v \in L$ as an example (the same for $v \in R$), each butterfly it participates in has one other vertex $w \in L(w \neq v)$ and two vertices $\{u, x\} \in R$. By definition, $w \in N_v^2$. In order to find the number of u, x pairs that v and w form a butterfly, we compute the intersection of the neighbor sets of v and w. We use $|N(v) \cap N(w)|$ to denote the number of common neighbors of v and w, the number of the intersection pairs is $\binom{|N(v) \cap N(w)|}{2}$. The butterfly degree equation

Algorithm 3 Butterfly Counting

Require: $B = (V_B, E_B)$.
Ensure: $\chi(v)$ for all vertices $v \in V_B$.
1: **for** $v \in V_B$ **do**
2:      $P \leftarrow$ *hashmap*; // initialized with zero
3:      **for** $u \in N(v)$ **do**
4:          **for** $w \in N(u)$ **do**
5:              $P[w] \leftarrow P[w] + 1$; //the number of 2-hop paths
6:      **for** $w \in P$ **do**
7:          $\chi(v) \leftarrow \chi(v) + \binom{P[w]}{2}$;
8: **return** $\{\chi(v) | \forall v \in V_B\}$;

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for v is as follows: $\chi(v) = \sum_{u \in N(v)} \sum_{w \in N(u) \setminus v} \binom{|N(v) \cap N(w)|}{2} = \sum_{w \in N_v^2} \binom{|N(v) \cap N(w)|}{2}$. Instead of performing a set intersection at each step, we count and store the number of paths from a vertex $v \in L$ to each of its distance-2 neighbor $w \in L$ by using a hash map P (lines 1-7) [32].

5.2.2 (k_1, k_2, b)-Butterfly-Core Maintenance. Algorithm 4 describes the procedure for maintaining G as a (k_1, k_2, b)-BCC after the deletion of vertices S from G. In Algorithm 1, $S = u^*$ (line 8). Generally speaking, after removing vertices S and their incident edges from G, G may not be a (k_1, k_2, b)-BCC any more, or Q may be disconnected. Thus, Algorithm 4 iteratively deletes vertices having degree less than k_1 (k_2), until G becomes a connected (k_1, k_2, b)-BCC containing Q. It firstly splits the vertex set S to two parts by their labels (line 1). Single vertex also works here, we only need to run Algorithm 4 on the corresponding side. Then run core maintenance algorithm on L and R respectively to maintain L (R) as a k_1-core (k_2-core) (lines 2-3). Next, we count butterfly degree again on the updated B with Algorithm 3 to check if it meets the butterfly constraint of the BCC model (line 4). Finally, Algorithm 4 produces a BCC (line 5).

5.3 Approximation and Complexity Analysis

We first analyze the approximation of Algorithm 1.

THEOREM 3. *Algorithm 1 achieves 2-approximation to an optimal solution H^* of the BCC-problem, that is, the obtained $\{k_1, k_2, b\}$-BCC O has $diam(O) \leq 2diam(H^*)$.*

PROOF. First We have $diam(O) = \max_{u \in O, v \in O} dist_O(u, v)$, and $dist_O(O, Q) = \max_{u \in O, v \in Q} dist_O(u, v)$, because of $Q \subseteq O$ then $dist_O(O, Q) \leq diam(O)$ holds. Suppose longest shortest path in O is from vertex s to t, i.e., $diam(O) = dist_O(s, t)$. For $\forall q \in Q$, $diam(O) = dist_O(s, t) \leq dist_O(s, q) + dist_O(q, t) \leq 2dist_O(O, Q)$.

Next, we prove $dist_O(O, Q) \leq dist_{H^*}(H^*, Q)$ motivated by [20]. Algorithm 1 outputs a sequence of intermediate graphs $\{G_0, ..., G_{l-1}\}$, which are BCC containing query vertices Q. O is the one with the smallest query distance, i.e., $dist_O(O, Q) \leq dist_{G_i}(G_i, Q), \forall i \in \{0, ..., l-1\}$. We consider two cases. (1) $H^* \nsubseteq G_{l-1}$. Suppose the first deleted vertex $v^* \in H^*$ happens in graph G_i, i.e., $H^* \subseteq G_i$, where $0 \leq i \leq l - 2$. The vertex v^* must be deleted because of the distance constraint but not the butterfly-core structure maintenance. Thus, $dist_{G_i}(G_i, Q) = dist_{G_i}(v^*, Q) = dist_{G_i}(H^*, Q)$. As a result, we have $dist_O(O, Q) \leq dist_{G_i}(G_i, Q) = dist_{G_i}(H^*, Q) \leq dist_{H^*}(H^*, Q)$, where the first inequality holds from O has the smallest query distance, and the second inequality holds for that H^* is a subgraph of

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Algorithm 4 Butterfly Core Maintenance(G, S)

Require: $G = L \cup B \cup R$, vertex set S to be removed, three integers $\{k_1, k_2, b\}$.
Ensure: A (k_1, k_2, b)-BCC graph.
1: Split S into S_L and S_R according to their labels;
2: Remove vertices S_L from L and maintain L as k_1-core, update B;
3: Remove vertices S_R from R and maintain R as k_2-core, update B;
4: Run butterfly Counting on B and check there exists one vertex on each side with butterfly degree larger than b;
5: **return** G;

G_i. (2) $H^* \subseteq G_{l-1}$. We prove $dist_{G_{l-1}}(G_{l-1}, Q) \leq dist_{G_{l-1}}(H^*, Q)$ by contradiction. This follows from the fact if $dist_{G_{l-1}}(G_{l-1}, Q) > dist_{G_{l-1}}(H^*, Q)$, then G_{l-1} will not be the last feasible BCC. There must exist a vertex $u^* \in G_{l-1} \setminus H^*$ with the largest query distance $dist_{G_{l-1}}(u^*, Q)$ so that $dist_{G_{l-1}}(u^*, Q) = dist_{G_{l-1}}(G_{l-1}, Q) > dist_{G_{l-1}}(H^*, Q)$. In the next iteration, Algorithm 1 will delete u^* from G_{l-1}, and maintain the butterfly-core structure of G_{l-1}. As H^* is a BCC, Algorithm 1 can find a feasible BCC G_l s.t. $H^* \subseteq G_l$, which contradicts that G_{l-1} is the last feasible BCC. Overall, $dist_O(O, Q) \leq dist_{G_{l-1}}(G_{l-1}, Q) \leq dist_{G_{l-1}}(H^*, Q) \leq dist_{H^*}(H^*, Q)$.

From above we have proved that $diam(O) \leq 2dist_O(O, Q)$ and $dist_O(O, Q) \leq dist_{H^*}(H^*, Q)$. We have $diam(O) \leq 2dist_O(O, Q) \leq 2dist_{H^*}(H^*, Q) \leq 2diam(H^*)$. □

To clarify, Algorithm 1 returns a BCC community O that has the minimum query distance $O = \arg\min_{G' \in \{G_0, ..., G_{l-1}\}} dist_{G'}(G', Q)$ (line 10), rather than the BCC community O' with the smallest diameter, i.e, $O' = \arg\min_{G' \in \{G_0, ..., G_{l-1}\}} diam_{G'}(G', Q)$. This can speedup the efficiency by avoiding expensive diameter calculation. It is intuitive that the BCC community O' also inherit 2-approximation of the optimal answer, i.e., $diam(O') \leq diam(O) \leq 2diam(H^*)$.

Complexity analysis. We analyze the time and space complexity of Algorithm 1. Let t be the number of iterations and $t \leq |V|$. We assume that $|V| - 1 \leq |E|$, w.l.o.g., considering that G is a connected graph.

THEOREM 4. *Algorithm 1 takes $O(t(\sum_{u \in V} d_u^2 + |E|))$ time and $O(|E|)$ space.*

PROOF. Let $|V|$ and $|E|$ denote vertices and edges number respectively, d_u is the vertex degree in the bipartite graph B. First, we consider the time complexity of finding G_0 in Algorithm 2 is $O(\sum_{u \in B} d_u^2 + |E|)$ which runs k-core computation once in $O(|E|)$ and calls Algorithm 3 once in $O(\sum_{u \in B} d_u^2)$[32]. Next, we analyze the time complexity of shrinking the community diameter. First, the computation of shortest distances by a BFS traversal starting from each query vertex $q \in Q$ takes $O(t|Q||E|)$ time for t iterations, here $|Q| = 2$ then $O(t|Q||E|)$ could be eliminated to $O(t|E|)$. Second, the time complexity of maintenance algorithm 4 is $O(t \sum_{u \in B} d_u^2 + |E|)$ in t iterations, i.e., t times butterfly counting but note that t times k-core maintain algorithm in total is $O(|E|)$. The number of removed edges is no less than $(min\{k_1, k_2\} - 1)$, thus the total number of iterations is $t \leq O(min\{|V| - max\{k_1, k_2\}, |E|/min\{k_1, k_2\}\})$ which is $O(|V|)$ since $|V| < |E|$. As a result, the time complexity of Algorithm 1 is $O(t(\sum_{u \in V} d_u^2 + |E|))$. Next, we analyze the space complexity of Algorithm 2. It takes $O(|V| + |E|)$ space to store graph G and $O(|V|)$ space to keep the vertex information including the

Algorithm 5 Fast Query Distance Computation(G_i, q, D_i)

Require: A graph G_i, query vertex q, a set of removal vertices D_i.
Ensure: The updated distance $dist(v, q)$ for all vertices v.
1: Remove all vertices D_i and their incident edges from G_i;
2: $d_{min} \leftarrow min_{v \in D_i} dist_{G_i}(v, q)$;
3: $S_s = \{v \in V(G_i) \setminus D_i \mid dist_{G_i}(v, q) = d_{min}\}$;
4: $S_u = \{v \in V(G_i) \setminus D_i \mid dist_{G_i}(v, q) > d_{min}\}$;
5: Apply the BFS traverse on G_i starting from S_s to update the query distance of vertices in S_u;
6: **return** all updated distance $dist(u, q)$ for $u \in S_u$;

coreness, butterfly degree, and query distance. Overall, Algorithm 2 takes $O(|V| + |E|) = O(|E|)$ space, due to a connected graph G with $|V| - 1 \leq |E|$. □

6 L²P-BCC: LEADER-PAIR BASED LOCAL BCC SEARCH

Based on our greedy algorithmic framework in Algorithm 1, we propose three methods for fast BCC search in this section. The first method is the fast computation of query distance, which only updates a partial of vertices with new query distances in Section 6.1. The second method fast identifies a pair of leader vertices, which both have the butterfly degrees no less than b_p in Section 6.2. The leader pair tends to have large butterfly degrees even after the phase of graph removal, which can save lots of computations in butterfly counting. The third method of local BCC search is presented in Section 6.3, which finds a small candidate graph for bulk removal refinement, instead of starting from whole graph G.

6.1 Fast Query Distance Computation

Here, we present a fast algorithm to compute the query distance for vertices in G. In line 4 of Algorithm 1, it needs to compute the query distance for all vertices, which invokes expensive computation costs. However, we observe that a majority of vertices keep the query distance unchanged after each phase of graph removal. This suggests that a partial update of query distances may ensure the updating exactness and speed up the efficiency.

The key idea is to identify the vertices whose query distances need to update. Given a set of vertices D_i deleted in graph G_i, let denote the distance $d_{min} = min_{v \in D_i} dist_{G_i}(v, q)$. Let the vertex set $S_u = \{v \in V(G_i) \setminus D_i | dist_{G_i}(v, q) > d_{min}\}$. We have two importantly useful observations as follows.
- For each vertex $v \in S_u$, we need to recompute the query distance $dist_{G_{i+1}}(v, q)$. It is due to the fact that the vertex u with $dist_{G_i}(u, q) <= d_{min}$ keeps the query distance unchanged in G_{i+1}. This is because no vertices that are along with the shortest paths between u and q, are deleted in graph G_i.
- For vertex $v \in S_u$, $dist_{G_{i+1}}(v, q) \geq dist_{G_i}(v, q)$ always holds, due to $G_{i+1} \subseteq G_i$. Thus, instead of traversing from query vertex q, we update the query distance $dist_{G_{i+1}}(v, q)$ by traversing from S_s, where $S_s = \{v \in V(G_i) \setminus D_i | dist_{G_i}(v, q) = d_{min}\}$.

We present the method of fast query distance computation in Algorithm 5. First, we remove a set of vertices D_i from graph G_i (line 1). We then calculate the minimum distance d_{min}, which is the minimum length of the shortest path from D_i to q (line 2). We select vertices S_s whose shortest path equals to d_{min} as the BFS

Table 2: The shortest distances of queries to other vertices. The symbol '−' represents the vertex set unchanged. The vertices with changed distance are depicted in bold, i.e., u_4 and u_7.

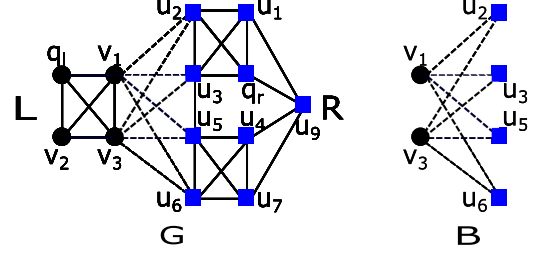| query	1	2	3	4
q_l	$\{v_1, v_2, v_3\}$	$\{u_2, u_3, u_5, u_6\}$	$\{q_r, u_1, u_4, u_7\}$	$\{u_9\}$
q_r	$\{u_1, u_2, u_3, u_9\}$	$\{v_1, v_3, u_4, u_5, u_7\}$	$\{q_l, v_2, u_6\}$	$\emptyset$
after the deletion of u_9				
q_l	−	−	−	$\emptyset$
q_r	$\{u_1, u_2, u_3\}$	$\{v_1, v_3, u_5\}$	$\{q_l, v_2, u_6, \mathbf{u_4}, \mathbf{u_7}\}$	$\emptyset$

Figure 3: An example of labeled graph G and its bipartite sub-graph B.

starting points (line 3). Let a set of vertices to be updated as S_u whose shortest path is larger than d_{min} (lines 4). Then, we run the BFS algorithm starting from vertices in S_s, we treat S_u as unvisited and all the other vertices as visited (line 5). The algorithm terminates until S_u are visited or the BFS queue becomes empty. Finally, we return the shortest path to q for all vertices (line 6). Note that in each iteration of Algorithm 1 it always keeps one query vertex's distance to other vertices unchanged, only needs to update the distances of another query vertex, because $S_u = \emptyset$ always holds for one of the query vertices.

EXAMPLE 4. *Consider the graph G in Figure 3 as G_i and $Q = \{q_l, q_r\}$. From Table 2 we know the vertex u_9 has the maximum distance to Q, i.e., $dist_G(u_9, Q) = 4$ (line 5 in Algorithm 1). Thus, the removal vertex set is $D_i = \{u_9\}$. Now, we apply Algorithm 5 on G to update the query distance. (1) For q_l, the vertex u_9 is the farthest vertex to q_l, thus $S_u = \emptyset$ (line 4), which indicates no vertices' distances to q_l need to update. (2) For q_r, $D_i = \{u_9\}$ and $d_{min} = min_{v \in \{u_9\}} dist_G(v, q_r) = dist_G(u_9, q_r) = 1$ (line 2). The vertex set to update is $S_u = \{v \in V(G) \setminus \{u_9\} | dist_G(v, q_r) > 1\} = \{v \in V(G) \setminus \{u_9\} | dist_G(v, q_r) = 2\} \cup \{v \in V(G) \setminus \{u_9\} | dist_G(v, q_r) = 3\} = \{v_1, v_3, u_4, u_5, u_7\} \cup \{q_l, v_2, u_6\} = \{q_l, v_1, v_2, v_3, u_4, u_5, u_6, u_7\}$ (line 4). Then, we apply BFS search starting from the vertex set $S_s = \{v \in V(G) \setminus \{u_9\} | dist_G(v, q_r) = 1\} = \{u_1, u_2, u_3\}$ (line 3), and update the shortest distance for vertices in S_u (line 5). The updated distances are also shown in Table 2.*

6.2 Leader Pair Identification

Next, we present efficient algorithms for leader pair identification, to improve the efficiency of the butterfly counting part.

Leader pair identification. The butterfly counting in Algorithm 3 needs to be invoked once a graph G_i is updated in Algorithm 1. This may lead to a large number of butterfly counting, which is very time costly. Recall that the definition of BCC requires a pair of vertices $\{v_l \in L, v_r \in R\}$ such that $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$. This motivates us to find a good pair of leader vertices, whose butterfly degrees are

Algorithm 6 Leader Pair Identification (G, q, η, b)

Require: A graph $G = L$ (or R), a query vertex $q = q_l$ (or q_r), η, b.
Ensure: A lead vertex $p = v_l$ (or v_r).
1: $p \leftarrow q$; //Initiate as query vertex.
2: $b_{max} \leftarrow max_{v \in G} \chi(v)$;
3: $b_p \leftarrow b_{max}/2$;
4: **if** $\chi(p) > b_p$ **then**
5:     **return** p; //Query vertex has a large enough butterfly degree.
6: **else**
7:     **while** $b_p \geq b$ **do**
8:         $d \leftarrow 1$; //Search from the 1-hop neighbors of q.
9:         **while** $d \leq \eta$ **do**
10:           $S \leftarrow \{u \mid dist_G(u, q) = d, u \in G\}$;
11:           **if** $\exists s \in S$ that $\chi(s) \geq b_p$ **then**
12:             **return** s;
13:           **else**
14:             $d \leftarrow d + 1$; //Increase the search distance d.
15:         $b_p \leftarrow b_p/2$;
16: **return** p;

large enough w.r.t. b_p in two groups L and R, even after a number of graph removal iterations. Thus, it can avoid finding a new leader pair and save time cost. In the following, we show two key observations to find a good leader pair (v_l, v_r) where $v_l \in L$ and $v_r \in R$.

OBSERVATION 1. *The leader pair P should have large butterfly degrees $\chi(v_l)$ and $\chi(v_r)$, which do not easily violate the constraint of cross-over interactions.*

OBSERVATION 2. *The leader pair P should have small query distances $dist(v_l, Q)$ and $dist(v_r, Q)$, which are close to query vertices Q and not easily deleted by graph removal.*

We present the algorithm of leader pair identification in Algorithm 6. Here, we use the graph G to represent a graph L/R and find a leader vertex v_l/v_r, η denotes to search leaders within η-hops neighbors of the query vertex q. We first initiate p as the query vertex q since it is the closest with distance 0 (line 1). This is especially effective when the input query vertex is leader biased who contains the largest butterfly degree. If the degree number is large enough, i.e., greater than $b_{max}/2$, we return p as the leader vertex (lines 2-5); Otherwise, we find the leader vertex p in the following manner. We first increase d from 1 to η (line 14), and decrease b_p in $\{b_{max}/2, b_{max}/4, ..., b\}$ (lines 7-15). We get the set of vertices S whose distance to the query is d (line 10). Then, we identify one vertex $s \in S$ with $\chi(s) \geq b_p$ and return as the leader vertex otherwise we increase d by 1 to search the next hop (lines 9-14). Note that we return an initial p if no better answer is identified (line 16).

EXAMPLE 5. *We apply Algorithm 6 on G in Figure 3 for $Q = \{q_l, q_r\}$ and select $\eta = 3$. In the graph G, the non-zero butterfly degree of vertices are $\chi(v_1) = \chi(v_3) = 6$ and $\chi(u_2) = \chi(u_3) = \chi(u_5) = \chi(u_6) = 3$. (1) For the subgraph L, first it initialize p as the query vertex q_l (line 1). Since $b_{max} = 6$ and $b_p = b_{max}/2 = 3$ (lines 2-3), where $\chi(p) = 0$ which is less than $b_p = 3$ then it goes to line 6. Then, it starts from $d = 1$ (line 8) to search q_l's 1-hop neighbors, i.e., $S = \{v_1, v_2, v_3\}$ (line 10), finally finds there exists vertex $s = v_1$ such that $\chi(s) \geq b_p = 3$ and returns v_1 as the leader vertex (lines 11-12). (2) For the subgraph R, it follows a similar*

Algorithm 7 Butterfly Degree Update for Leader Pair (B, p, v)

Require: A graph $B = (V_B, E_B)$, a leader vertex p, a deletion vertex v.
Ensure: Butterfly degree $\chi(p)$.
1: **if** $\ell(p) = \ell(v)$ **then**
2:     $\alpha \leftarrow |N(v) \cap N(p)|$; //The number of common neighbors of p, v.
3:     $\chi(p) \leftarrow \chi(p) - \binom{\alpha}{2}$;
4: **else**
5:     **if** $v \in N(p)$ **then**
6:         **for** $u \in N(v)$ and $u \neq p$ **do**
7:           $\beta \leftarrow \beta + |N(u) \cap N(p)| - 1$; //Add the count of removed butterflies.
8:         $\chi(p) \leftarrow \chi(p) - \beta$;
9: **return** $\chi(p)$;

process where $b_{max} = 3$ and $b_p = b_{max}/2 = 1.5$, returns u_2 as the leader vertex. Finally, we obtain $\{v_1, u_2\}$ as the leader pair.

Butterfly degree update for leader pair. Here, we consider how to efficiently update the butterfly degrees of leader pair vertices $\chi(v_l)$ and $\chi(v_r)$ after graph removal in Algorithm 1.

The algorithm of updating the butterfly degrees of the leader pair is outlined in Algorithm 7. First, we check the labels of p and v (line 1). If $\ell(p) = \ell(v)$, we find the common neighbors $N(v) \cap N(p)$ shared by p and v, its number denoted as cn, then the number of butterflies containing p and v is $\binom{\alpha}{2}$. Then, $\chi(p)$ decreases by $\binom{\alpha}{2}$ (lines 2-3). If $\ell(p) \neq \ell(v)$, firstly there exists no butterflies if p and v do not connect (line 5). We enumerate each vertex u, i.e., v's neighbors, and check their common neighbors with p, i.e., $N(u) \cap N(p)$. Note that β keeps the number of butterflies involving v, so we directly update $\chi(p)$ by decreasing β (lines 6-8).

EXAMPLE 6. *We apply Algorithm 7 on G in Figure 3 to update the butterfly degree of leader pair $\{v_1, u_2\}$. First, the deletion of vertex u_9 has no influence on the butterfly degree. Next vertex u^* to delete is selected from $\{v_2, u_1, u_4, u_6, u_7\}$, which has the maximum query distance $dist_G(u^*, Q) = 3$. To illustrate, we assume to delete u_6. (1) For u_2, $\chi(u_2) = 3$ before the deletion. Since $\ell(u_2) = \ell(u_6)$ (line 1), their common neighbors $N(u_2) \cap N(u_6) = \{v_1, v_3\}$ and $\alpha = |\{v_1, v_3\}| = 2$ (line 2). The updated butterfly degree is $\chi(u_2) = 3 - \binom{\alpha}{2} = 3 - 1 = 2$ (line 3). (2) For v_1, $\chi(v_1) = 6$ before the deletion. Since $\ell(v_1) \neq \ell(u_6)$ and $u_6 \in N(v_1)$ (lines 4-5), we enumerate u_6's neighbors except v_1, i.e., $\{v_3\}$ (line 6). Since $\beta = \sum_{u \in \{v_3\}}(|N(u) \cap N(v_1)| - 1) = |N(v_3) \cap N(v_1)| - 1 = |\{u_2, u_3, u_5, u_6\}| - 1 = 3$ (line 7), then the updated butterfly degree is $\chi(v_1) = 6 - 3 = 3$ (line 8).*

Complexity analysis. Next, we analyze the time and space complexity of leader pair identification and update in Algorithm 6 and 7. First, Algorithm 6 takes $O(|E| \log (b_{max} - b))$ time in $O(|E|)$ space, which identifies the leader pair using a binary search of approximate butterfly degree $b_p \in [b, b_{max}]$ within the query vertex's neighborhood. Next, we analyze the leader pair update in Algorithm 7.

THEOREM 5. *One run of butterfly degree update for leader pair in Algorithm 7 takes $O(d_u^2)$ time and $O(|E|)$ space.*

PROOF. The degree of leader vertex and delete vertex are d_p and d_v, the time complexity of Algorithm 7 is $O(min(d_p, d_v))$ if $\ell(p) = \ell(v)$; $O(d_p d_v)$ if $\ell(p) \neq \ell(v)$. Assume that the maximum degree in bipartite graph B is d_u. Thus, the total complexity of

Algorithm 8 Index-based Local Exploration

Require: $G = L \cup B \cup R$, $Q = \{q_l, q_r\}$, three integers $\{k_1, k_2, b\}$.
Ensure: A connected (k_1, k_2, b)-BCC R with a small diameter.
1: Compute a shortest path P connecting Q using the butterfly-core path weight;
2: $k_l \leftarrow \min_{v \in V_L} \delta(v)$; $k_r \leftarrow \min_{v \in V_R} \delta(v)$;
3: Iteratively expand P into graph $G_t = \{v \in L | \delta(v) \geq k_l\} \cup \{v \in R | \delta(v) \geq k_r\}$ by adding adjacent vertices v, until $|V(G_t)| > \eta$;
4: Compute a connected (k_1, k_2, b)-BCC containing Q of G_t with the largest coreness on each side;
5: Remove disqualified subgraphs on G_t to identify the final BCC;

Algorithm 7 is $O(d_u^2)$. In addition, we analyze the space complexity. Algorithms 7 take $O(|V| + |E|) = O(|E|)$ space to store the vertices and their incident edges in graph. □

In summary, a successful leader pair identification by Algorithm 6 can significantly reduce the times of calling the butterfly counting in Algorithm 3. In addition, it only needs to update the butterfly degree of leader vertices using Algorithm 7 but not the entire vertex set in BCC candidate graph, which is also an improvement of butterfly counting. This butterfly computing strategy for leader pair identification and update is very efficient for BCC search, as validated in Exp-5 in Section 8.

6.3 Index-based Local Exploration

In this section, we develop a query processing algorithm which efficiently detects a small subgraph candidate around query vertices Q, which tends to be densely connected both in its own label subgraph and bipartite graph.

First, we construct the BCindex for all vertices in G. The data structure of BCindex consists of two components: the coreness and butterfly number. For k-core index construction of each vertex, we apply the existing core decomposition [3] on graph G to construct its coreness. The k-core has a nested property, i.e., the k_1-core of graph G must be a subgraph of k_2-core if $k_1 > k_2$. The offline k-core index could efficiently find k-core subgraphs from G, meanwhile reduce the size of bipartite graph B. Moreover, we keep the butterfly degree number index of each vertex on the bipartite graph with different labels using Algorithm 3.

Based on the obtained BCindex, we present our method of index-based local exploration, which is briefly presented in Algorithm 8. The algorithm starts from the query vertices and finds the shortest path connecting two query vertices. A naive method of shortest path search is to find a path is the minimum number of edges, while this may produce a path along with the vertices of small corenesses and small butterfly degrees. To this end, we give a new definition of butterfly-core path weight as follows.

DEFINITION 6 (BUTTERFLY-CORE PATH WEIGHT). *Given a path P between two vertices s and t in G, the butterfly-core weight of path P is defined as* $\underline{dist}(s,t) = dist_G(s,t) + \gamma_1(\delta_{max} - \min_{v \in P} \delta(v)) + \gamma_2(\chi_{max} - \min_{v \in P} \chi(v))$, *where $\delta(v)$ is the coreness of vertex v, $\chi(v)$ is the butterfly degree of v, δ_{max} and χ_{max} are the maximum coreness and the maximum butterfly degree in G.*

The value of $(\delta_{max} - \min_{v \in P} \delta(v))$ and $(\chi_{max} - \min_{v \in P} \chi(v))$ respectively measures the shortfall in the coreness and butterfly degree of vertices in path P w.r.t. the corresponding maximum value

in G. Smaller the shortfall of a path, lower its distance. Here, γ_1 controls the extent to which a small vertex coreness is penalized, and γ_2 controls the penalized extent of a small butterfly degree. Using BCindex, for any vertex v we can access the structural coreness $\delta(v)$ and the butterfly degree $\chi(v)$ in $O(1)$ time. Thus, the shortest path P can be extracted based on the butterfly-core path weight definition.

We then expand the extracted path P to a large graph G_t as a candidate BCC by involving the local neighborhood of query vertices. We start from vertices in P, split the vertices by their labels into V_L and V_R, obtain the minimum coreness of vertices in each side as $k_l = \min_{v \in V_L} \delta(v)$ and $k_r = \min_{v \in V_R} \delta(v)$ (line 2). Due to the different density of two groups, we expand P in different core values. We then expand the path by iteratively inserting adjacent vertices with coreness no less than k_l and k_r respectively, in a BFS manner into G_t, until the vertex size exceeds a threshold η, i.e., $|V(G_t)| > \eta$, where η is empirically tuned. After that, for each vertex $v \in V(G_t)$, we add all its adjacent edges into G_t (line 3). Since G_t is a local expansion, the coreness of L, R will be at most k_l and k_r. Based on G_t, we extract the connected (k_1, k_2, b)-BCC containing Q. If input parameters k_1 and k_2 are not supplied, they are set automatically with the largest coreness on each side (line 4). Then it iteratively removes the farthest vertices from this BCC. Moreover, since $dist_{G_i}(u, Q)$ is monotone non-decreasing with decreasing graphs, to reduce the iteration number of butterfly-core maintenance, we propose to delete a batch of vertices with the same farthest query distance, i.e., $S = \{u^* | dist_{G_i}(u^*, Q) \geq d_{max}\}$, which can further improve the search efficiency (line 5). Although Algorithm 8 does not preserve 2-approximation guarantee of optimal answers, it achieves the results of good quality fast in practice, as validated in our experiments.

7 HANDLE MULTI-LABELED BCC SEARCH

In this section, we generalize the BCC model from 2 group labels to multiple m group labels where $m \geq 2$, and formulate the multi-labeled BCC-problem. Then, we discuss how to extend our existing techniques to handle a BCC search query with multiple labels.

Multi-labeled BCC search. First, we extend the *cross-group interaction* to a new definition of *cross-group connectivity*. According to the 4th constraint of Def. 4, two groups labeled with A_l and A_r have cross-group interaction *iff* $\exists v_l \in V_L$ and $\exists v_r \in V_R$ such that butterfly degree $\chi(v_l) \geq b$ and $\chi(v_r) \geq b$ in the induced subgraph of H by $V_L \cup V_R$. We say that two labels A_l and A_r have cross-group interaction, denoted as (A_l, A_r).

DEFINITION 7 (CROSS-GROUP CONNECTIVITY). *Two labels A_l and A_r have cross-group connectivity if and only if there exists a cross-group path $P = \{A_1, \ldots, A_j\}$ where $A_1 = A_l$, $A_j = A_r$, and (A_i, A_{i+1}) has cross-group interaction for any $1 \leq i < j$.*

To model group connection in a multi-labeled BCC, it requires that for any two labels A_l and A_r, there exists either a direct *cross-group interaction* or a *cross-group path* between two group core structures, implying the strong connection between two different groups within a BCC community. Specifically, we extend the definition of multi-labeled BCC (mBCC) model as follows.

DEFINITION 8 (MULTI-LABELED BUTTERFLY-CORE COMMUNITY). *Given a labeled graph G, an integer $m \geq 2$, group core*

Algorithm 9 Multi-labeled BCC Search Framework (G, Q)

Require: $G = (V, E, \ell)$, multi-labeled query $Q = \{q_1, \ldots, q_m\}$, group core parameters $\{k_i : 1 \leq i \leq m\}$, butterfly degree parameter b.
Ensure: A connected mBCC O with a small diameter.
1: Find a maximal connected subgraph of multi-labeled BCC containing Q as G_0 by Def. 8; //using Alg. 2
2: $l \leftarrow 0$;
3: **while** $connect_{G_l}(Q) = $ **true do**
4:      Delete vertex $u^* \in V(G_l)$ with the largest $dist_{G_l}(u^*, Q)$ from G_l; //using Alg. 5
5:      Maintain each labeled group as a k_i-core, $\forall i \in \{1, 2, \ldots, m\}$;
6:      Update the leader pairs and check the cross-group connectivity among m labeled groups by Def. 7; //Alg. 3 & 4, optimized by Alg. 6 & 7
7:      $G_{l+1} \leftarrow G_l$; $l \leftarrow l + 1$;
8: $O \leftarrow \arg\min_{G' \in \{G_0, \ldots, G_{l-1}\}} dist_{G'}(G', Q)$;

---

parameters $\{k_i : 1 \leq i \leq m\}$, a multi-labeled butterfly-core community (mBCC) $H \subseteq G$ satisfies the following conditions:

1. Multiple labels: there exist m different labels $A_1, A_2, \ldots, A_m \in \mathcal{A}$ such that $\forall 1 \leq i \leq m$, $V_i = \{v \in V(H) : \ell(v) = A_i\}$ and $V_i \cap V_j = \emptyset$ for $j \neq i$, and $V_1 \cup V_2 \cup \ldots \cup V_m = V_H$;

2. Core groups: the induced subgraph of H by V_i is k_i-core where $1 \leq i \leq m$;

3. Cross-group connectivity: for $1 \leq i, j \leq m$, any two labels A_i and A_j have cross-group connectivity in H.

For $m = 2$, this definition is equivalent to our butterfly-core community model in Def. 4. From the condition (1) and (2) the mBCC has exactly m labeled groups and each group is a k_i-core. The condition (3) requires that these m groups could be cross-group connected by the bipartite interactions. To ensure the cross-group connectivity, each group should have one leader vertex whose butterfly degree has no less than b. Note that the accumulated number of butterflies from multiple bipartite graphs does not all count into the butterfly degree. Specifically, we only count on the butterfly degree among two labeled bipartite graph.

mBCC-search problem. Based on the multi-labeled BCC community model, the corresponding mBCC-search problem can be similarly defined as Problem 1 as follows. Consider a query $Q = \{q_1, \ldots, q_m\}$ where each query vertex q_i has a distinct label A_i, the group core parameters $\{k_i : 1 \leq i \leq m\}$, and the butterfly degree parameter b, the *mBCC-search problem* is to find a connected mBCC containing Q with the smallest diameter.

Algorithm. We propose a multi-labeled BCC search framework in Algorithm 9, which leverages our previous techniques of search framework in Algorithm 1 and fast strategies in Section 6. The algorithm first finds a maximal connected mBCC containing all query vertices Q as G_0 (line 1) and then iteratively removes the farthest vertex u^* from the graph (lines 3-7). The mechanism is to maintain each labeled group as a k_i-core (line 5) and update the leader pairs (line 6). Based on the identified leader pairs and cross-group interactions, the algorithm can check the cross-group connectivity for m labeled groups in G_l by Def. 7 as follows. We first construct a new graph H_m with m isolated vertices, where each vertex represents a labeled group. Then, we insert into H_m an edge between two vertices if two labeled groups have a cross-group interaction. Finally, the cross-group connectivity of G_l is equivalent

Table 3: Network statistics (K = 10^3 and M = 10^6).

Network	$\|V\|$	$\|E\|$	*Labels*	k_{max}	d_{max}
Baidu-1	30K	508K	383	43	12
Baidu-2	41K	2M	346	189	13
Amazon	335K	926K	2	6	549
DBLP	317K	1M	2	113	342
Youtube	1.1M	3M	2	51	28,754
LiveJournal	4M	35M	2	360	14,815
Orkut	3.1M	117M	2	253	33,313

to the graph connectivity of H_m. In addition, our fast query distance computation in Alg. 5 and local search strategies in Alg. 6 and Alg. 7 can be also easily extended to improve the efficiency.

Complexity analysis. We analyze the complexity of Algorithm 9. At each iteration of Algorithm 9, the shortest path computation takes $O(|Q| \cdot (|V| + |E|)) = O(m|E|)$ time, due to $|V| - 1 \leq |E|$. Moreover, it takes $O(|V| + |E|)$ time to find and maintain all k_i-cores for different m labels. To identify the leader pairs, it runs the butterfly counting over the whole graph once in $O(\sum_{u \in V} d_u^2)$ time. The extra cost of checking cross-group connectivity takes $O(m^2) \subseteq O(m|E|)$ time, due to the queries $Q \subseteq V$ and $|V| - 1 \leq |E|$. Actually, the checking cross-group connectivity can be further optimized in $O(m)$ time using the union-find algorithm in any conceivable application [9]. Thus, for a query Q with m different labeled vertices, Algorithm 9 takes $O(t(m|E| + \sum_{u \in V} d_u^2))$ time for t iterations and $O(|V| + |E|)$ space.

8 EXPERIMENTS

In this section, we conduct experiments to evaluate our proposed model and algorithms.

Datasets. We use seven real datasets as shown in Table 3. Two new real-world datasets of labeled graphs are collected from Baidu which is a high tech company in China. They are IT professional networks with the ground-truth communities of joint projects between two department teams, denoted as *Baidu-1* and *Baidu-2*. In both graphs, each vertex represents an employee and the label represents the working department. An edge exists between two employees if they have communication through the company's internal instant messaging software. Baidu-1 and Baidu-2 are generated based on data logs for three months and one whole year, respectively. In addition, we use five graphs with ground-truth communities from SNAP, namely Amazon, DBLP, Youtube, LiveJournal and Orkut, with randomly added synthetic vertex labels into them. Specifically, we split the vertices based on communities into two parts, assigned all vertices in each part with one label. We also generated the query pairs by picking any two vertices with different labels. To add cross edges within communities, we randomly assigned vertices with 10% cross edges to simulate the collaboration behaviors between two communities. Moreover, we added 10% noise data of cross edges globally on the whole graph.

Compared methods. We compare our three BCC search approaches with two community search competitors as follows.
- CTC: finds a closest k-truss community containing a set of query vertices [20].
- PSA: progressively finds a minimum k-core containing a set of query vertices [23].
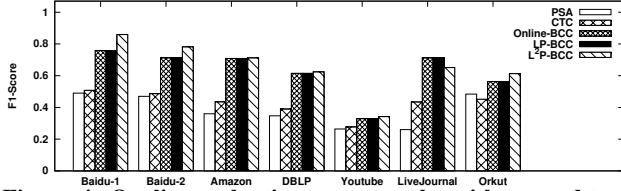- Online-BCC: our online BCC search in Algorithm 1.

Figure 4: Quality evaluation on networks with ground-truth communities.
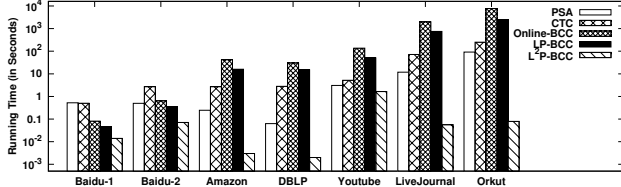

Figure 5: Efficiency evaluation on networks with ground-truth communities.

- LP-BCC: our Online-BCC method equipped with two accelerate strategies, i.e., fast query distance computation in Algorithm 5 and leader pair identification in Algorithms 6 & 7.

- L^2P-BCC: our local leader-pair BCC search described in Algorithm 8, which is also equipped with Algorithm 5 and Algorithms 6 & 7.

Note that all our methods use bulk deletion that removes a batch of vertices with the farthest distances from the graph. We use the default parameter setting of CTC [20] and PSA [23]. For L^2P-BCC, we set $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$.

Evaluation metrics. To evaluate the efficiency, we report the running time in seconds and treat a query search as infinite if it exceeds 30 minutes. To evaluate the community quality, we use the F1-score metric to measure the alignment between a discovered community C and a ground-truth community $\hat{C}$. Here, F1 is defined as $F1(C, \hat{C}) = \frac{2 prec(C,\hat{C}) recall(C,\hat{C})}{prec(C,\hat{C}) + recall(C,\hat{C})}$ where $prec(C, \hat{C}) = \frac{|C \cap \hat{C}|}{|C|}$ and $recall(C, \hat{C}) = \frac{|C \cap \hat{C}|}{|\hat{C}|}$.

Queries and parameters. We generate a BCC query $Q = \{q_1, \ldots, q_m\}$ and set $m = 2$ by default. For a default 2-labeled query $Q = \{q_l, q_r\}$, we set k_1 and k_2 as the coreness value of query vertices q_l and q_r, and $b = 1$, respectively. We vary two parameters, degree rank Q_d and inter-distance l, to generate different query sets of pair vertices. For Q_d, we sort all vertices in ascending order of their degrees in a network. A vertex is regarded to be with degree rank of $X\%$, if it has top highest $X\%$ degree in the network. The default value of Q_d is 80%, which means that a query vertex has a degree higher than the degree of 80% vertices in the whole network. The inter-distance l is the shortest path between two query vertices. The default $l = 1$ indicates that the two query vertices are directly connected in the network.

8.1 Quality, Efficiency, and Parameter Evaluations

Exp-1: Quality evaluation with ground-truth communities. We evaluate the effectiveness of different community search models over labeled graphs. Figure 4 reports the averaged F1-scores of all

Table 4: A comparison of Online-BCC **and** LP-BCC **on DBLP, in terms of query distance calculation and butterfly computation. The '#butterfly counting' denotes the calling times of butterfly counting procedure in Algorithm 3. Other steps are measured by their running time in seconds.**

Methods	Online-BCC	LP-BCC	Speedup
Query distance calculation	1.58	0.75	2.1**x**
Leader pair update	4.98	0.46	10.8**x**
#butterfly counting	34.86	1.21	28.8**x**
Total time	11.5	4.04	2.8**x**

methods over 1000 random queries on seven networks. We observe that our approaches achieve the highest F1-score on all networks against CTC [20] and PSA [23]. L^2P-BCC is better than Online-BCC and LP-BCC on most datasets except for LiveJournal. All methods cannot have good results on Youtube.

Exp-2: Efficiency evaluation of all methods. We evaluate the efficiency performance of different community search models. Figure 5 shows the running time results of all methods. All methods finish the queries processing within 30 minutes, except Online-BCC and LP-BCC on Orkut as they generate a large candidate graph G_0. Most other methods use the local search strategy and run much faster than LP-BCC. Interestingly, Online-BCC and LP-BCC run slightly faster than CTC and PSA on Baidu-1 and Baidu-2. Overall, L^2P-BCC achieves the best efficiency performance, which can deal with one BCC search query within 1 second on most datasets.

Exp-3: Efficiency evaluation on different queries. To evaluate the efficiency of our proposed improved strategies, we compare Online-BCC, LP-BCC, and L^2P-BCC. Figure 6 and Figure 7 show the running time on different networks, varying the vertex degree rank and inter-distance respectively. The most efficient method is L^2P-BCC. With the increased vertex degree, Online-BCC and LP-BCC become faster on DBLP and LiveJournal in Figure 6(c) and (d) as the vertices with a higher degree have denser and smaller induced graphs; it is different on Baidu-1 and Baidu-2 because of the dense structure of the graph. With the increased query vertices inter-distance l, the running time also increases on DBLP and LiveJournal. The reason may be that two query vertices are not the leader pair, leading to the low efficiency of our leader identification strategy to find the leaders satisfying the butterfly degree constraint.

Exp-4: Parameter sensitivity evaluation. We evaluate the parameter sensitivity of k_1, k_2, b on efficiency performance. When we test one parameter, the other two parameters are fixed. Moreover, we test one parameter of k_1 and k_2 as k due to their symmetry parameter property. Figure 8 shows the running time by varying the core value k. We observe that the larger k results in less running time since it generates a smaller G_0 for a larger k. Figure 9 shows the running time by varying the butterfly degree b. We observe that our approach achieves a stable efficiency performance on different values b.

Exp-5: Efficiency evaluation of query distance and butterfly computations. We evaluate the efficiency of our proposed fast strategies in query distance computation in Algorithm 5 and leader pair identification in Algorithms 6 and 7. We compare LP-BCC and Online-BCC on DBLP using 1000 queries and report the detailed results in Table 4. Algorithm 5 achieves 2.1**x** speedups than the baseline of query distance computation by LP-BCC. The leader pair identification in
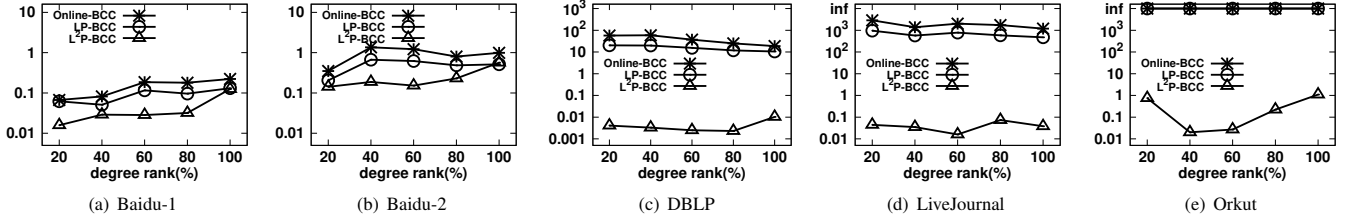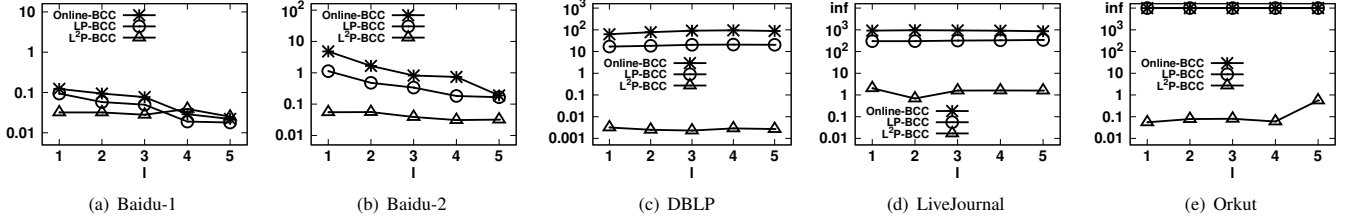
Figure 6: Varying vertex degree rank: Query Time.

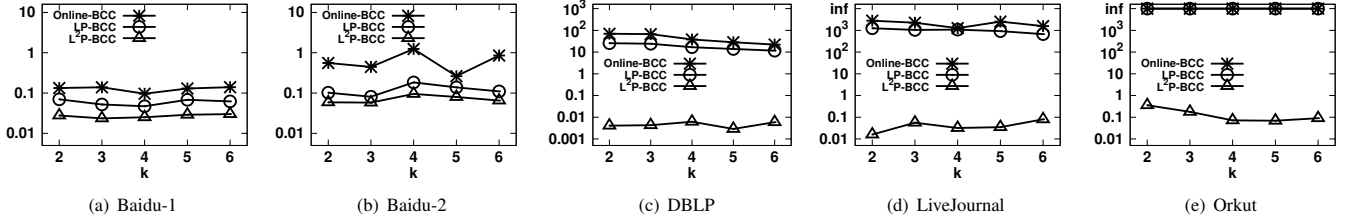**Figure 7: Varying inner distance l: Query Time.**

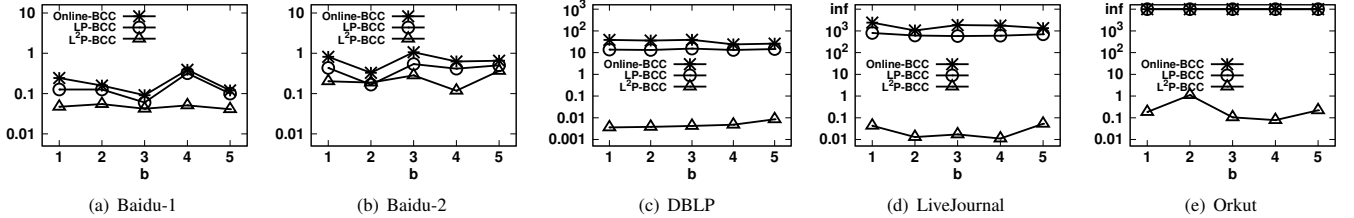

Figure 8: Varying core value k: Query Time.

Figure 9: Varying butterfly value b: Query Time.

Algorithms 6 and 7 greatly reduces the calling times of butterfly counting, validating that our proposed leader pair identifications can target a pair of stable leaders with large butterfly degrees even for graph removals. The leader pair identification achieves 28.8**x** and 10.8**x** speedups on #butterfly counting and the running time of leader pair update, respectively. In terms of total running time, Online-BCC integrating fast strategies achieves 2.8**x** speedups against LP-BCC.

8.2 Three Real-world Case Study Comparisons

To evaluate the effectiveness of the BCC model, we conduct case studies on three real-world networks including a global flight network [1], an international trade network [2] and the J. K. Rowling's Harry Potter series fiction network [3]. We compare our method LP-BCC with the closest truss community search (CTC [20]). We set the parameter $b = 3$ and the values of k_1, k_2 as the coreness of query

vertices q_l and q_r for LP-BCC. The method CTC uses its default setting [20].

Exp-6: A case study on flight networks. Here we study a labeled graph of flight network with 238 different labels, where each vertex represents a city with a label of the home country. The flight network has $3,334$ vertices and $19,205$ edges. Note that this graph is a undirected single-edge graph generated from the source data[1], which generates an edge between two cities if there exit more than one airline between them. Thus, edges between the same labeled vertices are domestic airlines. On the other hand, international airlines are represented by the edges between different labeled vertices. We set the query vertices $Q =$ {"Toronto", "Frankfurt"} to discover the dense flight transportation networks between them. The cities "Toronto" and "Frankfurt" have a label of "Canada" and "Germany" respectively. Figure 11(a) depicts our result of BCC. The flight community consists of a 6-core in blue, butterfly degree is 3 in gray, and a 5-core in red. A butterfly forms by four cities "Toronto", "Vancouver", "Frankfurt", and "Munich", which are the transport hubs

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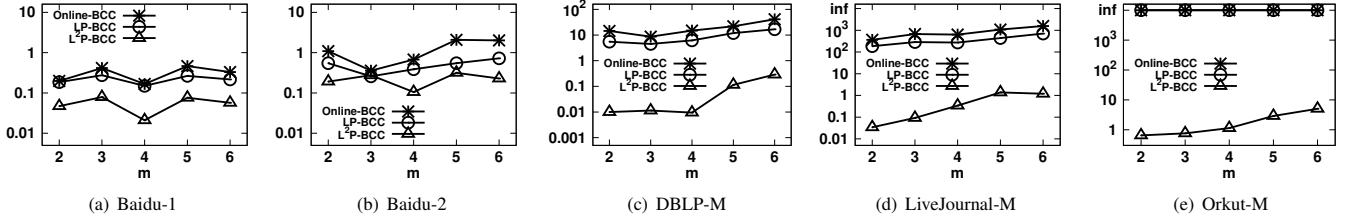
[1] https://raw.githubusercontent.com/jpatokal/openflights/master/data/routes.dat

[2] https://wits.worldbank.org/datadownload.aspx?lang=en

[3] https://github.com/efekarakus/potter-network

(a) Baidu-1   (b) Baidu-2   (c) DBLP-M   (d) LiveJournal-M   (e) Orkut-M

Figure 10: Varying m of multi-labeled query $Q = \{q_1, \ldots, q_m\}$: Query Time.

(a) Our Butterfly-Core Community   (b) CTC [20]

Figure 11: Case study of two community search models on flight networks for query Q ={"Toronto", "Frankfurt"}.

(a) Our Butterfly-Core Community   (b) CTC [20]
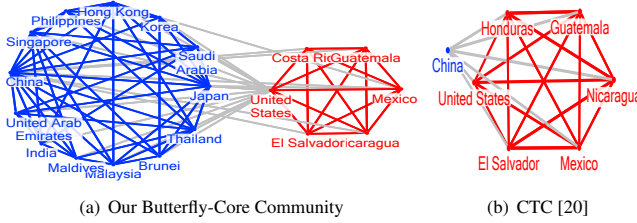
Figure 12: Case study of two community search models on trade networks for query Q ={"United State", "China"}.
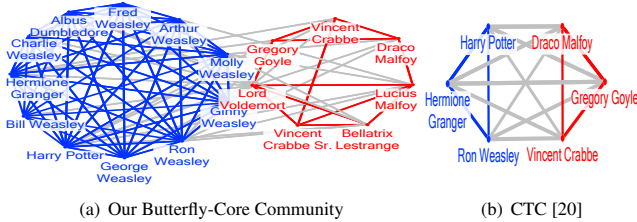

(a) Our Butterfly-Core Community   (b) CTC [20]

Figure 13: Case study of two community search models on Harry Potter fiction networks for query Q ={"Ron Weasley", "Draco Malfoy"}.

(a) Baidu-1   (b) Baidu-2

Figure 14: Quality evaluation on Baidu-1 and Baidu-2 with multi-labeled ground-truth communities.

Q ={"United State", "China"}. Figure 12(a) reports our BCC community result. It consists of dense trade subnetworks in "Asia" and "North America", and also the trade leaders, i.e., "United State" and "China" have the most transcontinental trades. Figure 12(b) shows the CTC community [20] fails to find the other major trade partners in "Asia".

Exp-8: A case study on Harry Potter fiction networks. The Harry Potter network is a 2-labeled graph, where each vertex represents a character. Each character has a label representing his camp justice or evil. An edge is added between two characters if they have intersections in the fiction. The edges between two same labeled vertices denote the family and ally in the same camp, while the edges between two different labeled vertices denote hostility enemies in different camps. There are 65 vertices and 513 edges. We set the query Q ={"Ron Weasley", "Draco Malfoy"}. Figure 13(b) shows that the CTC community only finds "Ron Weasley" with his best friends of Harry and Hermione in the Gryffindor house, Malfoy and his two cronies belonging in the Slytherin house. However, it misses the main leader in the evil camp, i.e., "Lord Voldemort". What's more, it also fails to find Ron's huge family, including his parents and brothers, as our BCC result is shown in Figure 13(a).

In summary, existing community models [20, 23] which ignore the graph labels, are hard to discover those communities across over two different labeled groups, which make a huge difference from our BCC model in three aspects: (1) they ignore the different dense level of two labeled groups; (2) they confuse the semantics of different edge types (e.g., international and domestic airlines); (3) they may miss some members of one group due to the strong structural required by models. On other hand, our BCC model aims to mine out the whole dense teams including the leader pairs, inner edges within a group, and cross edges between two groups.

8.3 Multi-labeled BCC Search Evaluation

Exp-9: Quality evaluation of multi-labeled BCC search on ground-truth communities. We conduct quality evaluations of multi-labeled BCC search on two datasets of Baidu-1 and Baidu-2, which consists

for transnational airlines. The core subgraphs in blue and red reflect the complex and dense networks of domestic airlines in Canada and Germany. Figure 11(b) shows the community result of CTC [20]. It consists of a dense airline community involving "Toronto" and "Frankfurt". However, most discovered vertices in Figure 11(b) are the cities in Canada, which fails to find the international airline community as our BCC model in Figure 11(a).

Exp-7: A case study on trade networks. The international trade network describes trade relations between countries/regions. Each vertex represents a country/region with its located continent as a label, e.g., the label of "China" is "Asia". The trade labeled graph has seven labels. We add an edge between two countries/regions if one is the top-5 import or export trade partners to the other in 2019. There are 249 vertices and 1281 edges. We set query vertices
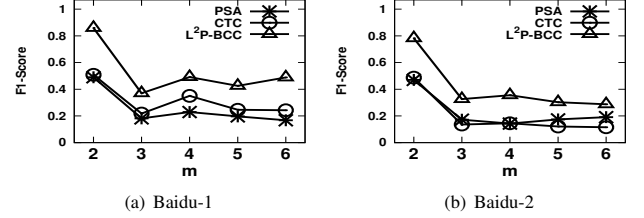
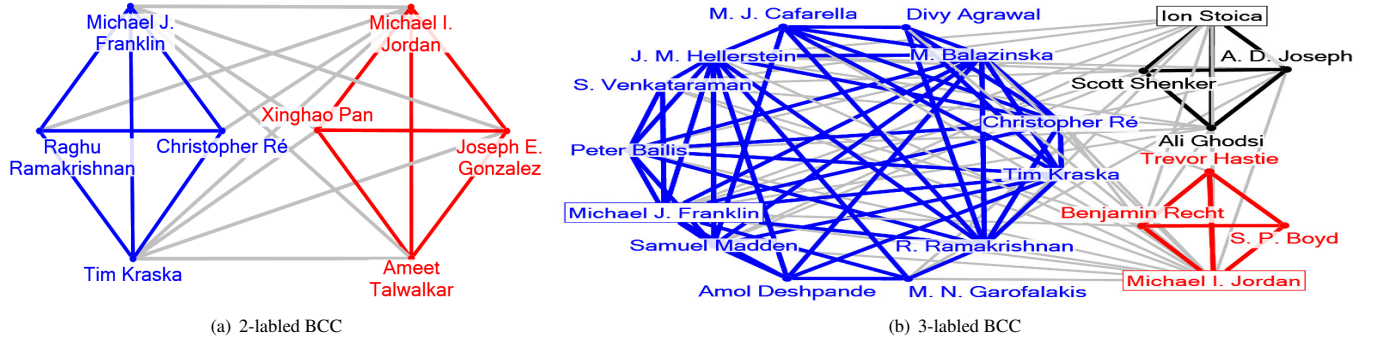(a) 2-labled BCC

(b) 3-labled BCC

Figure 15: Case study of multi-labeled BCC search on academic collaboration networks for interdisciplinary research group discovery using two queries Q_1 and Q_2. Here, Q_1 = {"Tim Kraska", "Michael I. Jordan"} and Q_2 = {"Michael J. Franklin", "Michael I. Jordan", "Ion Stoica"}.

of $3,000$ and $1,000$ ground-truth multi-labeled cross-group communities. Each cross-group community represents a joint project among multiple department teams in enterprise. We vary the number of query labels m in Q and report F1-score on the average of 100 queries. Figure 14 reports the results of CTC [20], PSA [23], and our local method L^2P-BCC for multi-labeled BCC search in Algorithm 9. We observe that all methods achieve worse performance with the increased query labels m, indicating that a more challenging task for identifying multi-labeled BCC communities. Nevertheless, our approach L^2P-BCC consistently performs better than the competitors CTC and PSA for different parameter m in Baidu-1 and Baidu-2.

Exp-10: Efficiency evaluation for multi-labeled BCC search. We evaluate the efficiency of multi-labeled BCC search methods. Following the mBCC search framework in Algorithm 9, we implement three extension approaches Online-BCC, LP-BCC and L^2P-BCC for multi-labeled BCC search accordingly. Beside two ground-truth datasets Baidu-1 and Baidu-2, we further generate and use three large graphs with multiple labels. Specifically, we assign six vertex labels for all vertices randomly in graphs DBLP, LiveJournal and Orkut, denoted as DBLP-M, LiveJournal-M, and Orkut-M respectively. Figure 10 shows the running time of three mBCC extension methods Online-BCC, LP-BCC and L^2P-BCC, varying by the number of query labels. Figure 10(a)-(b) show that all methods achieve stable efficiency performance for different query labels m on small graphs Baidu-1 and Baidu-2. On the other hand, all methods takes a longer running time slightly with the increased m on large graphs in Figure 10(c)-(d). The reason is that mBCC search takes more cost of shortest path computation for m queries, confirming the time complexity analysis of Algorithm 9 in Section 7. Importantly, our local method L^2P-BCC once again runs fastest among all the three methods, validating the effectiveness of our fast strategies even for multi-labeled BCC search.

Exp-11: Case study of multi-labeled BCC search for interdisciplinary collaboration groups on DBLP. We construct a real-world dataset of research collaboration network based on the "DBLP-Citation-network V12" at Aminer [39].[4] The collaboration network has $144,334$ vertices, $1,821,930$ edges, and a total of 7 vertex labels,

which is publicly available.[5] Each vertex represents an author. Two authors have an edge if they have at least one paper collaboration. For each vertex, we count on the author's published papers based on research topics, e.g., "Database" papers in the venues of SIGMOD, VLDB, and so on. Finally, we take the *vertex label* of an author as the research field of his/her most published papers, e.g., "Database", "Machine Learning", and so on. An edge between two authors with different research filed labels, is regarded as an *interdisciplinary collaboration*. We set the parameter $b = 3$ and the coreness values of $k_i = 3$ for all query vertices q_i. First, we query the 2-labeled BCC community containing Q_1 ={"Tim Kraska", "Michael I. Jordan"}, which is shown in Figure 14(a). This is an interdisciplinary research group crossover the fields of "Database" and "Machine Learning", which works on the machine learning techniques for database systems and vice versa (a.k.a. ML4DB and DB4ML). Homogeneous groups in red and blue are densely connected, forming the 3-core and indeed 4-clique, respectively. In terms of butterfly structure in bipartite graphs, two query vertices "Tim" and "Michael" have butterfly degree of 6 and 3 respectively, which has interdisciplinary collaborations with other research group and bridges these two research communities together. Second, we conduct multi-labeled BCC search, by querying three labeled vertices Q_2 ={"Michael J. Franklin", "Michael I. Jordan", "Ion Stoica"}, who are from the research fields of "Database", "Machine Learning", and "Systems and Networking", respectively. The result of 3-labeled cross-discipline community is shown in Figure 14(b). As can be seen, several well-known scholars of these fields appear in the BCC community and have close intra-group collaborations and interdisciplinary cooperations across different research fields of "Machine Learning", "Database", and "Systems and Networking". The database group is a 3-core and there are 13 vertices, the cross-group connectivity is formed by the *cross-group path P* ={"Machine learning", "Database", "Systems and Networking"}, since the database group both has cross-group interactions with "Machine Learning" and "Systems and Networking" groups.

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9 CONCLUSION

In this paper, we studied a new problem of butterfly-core community search over a labeled graph. We theoretically proved the problem's NP-hardness and non-approximability. To tackle it efficiently, we proposed a 2-approximation solution for BCC search. To further improve the efficiency, we developed a fast local method L^2P-BCC to quickly calculate query distances and identify leader-pair. We proposed the mBCC model and the extended framework to explore cross-group community search with multiple labels. Extensive experiments on eight real-world networks with ground-truth communities and interesting case studies validated the effectiveness and efficiency of our BCC and mBCC models and algorithms.

REFERENCES

[1] Ahmed Al-Baghdadi and Xiang Lian. 2020. Topic-based community search over spatial-social networks. *Proceedings of the VLDB Endowment* 13, 12 (2020), 2104–2117.
[2] Nicola Barbieri, Francesco Bonchi, Edoardo Galimberti, and Francesco Gullo. 2015. Efficient and effective community search. *DMKD* 29, 5 (2015), 1406–1433.
[3] Vladimir Batagelj and Matjaz Zaversnik. 2003. An O (m) algorithm for cores decomposition of networks. *arXiv preprint cs/0310049* (2003).
[4] Stephen P Borgatti and Martin G Everett. 1997. Network analysis of 2-mode data. *Social networks* 19, 3 (1997), 243–270.
[5] Cécile Bothorel, Juan David Cruz, Matteo Magnani, and Barbora Micenkova. 2015. Clustering attributed graphs: models, measures and methods. *Network Science* 3, 3 (2015), 408–444.
[6] Lu Chen, Chengfei Liu, Kewen Liao, Jianxin Li, and Rui Zhou. 2019. Contextual community search over large social networks. In *2019 IEEE 35th International Conference on Data Engineering (ICDE)*. IEEE, 88–99.
[7] Lu Chen, Chengfei Liu, Rui Zhou, Jianxin Li, Xiaochun Yang, and Bin Wang. 2018. Maximum co-located community search in large scale social networks. *Proceedings of the VLDB Endowment* 11, 10 (2018), 1233–1246.
[8] Lu Chen, Chengfei Liu, Rui Zhou, Jiajie Xu, Jeffrey Xu Yu, and Jianxin Li. 2020. Finding Effective Geo-social Group for Impromptu Activities with Diverse Demands. In *Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*. 698–708.
[9] Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, and Clifford Stein. 2009. *Introduction to algorithms*. MIT press.
[10] Wanyun Cui, Yanghua Xiao, Haixun Wang, Yiqi Lu, and Wei Wang. 2013. Online search of overlapping communities. In *SIGMOD*. 277–288.
[11] Wanyun Cui, Yanghua Xiao, Haixun Wang, and Wei Wang. 2014. Local search of communities in large graphs. In *SIGMOD*. 991–1002.
[12] Tyler Derr, Cassidy Johnson, Yi Chang, and Jiliang Tang. 2019. Balance in signed bipartite networks. In *CIKM*. 1221–1230.
[13] Yixiang Fang, Reynold Cheng, Siqiang Luo, and Jiafeng Hu. 2016. Effective community search for large attributed graphs. *PVLDB* (2016).
[14] Yixiang Fang, Xin Huang, Lu Qin, Ying Zhang, Wenjie Zhang, Reynold Cheng, and Xuemin Lin. 2019. A survey of community search over big graphs. *VLDBJ* (2019), 1–40.
[15] Yixiang Fang, Zhongran Wang, Reynold Cheng, Hongzhi Wang, and Jiafeng Hu. 2018. Effective and efficient community search over large directed graphs. *TKDE* 31, 11 (2018), 2093–2107.
[16] Yixiang Fang, Yixing Yang, Wenjie Zhang, Xuemin Lin, and Xin Cao. 2020. Effective and efficient community search over large heterogeneous information networks. *PVLDB* 13, 6 (2020), 854–867.
[17] Fangda Guo, Ye Yuan, Guoren Wang, Xiangguo Zhao, and Hao Sun. 2021. Multi-attributed Community Search in Road-social Networks. *arXiv preprint arXiv:2101.09668* (2021).
[18] Xin Huang and Laks VS Lakshmanan. 2017. Attribute-driven community search. *PVLDB* 10, 9 (2017), 949–960.
[19] Xin Huang, Laks VS Lakshmanan, and Jianliang Xu. 2019. *Community Search over Big Graphs*. Morgan & Claypool Publishers.
[20] Xin Huang, Laks VS Lakshmanan, Jeffrey Xu Yu, and Hong Cheng. 2015. Approximate closest community search in networks. *PVLDB* (2015).
[21] Xun Jian, Yue Wang, and Lei Chen. 2020. Effective and Efficient Relational Community Detection and Search in Large Dynamic Heterogeneous Information Networks. *PVLDB* 13, 10 (2020).
[22] Junghoon Kim, Tao Guo, Kaiyu Feng, Gao Cong, Arijit Khan, and Farhana M Choudhury. 2020. Densely connected user community and location cluster search in location-based social networks. In *Proceedings of the 2020 ACM SIGMOD International Conference on Management of Data*. 2199–2209.

[23] Conggai Li, Fan Zhang, Ying Zhang, Lu Qin, Wenjie Zhang, and Xuemin Lin. 2019. Efficient progressive minimum k-core search. *PVLDB* 13, 3 (2019), 362–375.
[24] Jianxin Li, Xinjue Wang, Ke Deng, Xiaochun Yang, Timos Sellis, and Jeffrey Xu Yu. 2017. Most influential community search over large social networks. In *ICDE*. 871–882.
[25] Rundong Li, Pinghui Wang, Peng Jia, Xiangliang Zhang, Junzhou Zhao, Jing Tao, Ye Yuan, and Xiaohong Guan. 2021. Approximately Counting Butterflies in Large Bipartite Graph Streams. *IEEE Transactions on Knowledge and Data Engineering* (2021).
[26] Rong-Hua Li, Lu Qin, Jeffrey Xu Yu, and Rui Mao. 2015. Influential Community Search in Large Networks. *PVLDB* 8, 5 (2015).
[27] Zhe Lin, Fan Zhang, Xuemin Lin, Wenjie Zhang, and Zhihong Tian. 2021. Hierarchical core maintenance on large dynamic graphs. *Proceedings of the VLDB Endowment* 14, 5 (2021), 757–770.
[28] Qing Liu, Minjun Zhao, Xin Huang, Jianliang Xu, and Yunjun Gao. 2020. Truss-based Community Search over Large Directed Graphs. In *SIGMOD*.
[29] Qing Liu, Yifan Zhu, Minjun Zhao, Xin Huang, Jianliang Xu, and Yunjun Gao. 2020. VAC: Vertex-Centric Attributed Community Search. In *ICDE*.
[30] Jiehuan Luo, Xin Cao, Xike Xie, Qiang Qu, Zhiqiang Xu, and Christian S Jensen. 2020. Efficient Attribute-Constrained Co-Located Community Search. In *2020 IEEE 36th International Conference on Data Engineering (ICDE)*. IEEE, 1201–1212.
[31] Garry Robins and Malcolm Alexander. 2004. Small worlds among interlocking directors: Network structure and distance in bipartite graphs. *Computational & Mathematical Organization Theory* 10, 1 (2004), 69–94.
[32] Seyed-Vahid Sanei-Mehri, Ahmet Erdem Sariyuce, and Srikanta Tirthapura. 2018. Butterfly Counting in Bipartite Networks. In *KDD*.
[33] Seyed-Vahid Sanei-Mehri, Yu Zhang, Ahmet Erdem Sariyüce, and Srikanta Tirthapura. 2019. FLEET: butterfly estimation from a bipartite graph stream. In *Proceedings of the 28th ACM International Conference on Information and Knowledge Management*. 1201–1210.
[34] Ahmet Erdem Sarıyüce and Ali Pinar. 2018. Peeling bipartite networks for dense subgraph discovery. In *Proceedings of the Eleventh ACM International Conference on Web Search and Data Mining*. 504–512.
[35] Stephen B Seidman. 1983. Network structure and minimum degree. *Social networks* 5, 3 (1983), 269–287.
[36] Mauro Sozio and Aristides Gionis. 2010. The community-search problem and how to plan a successful cocktail party. In *KDD*. 939–948.
[37] Longxu Sun, Xin Huang, Rong-Hua Li, and Jianliang Xu. 2019. Fast Algorithms for Intimate-Core Group Search in Weighted Graphs. In *WISE*. 728–744.
[38] Yizhou Sun and Jiawei Han. 2013. Mining heterogeneous information networks: a structural analysis approach. *ACM SIGKDD Explorations Newsletter* (2013).
[39] Jie Tang, Jing Zhang, Limin Yao, Juanzi Li, Li Zhang, and Zhong Su. 2008. Arnetminer: extraction and mining of academic social networks. In *Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining*. 990–998.
[40] Chaokun Wang and Junchao Zhu. 2019. Forbidden nodes aware community search. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 33. 758–765.
[41] Kai Wang, Xuemin Lin, Lu Qin, Wenjie Zhang, and Ying Zhang. 2019. Vertex priority based butterfly counting for large-scale bipartite networks. *PVLDB* 12, 10 (2019), 1139–1152.
[42] Kai Wang, Xuemin Lin, Lu Qin, Wenjie Zhang, and Ying Zhang. 2020. Efficient bitruss decomposition for large-scale bipartite graphs. In *2020 IEEE 36th International Conference on Data Engineering (ICDE)*. IEEE, 661–672.
[43] Yubao Wu, Ruoming Jin, Jing Li, and Xiang Zhang. 2015. Robust local community detection: on free rider effect and its elimination. *PVLDB* 8, 7 (2015), 798–809.
[44] Yixing Yang, Yixiang Fang, Maria E Orlowska, Wenjie Zhang, and Xuemin Lin. 2021. Efficient bi-triangle counting for large bipartite networks. *Proceedings of the VLDB Endowment* 14, 6 (2021), 984–996.
[45] Zhiwei Zhang, Xin Huang, Jianliang Xu, Byron Choi, and Zechao Shang. 2019. Keyword-Centric Community Search. In *ICDE*. 422–433.
[46] Dong Zheng, Jianquan Liu, Rong-Hua Li, Cigdem Aslay, Yi-Cheng Chen, and Xin Huang. 2017. Querying intimate-core groups in weighted graphs. In *IEEE International Conference on Semantic Computing*. 156–163.
[47] Yang Zhou, Hong Cheng, and Jeffrey Xu Yu. 2009. Graph clustering based on structural/attribute similarities. *PVLDB* (2009).