ECON20001 Intermediate Macroeconomics

Tutorial 8 (Week 9)

Zheng Fan

The University of Melbourne

Introduction

Zheng Fan

- Ph.D student in Economics at Unimelb
- Consultation & Ed discussion board (your first priority)
- Email coordinators for administrative issues
- Consult Stop 1 for special consideration
- Email: fan.z@unimelb.edu.au (last resort!)

Before asking any questions, make sure you have gone through the Ed discussion board, subject guide, announcements and Q&A on Canvas!!!

Last week lectures

Technological progress

- Exogenous: $\frac{A_{t+1}}{A_t} = 1 + g_A$
- Production function: $Y = F(K, AN)$
- Per effective worker: $k = \frac{K}{AN}$; $\quad y = \frac{Y}{AN}$

  → Intensive version of the production function $y = \cancel{f(k)}$

  → Growth rate of output per effective worker $g_Y - g_A - g_N$

$$g_A = \frac{A_{t+1} - A_t}{A_t} = \frac{A_{t+1}}{A_t} - 1$$

$$\log y_t = \log Y_t - \log A_t - \log N_t$$

$$\log y_{t-1} = \log Y_{t-1} - \log A_{t-1} - \log N_{t-1}$$

$$g_y = g_Y - g_A - g_N$$

Last week lectures

Solow model:

- Capital accumulation: $K_{t+1} - K_t = sY_t - \delta K_t$

$$\rightarrow (1 + g_A + g_N)(k_{t+1} - k_t) = sf(k_t) - (\delta + g_A + g_N)k_t$$

The proof is available in the Bonus slides lecture 5.

- Effective depreciation rate: $(\delta + g_A + g_N)$
- Balanced growth path: $g_K = g_Y = g_N + g_A$

A Cobb-Douglas example:

- $Y = K^\alpha (AN)^{1-\alpha} \rightarrow y = k^\alpha$
- Capital accumulation:
  $(1 + g_A + g_N)\underbrace{(k_{t+1} - k_t)}_{=0} = sk_t^\alpha - (\delta + g_A + g_N)k_t = 0$
- Steady state can be achieved by making: $k_{t+1} - k_t = 0$

Last week lectures

Convergence:

- Absolute convergence: same g_A, g_N and same $s, \alpha, \delta \to$ same output per worker in long run

$A \longrightarrow s, s_A$

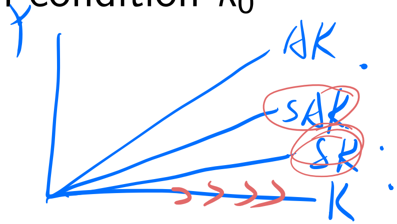
- Conditional convergence: same g_A, g_N but different $s, \alpha, \delta \to$ same growth rate but different long run levels of output per worker

$B \longrightarrow s, s_B$

- Long run is independent of initial condition k_0

Endogenous growth:

- AK growth model: $Y = AK$



$\to$ no diminishing returns to physical capital. Why?

- Capital accumulation: $K_{t+1} - K_t = sAK_t - \delta K_t$
- No steady state but a balanced growth path at the rate of $g^* = sA - \delta$

Last week lectures

$\alpha + \beta = 1$

$\alpha = \alpha$. $\beta = 1 - \alpha$.

Human capital accumulation

- Production function: $\boxed{Y = AK^{\alpha} H^{1-\alpha}}$
- s_K investment to physical capital and s_H investment to human capital
- Capital accumulation:

$$\text{①} \quad K_{t+1} - K_t = s_K Y_t - \delta K_t = gK_t \rightarrow (g+\delta)K_t = s_K Y_t \quad (=0)$$

$$\text{②} \quad H_{t+1} - H_t = s_H Y_t - \delta H_t = gH_t \rightarrow (g+\delta)H_t = s_H Y_t \quad (=0)$$

$$\frac{②}{①} \rightarrow \frac{s_H}{s_k} = \frac{H_t}{K_t} = \phi$$

- Then: $H_t = \phi^* K_t$ with $\dfrac{s_H}{s_K} = \phi^* \rightarrow Y_t = A(\phi^*)^{1-\alpha} K_t$

  $\rightarrow$ Mathematically equivalent to AK model, similar implication

- Solving for endogenous growth rate: $g^* = sA - \delta$ with $s = s_K^{\alpha} s_H^{1-\alpha}$

Last week lectures

$= \frac{H_t}{K_t}$.

Human capital accumulation

- Implications: Similar to AK model
- Transitional dynamics: $\phi_t \to \phi^*$ $= \frac{H_t^*}{K_t^*} = \frac{s_H}{s_K}$?
  - If initial stock of human capital is relatively low, $\phi_0 < \phi^*$, human capital accumulation is faster than physical capital accumulation and ϕ_t rises over time.

In-tutorial Sheet - Q2

2. Consider a Solow growth model with a human capital augmented production function

$$Y_t = AK_t^{\alpha} H_t^{\beta} N^{1-\alpha-\beta}$$

where both α and β are between zero and one and $\alpha + \beta < 1.$ Suppose that both physical and human capital are accumulated with constant savings rates s_K and s_H respectively and depreciate at the common rate δ, that is

$$K_{t+1} - K_t = s_K Y_t - \delta K_t$$

and

$$H_{t+1} - H_t = s_H Y_t - \delta H_t$$

There is no growth in productivity A or raw labor N.

- Solve the steady state y, k, h and ϕ.

$\left(\frac{K}{N}\right)^*$

In-tutorial Sheet - Q2

Dividing both sides of the production function by N we have the intensive version of the production function

$$y = \frac{Y}{N} = \frac{AK^\alpha H^\beta N^{1-\alpha-\beta}}{N} = \frac{AK^\alpha H^\beta \cancel{N^{1-\alpha-\beta}}}{N^\alpha N^\beta \cancel{N^{1-\alpha-\beta}}} = Ak^\alpha h^\beta$$

$h_t = \phi k_t$.

And similarly dividing both sides of the accumulation equations for physical and human capital by the constant N we have the per worker versions

$$\underbrace{k_{t+1} - k_t}_{=0} = s_K y_t - \delta k_t \rightarrow s_K y_t = \delta k_t \quad (1)$$

and

$$\underbrace{h_{t+1} - h_t}_{=0} = s_H y_t - \delta h_t \rightarrow s_H y_t = \delta h_t \quad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{s_H}{s_K} = \boxed{\frac{h_t}{k_t} = \phi}$$

In-tutorial Sheet - Q2

$$\boxed{y = \mathcal{A} k^{\alpha}}$$

$\mathcal{A} = A\phi^{\beta}$

$\alpha = \alpha + \beta$

In steady state $k_{t+1} = k_t$ and $h_{t+1} = h_t$, so we need to solve

$$s_K y = \delta k \Rightarrow k^*$$

$$s_H y = \delta h$$

Dividing the latter by the former we have

$$\boxed{\phi = \frac{h^*}{k^*} = \frac{s_H y}{s_K y} = \frac{s_H}{s_K}} \rightarrow h^* = \phi k^*$$

Then, using the production function, in steady state output per worker is related to capital per worker by

$$y = Ak^{\alpha}h^{\beta} = Ak^{\alpha}(\phi k)^{\beta} = A\phi^{\beta}k^{\alpha+\beta}$$

In-tutorial Sheet - Q2

To interpret most of what follows, observe that this steady state version of the production function is like a standard production function with productivity term $A\phi^{\beta}$ instead of just A and capital share $\alpha + \beta$ instead of just α.

We can then plug this formula for output per worker back into the steady state accumulation equation for capital per worker

$$\boxed{\phi = \frac{s_H}{s_K}}$$

$$s_K \underbrace{A\phi^{\beta} k^{\alpha+\beta}}_{y} = \delta k$$

with solution

$$k^* = \left(\frac{s_K A \phi^{\beta}}{\delta} \right)^{\dfrac{1}{1-\alpha-\beta}} \rightarrow h^*$$

$$\rightarrow y^*.$$

In-tutorial Sheet - Q2

$H = K$.
$\hookrightarrow s_H = s_k = \underline{s}$

$y = A k^{\alpha} h^{\beta}$
$= A k^{\underline{\alpha+\beta}}$ $\rightarrow A k^{1/3}$.

$\alpha = \beta = 1/6$.

$\rightarrow \boxed{y = A k^{\alpha}}$ $\alpha = \frac{1}{3}$.
$= A k^{1/3}$.

And since we already found that $\phi^* = s_H/s_K$, we can write this as

$$k^* = \left(\frac{\overbrace{s_K^{1-\beta} s_H^{\beta}}^{s} A}{\delta} \right)^{\frac{1}{1-\alpha-\beta}} = \boxed{\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}}.$$

To interpret most of what follows, observe that this solution for capital per worker is what we would obtain in a simple Solow model without human capital if we have "aggregate" saving rate $s = s_K^{1-\beta} s_H^{\beta}$ and "aggregate" capital share $\alpha + \beta$ instead of just α.

With the solution for k^* in hand, we recover y^* from the production function and $h^* = \phi^* k^*$.

In-tutorial Sheet - Q2

(a) Suppose $A = 1, \alpha = \beta = 1/6$, and $s_K = s_H = 0.2$ and $\delta = 0.1$. Let $y = Y/N, k = K/N$ and $h = H/N$ denote output per worker, physical capital per worker, and human capital per worker respectively. Let $\phi = h/k$ denote the intensity of human capital relative to physical capital. Calculate y, k, h and ϕ in steady state.

In-tutorial Sheet - Q2

(a) Suppose $A = 1, \alpha = \beta = 1/6$, and $s_K = s_H = 0.2$ and $\delta = 0.1$. Let $y = Y/N, k = K/N$ and $h = H/N$ denote output per worker, physical capital per worker, and human capital per worker respectively. Let $\phi = h/k$ denote the intensity of human capital relative to physical capital. Calculate y, k, h and ϕ in steady state.

Using the given parameters, the human capital intensity is

$$\phi^* = \frac{s_H}{s_K} = \frac{0.2}{0.2} = 1 \quad \Rightarrow \quad \frac{H^*}{K^*} = 1 \Rightarrow H^* = K^*$$

so there is an equal amount of human and physical capital. The aggregate saving rate is the geometric average of the saving rates in each type of capital

$$s = s_K^{1-\beta} s_H^{\beta} = (0.2)^{5/6}(0.2)^{1/6} = 0.2$$

In-tutorial Sheet - Q2

and the aggregate capital share is $\alpha + \beta = 1/6 + 1/6 = 1/3$. So capital per worker is

$$k^* = \left(\frac{s_K^{1-\beta} s_H^{\beta} A}{\delta} \right)^{\frac{1}{1-\alpha-\beta}} = \left(\frac{(0.2)1}{0.1} \right)^{3/2} = 2.828$$

For human capital per worker

$$h^* = \phi^* k^* = (1)(2.828) = 2.828$$

And output per worker is

$$y^* = A(k^*)^{\alpha}(h^*)^{\beta} = (1)(2.828)^{1/6}(2.828)^{1/6} = 2.828^{1/3} = 1.414$$

In-tutorial Sheet - Q2

(b) How does steady state output per worker in this economy compare to the steady state output per worker of the analogous simple Solow model without human capital (i.e., with $Y = AK^{\alpha}N^{1-\alpha}$ and $A = 1, \alpha = 1/3, s = 0.2$ and $\delta = 0.1$). Explain the differences.

$$k^* = \left(\frac{A \cdot s_H^{\beta} s_k^{1-\beta}}{\delta}\right)^{\frac{1}{1-\alpha-\beta}} \quad \alpha + \beta = \frac{1}{3}$$

$$= \left(\frac{A \cdot 0.2^{\beta} \cdot 0.2^{1-\beta}}{\delta}\right)^{\frac{1}{1-\frac{1}{6}-\frac{1}{6}}}$$

$$= \left(\frac{A \cdot 0.2}{\delta}\right)^{\frac{1}{1-\frac{1}{3}}}$$

$$y = Ak^{\alpha}$$

$$sy = \delta k$$

$$\hookrightarrow k^* = \left(\frac{A \cdot s}{\delta}\right)^{\frac{1}{1-\alpha}} \quad \alpha = \frac{1}{3}$$

$$= \left(\frac{A \cdot 0.2}{\delta}\right)^{\frac{1}{1-\frac{1}{3}}}$$

In-tutorial Sheet - Q2

(b) How does steady state output per worker in this economy compare to the steady state output per worker of the analogous simple Solow model without human capital (i.e., with $Y = AK^{\alpha}N^{1-\alpha}$ and $A = 1, \alpha = 1/3, s = 0.2$ and $\delta = 0.1$). Explain the differences.

Steady state output per worker is the same as the steady state output per worker of a Solow model with $A = 1, \alpha = 1/3, s = 0.2$ and $\delta = 0.1$.

Notice that this analogous Solow model has a physical capital share of $1/3$ not the $1/6$ used in part (a).

This is because human capital in part (a) also has a share of $\alpha = 1/6$ so that the "aggregate" capital share is $\alpha + \beta = 1/6 + 1/6 = 1/3$. Put differently, the share for labor N is $2/3$ in both models.

In-tutorial Sheet - Q2

(c) How would your answers to (a) and (b) differ if instead $\alpha = \beta = 1/3$? What about if $\alpha = 1/3$ and $\beta = 2/3$? Explain.

In-tutorial Sheet - Q2

(c) How would your answers to (a) and (b) differ if instead $\alpha = \beta = 1/3$? What about if $\alpha = 1/3$ and $\beta = 2/3$? Explain.

If the share of physical capital is 1/3 and the share of human capital is 1/3, then the aggregate capital share is $\alpha + \beta = 2/3$ and the share for labor is now only 1/3. Steady state capital per worker increases to

$$k^* = \left(\frac{s_K^{1-\beta} s_H^{\beta} A}{\delta} \right)^{\dfrac{1}{1-\alpha-\beta}} = \left(\frac{(0.2)1}{0.1} \right)^3 = 8$$

with human capital per worker

$$h^* = \phi^* k^* = (1)(8) = 8$$

and output per worker

$$y^* = A(k^*)^{\alpha}(h^*)^{\beta} = (1)(8)^{1/3}(8)^{1/3} = 8^{2/3} = 4$$

In-tutorial Sheet - Q2

The aggregate share of capital in this economy is much higher and steady state output per worker is higher than in the analogous Solow model in (b). Because the aggregate capital share is $2/3$ rather than $1/3$, the steady state production function is less concave and it takes longer for diminishing returns to capital to exert its grip.

If $\alpha = 1/3$ and now $\beta = 2/3$ then we have the economy discussed in Lecture 16 with constant returns to the accumulable factors(labor's share is $1 - \alpha - \beta = 0$). Hence the steady state version of the production function is of the form $y = Ak$ (linear in k) and there is no steady state. Instead, there is a balanced growth path with growth rate

$$g^* = sA - \delta$$

where $s = s_K^{\alpha} s_H^{1-\alpha}$.

The end

Thanks for your attention! 😄