Notes on efficient ways to compute loop integrals in the EFTofLSS

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Abstract

We compute QFT-like bubble and triangle integrals for general complex masses.

Contents

1 The problem: find expressions for bubble and triangle integrals

We are interested in two types of integrals, which we call the bubble integral and the triangle integral. The bubble integral has the form,

$$B(k^2, M_1, M_2) = \int \frac{d^3q}{\pi^{3/2}} \frac{1}{(q^2+M_1)((k-q)^2+M_2)} \tag{1}$$

and the triangle integral has the form,

$$T(k_1^2, k_2^2, k_3^2, M_1, M_2, M_3) = \int \frac{d^3q}{\pi^{3/2}} \frac{1}{(q^2+M_1)((k_1-q)^2+M_2)((k_2+q)^2+M_3)}, \tag{2}$$

where $k_1 + k_2 + k_3 = 0$. To compute these integrals, we use Feynman parameterization and dim reg. When the masses M_1, M_2 and M_3 are real, we do not have to worry about branch cuts and analytic continuation of the above integral in the complex plane. However, we will need to be more careful in the case of complex masses.

2 Preliminaries

In this section, we will state some useful ingredients that are helpful in our calculation.

2.1 Generalized multiplicative formulas for logs and square roots

Following the paper by t'Hooft and Veltman (1979), we use logs and square roots that have a branch cut in the negative real axis.

The rule for the log of a product is then:

$$\begin{aligned} \log(ab) &= \log(a) + \log(b) + \eta(a,b)\,, \\ \eta(a,b) &= 2\pi i\left[\theta(-\operatorname{Im} a)\theta(-\operatorname{Im} b)\theta(\operatorname{Im} ab) - \theta(\operatorname{Im} a)\theta(\operatorname{Im} b)\theta(-\operatorname{Im} ab)\right]\,, \end{aligned} \tag{3}$$

where θ is the Heaviside step function, and a and b are complex numbers. If a (and/or b) is a negative real, one should prescribe $\operatorname{Im} a = i\,\epsilon$, where $\epsilon > 0$. Thus in that case, we have $\theta(\operatorname{Im} a) = 1$ and $\theta(-\operatorname{Im} a) = 0$. For consistency, if a is a positive real, we prescribe $\theta(\operatorname{Im} a) = 0$ and $\theta(-\operatorname{Im} a) = 1$.

Useful consequences are:

$$\log(ab) = \log(a) + \log(b)\,,\ \text{if } \operatorname{Im} a \text{ and } \operatorname{Im} b \text{ have different sign,} \tag{4}$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)\,,\ \text{if } \operatorname{Im} a \text{ and } \operatorname{Im} b \text{ have the same sign.} \tag{5}$$

Defining the square root as $\sqrt{a} \equiv \exp(\log(a)/2)$, we obtain the analogous expressions:

$$\begin{aligned} &\sqrt{ab} = s(a,b)\sqrt{a}\sqrt{b}\,, \\ &s(a,b) \equiv \exp\{\eta(a,b)/2\} = (-1)^{\theta(-\operatorname{Im} a)\theta(-\operatorname{Im} b)\theta(\operatorname{Im} ab)}(-1)^{\theta(\operatorname{Im} a)\theta(\operatorname{Im} b)\theta(-\operatorname{Im} ab)}\,, \end{aligned} \tag{6}$$

and specifically:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}\,,\ \text{if } \operatorname{Im} a \text{ and } \operatorname{Im} b \text{ have different sign,} \tag{7}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\,,\ \text{if } \operatorname{Im} a \text{ and } \operatorname{Im} b \text{ have the same sign.} \tag{8}$$

As a quick example, let us look at $\sqrt{-z}$. Applying the product rule in the numerator, we obtain:

$$\sqrt{-z} = s(-1,z)\,i\sqrt{z} = (-1)^{\theta(\operatorname{Im} 1)\theta(-\operatorname{Im} z)}(-1)^{\theta(\operatorname{Im} -1)\theta(\operatorname{Im} z)}i\sqrt{z}\,, \tag{9}$$

and using our prescription, we get $\theta(\operatorname{Im} -1) = 1$ and $\theta(\operatorname{Im} 1) = 0$, and our result simplifies to

$$\sqrt{-z} = s(-1,z)\,i\sqrt{z} = (-1)^{\theta(\operatorname{Im} z)}i\sqrt{z}\,, \tag{10}$$

which means that $\sqrt{-z} = i\sqrt{z}$ only in the lower complex plane and in the positive real axis, while $\sqrt{-z} = -i\sqrt{z}$ in the upper complex plane and in the negative real axis.

2.2 Schwinger parametrization

Let us derive the Feynman parametrization for the bubble integral using Schwinger parameters. We will use two important (equivalent) identities:

$$\frac{i}{A} = \int_0^\infty ds(1+i\epsilon)\ \exp(iA(1+i\epsilon)s)\,, \tag{11}$$

if $\operatorname{Im} A > 0$, and another analogous result

$$-\frac{i}{A} = \int_0^\infty ds(1 - i\epsilon)\, \exp(-iA(1 - i\epsilon)s)\,, \tag{12}$$

if $\operatorname{Im} A < 0$. We will constrain the value of ϵ in both cases to ensure the integrals are convergent. Eqs. (11) and (12) are convergent for $\epsilon = 0$ but it will become important later that ϵ is positive and non-zero. Let us now consider the case of $\operatorname{Im} m > 0$. Eq. (11) is convergent if,

$$\operatorname{Re}\left[i(\operatorname{Re} A + i \operatorname{Im} A)(1 + i\epsilon)\right] = -\operatorname{Im} A - \operatorname{Re} A\epsilon < 0. \tag{13}$$

Taking the form of A to be $q^2 + m$, the conditions on the imaginary part of A become the conditions on the imaginary part of m since q is real. Eq. (13) then becomes,

$$-\operatorname{Im} m - (q^2 + \operatorname{Re} m)\epsilon < 0, \tag{14}$$

which now depends on the sign of $\operatorname{Re} m$. If $\operatorname{Re} m < 0$,

$$\epsilon < \frac{\operatorname{Im} m}{|\operatorname{Re} m|}, \tag{15}$$

where we are free to choose ϵ to be arbitrarily small but positive. If $\operatorname{Re} m > 0$,

$$\epsilon > -\frac{\operatorname{Im} m}{\operatorname{Re} m}, \tag{16}$$

we are again free to choose ϵ_- to be arbitrarily small but positive.

Now consider the case for $\operatorname{Im} m < 0$, then Eq. (12) is convergent if,

$$\operatorname{Im} m - (q^2 + \operatorname{Re} m)\epsilon < 0. \tag{17}$$

The inequality is bounded by the case when $q^2 = 0$, so if $\operatorname{Re} m < 0$,

$$\epsilon < \frac{|\operatorname{Im} m|}{|\operatorname{Re} m|}, \tag{18}$$

and if $\operatorname{Re} m > 0$,

$$\epsilon > \frac{\operatorname{Im} m}{\operatorname{Re} m}. \tag{19}$$

In both cases, we are still free to take ϵ to be positive and arbitrarily small.

3 Bubble integral using Schwinger parameters

3.1 Masses with the same imaginary part sign

3.1.1 Finding a general expression for the integral

Looking at the bubble integral, we note that the imaginary parts of each term in the denominator is the same as the imaginary part of the corresponding mass. Let us then first look at the case where both masses have a positive imaginary part. Using our identities, we obtain straightforwardly:

$$B(k^2, M_1, M_2) = -\int_0^\infty ds_1 \int_0^\infty ds_2 \int \frac{d^3q}{\pi^{3/2}} e^{i(q^2+M_1)s_1} e^{i((k-q)^2+M_2)s_2}\,. \tag{20}$$

Simplifying this, we obtain:

$$\begin{aligned} B(k^2, M_1, M_2) = &-\int_0^\infty ds_1 \int_0^\infty ds_2 \int \frac{d^3q}{\pi^{3/2}} (1 + i\epsilon^1)(1 + i\epsilon^2) \\ &\exp\left\{ iS_+(q + \frac{s_2(1 + i\epsilon^2)}{S_+}k)^2 + i\frac{s_1 s_2(1 + i\epsilon^1)(1 + i\epsilon^2)}{S_+}k^2 + i(M_1 s_1(1 + i\epsilon^1) + M_2 s_2(1 + i\epsilon^2)) \right\}, \end{aligned} \tag{21}$$

where $S_+ = (1 + i\epsilon^1)s_1 + (1 + i\epsilon^2)s_2$.

We can then do 2 things that give the same result:

- We do directly the Gaussian integral in q, then do a change of variable $s_1 = \tau x$, $s_2 = \tau(1-x)$ and then integrate in τ, or alternatively
- We do a change of variable $s_1 = \tau x$, $s_2 = \tau(1-x)$, integrate in τ, and then do the q integral in the standard dim reg way.

Let us pick the first way for concreteness. We can now set $\epsilon^1 = \epsilon^2 = \epsilon$, and guarantee that the Gaussian integral converges (because we have $S_+ = s_1 + s_2 + i\tilde{\epsilon}$, where we redefined the ϵ). After doing the Gaussian q integral, and setting $\epsilon \to 0$ where no poles show up, we get

$$B(k^2, M_1, M_2) = -\int_0^\infty ds_1 \int_0^\infty ds_2 \, \frac{1}{(-i(s_1+s_2+i\epsilon))^{3/2}} \exp\left\{ i\frac{s_1 s_2}{s_1+s_2+i\epsilon}k^2 + i(M_1 s_1 + M_2 s_2)\right\}, \tag{22}$$

where we used the result

$$\int d^d q \exp\left(iaq^2\right) = \frac{\pi^{d/2}}{(-ia)^{d/2}}, \tag{23}$$

if $\operatorname{Im} a > 0$. Now, doing the change of variable described before: $s_1 = \tau x$, $s_2 = \tau(1-x)$, and noting that the Jacobian of the transformation is τ, we get

$$B(k^2, M_1, M_2) = -\frac{1}{(-i)^{3/2}} \int_0^1 dx \int_0^\infty d\tau \, \tau(\tau+i\epsilon)^{-3/2} \exp\left\{ i\frac{\tau^2}{\tau+i\epsilon}x(1-x)k^2 + i\tau(M_1 x + M_2(1-x))\right\}, \tag{24}$$

and since the integral is convergent when $\epsilon \to 0$, we get:

$$B(k^2, M_1, M_2) = -\frac{1}{(-i)^{3/2}} \int_0^1 dx \int_0^\infty d\tau \, \tau^{-1/2} \exp\left\{ i\tau x(1-x)k^2 + i\tau(M_1 x + M_2(1-x))\right\}, \tag{25}$$

and doing the τ integral yields the standard Feynman parameter integral:

$$B(k^2, M_1, M_2) = \frac{\Gamma(1/2)}{(-i)^{3/2}} \int_0^1 dx \frac{(-i)^{3/2}}{\sqrt{x(1-x)k^2 + M_1 x + M_2(1-x)}} \tag{26}$$

$$= \sqrt{\pi} \int_0^1 dx \frac{1}{\sqrt{x(1-x)k^2 + M_1 x + M_2(1-x)}}, \tag{27}$$

which is our familiar Feynman integral. Note that in this case the square root does not have any branch cut, because its argument always has a positive imaginary part, by hypothesis. We were thus able to find the Feynman parameter integral using Schwinger parameters for this case. For two masses with negative imaginary parts, the exact same steps apply, and we obtain the same result.

Solving this integral yields:

$$B(k^2, M_1, M_2) = \frac{\sqrt{\pi}}{k} \left[i \log\left(2\sqrt{x(1-x) + m_1 x + m_2(1-x)} + i(m_1 - m_2 - 2x + 1)\right)\right]_{x=0}^{x=1}, \tag{28}$$

where $m_1 = M_1/k^2$ and $m_2 = M_2/k^2$. To use this expression, we need to check if the argument of the log crosses its branch cut (the negative real axis). If it does, we need to add/subtract $2\pi i$, depending on the direction of the crossing.

3.1.2 Analyzing the branch cut crossings

We can finally analyze under what conditions a log branch cut crossing happens. Let us define

$$A(x, m_1, m_2) \equiv 2\sqrt{x(1-x) + m_1 x + m_2(1-x)} + i(m_1 - m_2 - 2x + 1), \tag{29}$$

then, we have a branch cut crossing when $A(x, m_1, m_2) = -t$, where $t > 0$, for $x \in]0, 1[$ Solving for x yields

$$2\sqrt{x(1-x) + m_1 x + m_2(1-x)} + i(m_1 - m_2 - 2x + 1) = -t \tag{30}$$

$$\Rightarrow 2\sqrt{x(1-x) + m_1 x + m_2(1-x)} = -t - i(m_1 - m_2 - 2x + 1) \tag{31}$$

$$\Rightarrow 4x(1-x) + 4m_1 x + 4m_2(1-x) = t^2 + 2it(m_1 - m_2 - 2x + 1) - (m_1 - m_2 - 2x + 1)^2 \tag{32}$$

$$\Rightarrow \Delta(m_1, m_2) = t^2 + 2it(m_1 - m_2 + 1) - 4itx, \tag{33}$$

where

$$\begin{aligned}\Delta(m_1, m_2) &\equiv m_1^2 - 2m_1m_2 + 2m_1 + m_2^2 + 2m_2 + 1 = (m_1 - m_2)^2 + 2(m_1 + m_2) + 1 \\ &= (m_2 - m_1 + 1)^2 + 4m_1 = (m_1 - m_2 + 1)^2 + 4m_2\,.\end{aligned} \tag{34}$$

which gives two constraints: one for the real part, and another for the imaginary part. The real part equation gives us directly t

$$t_\pm = \mathrm{Im}(m_1) - \mathrm{Im}(m_2) \pm \sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}\,, \tag{35}$$

where $\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2)) > 0$ in order for t to be real. The imaginary part equation gives us the two possible solutions for x corresponding to the two solutions for t:

$$x_+ = \frac{-2\,\mathrm{Im}(m_2) + (\mathrm{Re}(m_1) - \mathrm{Re}(m_2) + 1)\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}}{2\left(\mathrm{Im}(m_1) - \mathrm{Im}(m_2) + \sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}\right)}\,, \tag{36}$$

$$x_- = \frac{-2\,\mathrm{Im}(m_2) - (\mathrm{Re}(m_1) - \mathrm{Re}(m_2) + 1)\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}}{2\left(\mathrm{Im}(m_1) - \mathrm{Im}(m_2) - \sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}\right)}\,. \tag{37}$$

The solution for x as a function of t is

$$x_t = \frac{1}{2} + \frac{\mathrm{Re}(m_1 - m_2)(-\mathrm{Im}(m_1) + \mathrm{Im}(m_2) + t) - \mathrm{Im}(m_1 + m_2)}{2t} \tag{38}$$

However, only one of these solutions, x_- corresponds to $A(x, m_1, m_2)$ being a real number. One way to see this is by looking at Eq. (31). Taking the real part of both sides, we see that $\mathrm{Re}(l.h.s.) > 0$ and that $\mathrm{Re}(r.h.s.) = \mp\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}$, corresponding to $t_\pm$. Therefore only the solution t_- (and thus x_-) corresponds to a positive real part, and can satisfy the equation.

This means that there is at most one branch cut crossing of the log in Eq. (28). In fact, there is a branch cut if $x_- \in]0, 1[$ and $t_- > 0$.

Let us check what are the conditions that m_1 and m_2 have to satisfy in order for $x_- \in]0, 1[$ and $t_- > 0$. Solving $x_- > 0$ and $x_- < 1$ yields, respectively:

$$\mathrm{Im}(m_2) < -\frac{1}{2}(1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} \tag{39}$$

$$\mathrm{Im}(m_1) > \frac{1}{2}(1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}\,. \tag{40}$$

We can now identify two cases:

- If $1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2) > 0$, then, defining $\kappa \equiv \frac{1}{2}(1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} > 0$, we have $\mathrm{Im}(m_1) > \kappa > 0$ and $\mathrm{Im}(m_2) < -\kappa < 0$. So if $1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2) > 0$, we do not have to worry about branch cuts if $\mathrm{Im}(m_1)$ and $\mathrm{Im}(m_2)$ have the same sign.
- If $1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2) < 0$, then, defining $\kappa \equiv -\frac{1}{2}(1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} > 0$, we have $\mathrm{Im}(m_1) > -\kappa$ and $\mathrm{Im}(m_2) < \kappa$. This is possible to have if $\mathrm{Im}(m_1)$ and $\mathrm{Im}(m_2)$ have the same sign, while also keeping $t_- > 0$ if $\mathrm{Im}(m_1 - m_2) > \sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}$.

Finally, we can prove that if there is a branch cut crossing, then it always goes in the same direction. To do this, we assume that $A(x, m_1, m_2)$ defined in Eq. (29) is equal to $-t$, where $t > 0$. Differentiating A with respect to x, we obtain:

$$\begin{aligned}\frac{dA}{dx} &= \frac{m_1 - m_2 - 2x + 1}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}} - 2i \\ &= \frac{m_1 - m_2 - 2x + 1 - 2i\sqrt{x(m_1 - m_2 - x + 1) + m_2}}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}} \\ &= -i\frac{A(x, m_1, m_2)}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}} \\ &= \frac{it}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}}\,,\end{aligned} \tag{41}$$

and, since $\mathrm{Re}(\sqrt{z}) \geq 0$, we obtain $\mathrm{Re}(t/\sqrt{x(m_1 - m_2 - x + 1) + m_2}) \geq 0$. This implies that $\mathrm{Im}(dA/dx) > 0$ at the branch cut crossing, which shows that A crosses the branch cut always from the negative imaginary plane to the positive imaginary plane, and thus that $\mathrm{Im}(A(1, m_1, m_2)) > 0$ and $\mathrm{Im}(A(0, m_1, m_2)) < 0$.

Conversely, if $\mathrm{Im}(A(1, m_1, m_2)) > 0$ and $\mathrm{Im}(A(0, m_1, m_2)) < 0$, that means there was one, and only one (see above) crossing of the imaginary axis, at x_{cross}. Let's call $-t \equiv A(x_{\mathrm{cross}}, m_1, m_2)$. There is a branch cut crossing if $t > 0$. At $x = x_{\mathrm{cross}}$,

$$\begin{aligned}
\frac{dA}{dx} &= \frac{it}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}} \\
\Rightarrow \mathrm{Im}\left(\frac{dA}{dx}\right) &= t\,\mathrm{Re}\left(\frac{1}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}}\right) \\
\Rightarrow t &= \frac{\mathrm{Im}\left(\frac{dA}{dx}\right)}{\mathrm{Re}\left(\frac{1}{\sqrt{x(m_1 - m_2 - x + 1) + m_2}}\right)} > 0\,,
\end{aligned} \tag{42}$$

therefore, we have necessarily a branch cut. This concludes the proof that

$$\begin{aligned}
&\mathrm{Im}(A(1, m_1, m_2)) > 0 \text{ and } \mathrm{Im}(A(0, m_1, m_2)) < 0 \\
\Leftrightarrow\ &\text{there is one, and only one, branch cut crossing between } x = 0 \text{ and } x = 1\,.
\end{aligned} \tag{43}$$

This means that, if $\mathrm{Im}(A(1, m_1, m_2)) > 0$ and $\mathrm{Im}(A(0, m_1, m_2)) < 0$, Eq. (28) becomes

$$B(k^2, M_1, M_2) = \frac{\sqrt{\pi}}{k} i \left\{ \left[\log\left(2\sqrt{x(1-x) + m_1 x + m_2(1-x)} + i(m_1 - m_2 - 2x + 1)\right)\right]_{x=0}^{x=1} - 2\pi i \right\}. \tag{44}$$

3.2 Masses with opposite imaginary part sign

3.2.1 Directly solving the integral

Let us focus now on the case where the two masses have a different sign in the imaginary part. For concreteness, we assume $\mathrm{Im}\, M_1 > 0$ and $\mathrm{Im}\, M_2 < 0$.

In this case we need non-zero ϵ insertions with Eqs. (11) and (12) since we will develop poles without them. Eq. (20) then becomes,

$$\begin{aligned}
B(k^2, M_1, M_2) = \int_0^\infty & ds_1 ds_2 \frac{d^3q}{\pi^{3/2}} (1 + i\epsilon^1)(1 - i\epsilon^2) \exp\left(iS \times \left(q + \frac{k(1 - i\epsilon^2)s_2}{S}\right)^2 - i\frac{k^2(1 + i\epsilon^1)s_1(1 - i\epsilon^2)s_2}{S}\right. \\
&\left. + i\left((1 + i\epsilon^1)s_1 M_1 - (k^2 + M_2)(1 - i\epsilon^2)s_2\right)\right),
\end{aligned} \tag{45}$$

where $S = (1 + i\epsilon^1)s_1 - (1 - i\epsilon^2)s_2 = s_1 - s_2 + i\tilde{\epsilon}$. The q integral will be convergent for $\epsilon^1 > 0$ and $\epsilon^2 > 0$. Doing the momentum integral and taking $\epsilon \to 0$ where possible, we obtain,

$$B(k^2, M_1, M_2) = \frac{1}{(-i)^{3/2}} \int ds_1 ds_2 \frac{1}{S^{3/2}} e^{iI} \tag{46}$$

where $I = M_1 s_1 - M_2 s_2 - k^2 s_1 s_2 / S$. Now we make the change of variables, $s_1 = \tau x$ and $s_2 = \tau(1 - x)$ and also take $\epsilon^1 = \epsilon^2 = \epsilon$,

$$B(k^2, M_1, M_2) = \frac{1}{-i^{3/2}} \int_0^1 dx \int_0^\infty d\tau \frac{\tau^{-1/2}}{(2x - 1 + i\epsilon)^{3/2}} \exp\left(i\tau\left(M_1 x - M_2(1 - x) - \frac{k^2 x(1 - x)}{2x - 1 + i\epsilon}\right)\right) \tag{47}$$

The τ integration then gives,

$$B(k^2, M_1, M_2) = -\sqrt{\pi} \int_0^1 dx \frac{1}{(2x - 1 + i\epsilon)^{3/2}} \frac{1}{\sqrt{-\frac{k^2 x(1-x)}{2x-1+i\epsilon} + M_1 x - M_2(1 - x)}}. \tag{48}$$

We evaluate this integral by taking the principal value using the identity,

$$\frac{1}{X - X_0 + i\epsilon} = P.V. \frac{1}{X - X_0} - i\pi\delta(X - X_0). \tag{49}$$

We obtain,

$$B(k^2, M_1, M_2) = -\sqrt{\pi} \left(P.V. \int_0^1 dx \frac{1}{2x-1} \frac{1}{\sqrt{2x-1}\sqrt{-\frac{k^2 x(1-x)}{2x-1} + M_1 x - M_2(1-x)}} - \frac{i\pi}{2}(-2i) \right). \tag{50}$$

The principal value term can be evaluated by using arctan and carefully choosing a specific Riemann sheet. Specifically, one obtains:

$$B(k^2, M_1, M_2) = -\frac{\sqrt{\pi}}{k} \left(-\frac{2i\sqrt{-y_1}\sqrt{-y_2}}{\sqrt{2y_1 - 1}\sqrt{1-2y_2}\sqrt{-(1+2(m_1+m_2))y_1 y_2}} \left[\arctan\left(\frac{\sqrt{1-2y_2}\sqrt{x-y_1}}{\sqrt{2y_1-1}\sqrt{x-y_2}} \right) \right]_{x=0}^{x=1} -\pi \right), \tag{51}$$

where y_1 and y_2 are the roots of the second-degree polynomial $P(x)$ given by

$$P(x) = (x-1)x + (2x-1)\, x\, m_1 + \left(2x^2 - 3x + 1\right) m_2\,, \tag{52}$$

giving for y_1 and y_2:

$$y_1 = \frac{-\sqrt{m_1^2 - 2m_1 m_2 + 2m_1 + m_2^2 + 2m_2 + 1} + m_1 + 3m_2 + 1}{2(2m_1 + 2m_2 + 1)} \tag{53}$$

$$y_2 = \frac{\sqrt{m_1^2 - 2m_1 m_2 + 2m_1 + m_2^2 + 2m_2 + 1} + m_1 + 3m_2 + 1}{2(2m_1 + 2m_2 + 1)}\,. \tag{54}$$

Again, the crossings of the arctan branch cuts need to be taken into account carefully (by adding or subtracting π, or $\pi/2$ in the case the crossing happens exactly at i or $-i$, for example when $x = 1/2$). This expression agrees with the numerical momentum integral.

3.2.2 Relating this case with the case where masses have the same imaginary part sign

We can also take the expression in Eq. (50) and do a change of variable to put it in a more familiar form. Indeed, by introducing $\hat{x} \equiv \frac{x}{2x-1}$, we get the mapping

$$x \in \left[0, \frac{1}{2} - \frac{\epsilon}{4}\right] \cup \left[\frac{1}{2} + \frac{\epsilon}{4}, 1\right] \Rightarrow \hat{x} \in \left[0, \frac{1}{2} - \frac{1}{\epsilon}\right] \cup \left[\frac{1}{2} + \frac{1}{\epsilon}, 1\right]\,, \tag{55}$$

with ∞ in $\hat{x}$ corresponding to $\frac{1}{2}$ in x. Applying this change of variable to Eq. (50) yields:

$$B(k^2, M_1, M_2) = \sqrt{\pi} \left(\int_0^{\frac{1}{2}-\frac{1}{\epsilon}} d\hat{x} \frac{i}{\sqrt{-\hat{x}(1-\hat{x})k^2 - M_1\hat{x} - M_2(1-\hat{x})}} + \int_{\frac{1}{2}+\frac{1}{\epsilon}}^1 d\hat{x} \frac{1}{\sqrt{\hat{x}(1-\hat{x})k^2 + M_1\hat{x} + M_2(1-\hat{x})}} + \pi \right)\,, \tag{56}$$

which is a very familiar integrand (see Eq. (27)).

We can further simplify this by noticing that

$$s(-1, \hat{x}(1-\hat{x})k^2 + M_1\hat{x} + M_2(1-\hat{x})) = 1\,, \tag{57}$$

if $x < 0$, $\mathrm{Im}(M_1) > 0$, and $\mathrm{Im}(M_2) > 0$ (because the second argument of s always has a negative imaginary part in this case). Therefore, we can simplify (56), obtaining

$$B(k^2, M_1, M_2) = \sqrt{\pi} \left(\int_0^{\frac{1}{2}-\frac{1}{\epsilon}} \frac{d\hat{x}}{\sqrt{\hat{x}(1-\hat{x})k^2 + M_1\hat{x} + M_2(1-\hat{x})}} + \int_{\frac{1}{2}+\frac{1}{\epsilon}}^1 \frac{d\hat{x}}{\sqrt{\hat{x}(1-\hat{x})k^2 + M_1\hat{x} + M_2(1-\hat{x})}} + \pi \right). \tag{58}$$

Now both components have the same integrand. Let us define

$$I \equiv \int_0^{\frac{1}{2}-\frac{1}{\epsilon}} \frac{d\hat{x}}{\sqrt{\hat{x}(1-\hat{x})+m_1\hat{x}+m_2(1-\hat{x})}}\,, \tag{59}$$

$$II \equiv \int_{\frac{1}{2}+\frac{1}{\epsilon}}^{1} \frac{d\hat{x}}{\sqrt{\hat{x}(1-\hat{x})+m_1\hat{x}+m_2(1-\hat{x})}}\,. \tag{60}$$

As written above in Eq. (28), the antiderivative corresponding to I and II is

$$G(x) = i\log\left(2\sqrt{x(1-x)+m_1x+m_2(1-x)}+i(m_1-m_2-2x+1)\right)\,, \tag{61}$$

where we have dropped the hat for clarity. Therefore, we have for $I+II$:

$$I+II = G(1)-G(0)+i\lim_{\epsilon\to 0}\left(\log\left(\frac{1}{4}i\epsilon\Delta(m_1,m_2)\right)-\log\left(-\frac{1}{4}i\epsilon\Delta(m_1,m_2)\right)\right)\,, \tag{62}$$

where

$$\Delta(m_1,m_2) = m_1^2-2m_1m_2+2m_1+m_2^2+2m_2+1\,. \tag{63}$$

Simplifying Eq. (62) yields

$$\begin{aligned} I+II &= G(1)-G(0)+i\left(-\log(-1)-\eta(-1,\frac{1}{4}i\epsilon\Delta(m_1,m_2))\right) \\ &= G(1)-G(0)+\pi-i\,\eta(-1,i\Delta(m_1,m_2))\,, \end{aligned} \tag{64}$$

so Eq. (58) simplifies to

$$B(k^2,M_1,M_2) = \sqrt{\pi}\left[G(1)-G(0)+2\pi-i\,\eta(-1,i\Delta(m_1,m_2))\right]\,. \tag{65}$$

Note that $i\,\eta$ is either 2π or 0. The simplified expression for $i\eta(-1,i\Delta(m_1,m_2))$ is:

$$i\eta(-1,i\Delta(m_1,m_2)) = 2\pi\times\theta\left(\mathrm{Re}(\Delta(m_1,m_2))\right)\,, \tag{66}$$

where θ is the Heaviside theta function. Note that $\mathrm{Re}(\Delta(m_1,m_2)) = \Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))-(\mathrm{Im}(m_1-m_2))^2$. Notice also that Eq. (65) is almost identical to Eq. (28).

Of course we have to take into account the branch cut crossings of the antiderivative that can occur in the integration region. We will deal with them now.

Following the steps of the previous case where the imaginary part of the masses have the same sign, we verify that our only branch cut crossing happens at x_- given by Eq. (37).

$$x_- = \frac{-2\,\mathrm{Im}(m_2)-(\mathrm{Re}(m_1)-\mathrm{Re}(m_2)+1)\sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))}}{2\left(\mathrm{Im}(m_1)-\mathrm{Im}(m_2)-\sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))}\right)}\,. \tag{67}$$

The question here is whether $x_- < 0$ or $x_- > 1$ (while keeping $t_- > 0$). The constraints we get for m_1 and m_2 are analogous to the ones obtained in Eqs. (39) and (40):

$$x_- < 0 \Rightarrow \quad \mathrm{Im}(m_2) > -\frac{1}{2}(1+\mathrm{Re}(m_1)-\mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))} \tag{68}$$

$$x_- > 1 \Rightarrow \quad \mathrm{Im}(m_1) < \frac{1}{2}(1+\mathrm{Re}(m_1)-\mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))}\,. \tag{69}$$

As in the previous situation, we can now identify two cases:

- Case $1+\mathrm{Re}(m_1)-\mathrm{Re}(m_2) > 0$: Defining $\kappa \equiv \frac{1}{2}(1+\mathrm{Re}(m_1)-\mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))} > 0$, we have $\mathrm{Im}(m_1) < \kappa$ or $\mathrm{Im}(m_2) > -\kappa$, both of which can satisfy the conditions $\mathrm{Im}(m_1) > 0$ and $\mathrm{Im}(m_2) < 0$. So if $\mathrm{Im}(m_1)-\mathrm{Im}(m_2) > \sqrt{\Delta(\mathrm{Re}(m_1),\mathrm{Re}(m_2))}$, there can be branch cut crossings.

- Case $1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2) < 0$: Defining $\kappa \equiv -\frac{1}{2}(1 + \mathrm{Re}(m_1) - \mathrm{Re}(m_2))\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} > 0$, we have $\mathrm{Im}(m_1) < -\kappa < 0$ and $\mathrm{Im}(m_2) > \kappa > 0$. So in this case it is impossible to satisfy the conditions $\mathrm{Im}(m_1) > 0$ and $\mathrm{Im}(m_2) < 0$. Also, t_- is necessarily negative in this situation, so there can be no branch cut crossings.

Therefore, in order to have branch cut crossings for $\mathrm{Im}(m_1) > 0$ and $\mathrm{Im}(m_2) < 0$, we must have $\mathrm{Re}(m_2 - m_1) < 1$. Let us analyze other conditions for branch cuts.

We can analyze how the imaginary part of $A(x, m_1, m_2)$ varies. Direct computation shows that (assuming $\mathrm{Im}(m_1) > 0$ and $\mathrm{Im}(m_2) < 0$)

$$\begin{aligned} A\left(\frac{1}{2} - \frac{1}{\epsilon}, m_1, m_2\right) &= \frac{1}{4} i\Delta(m_1, m_2)\epsilon + o(\epsilon^2)\,, \\ A\left(\frac{1}{2} + \frac{1}{\epsilon}, m_1, m_2\right) &= -\frac{1}{4} i\Delta(m_1, m_2)\epsilon + o(\epsilon^2) \end{aligned} \tag{70}$$

Thus, $\mathrm{Im}\left(A\left(\frac{1}{2} - \frac{1}{\epsilon}, m_1, m_2\right)\right) = \frac{\epsilon}{4}\mathrm{Re}(\Delta(m_1, m_2))$, and $\mathrm{Im}\left(A\left(\frac{1}{2} + \frac{1}{\epsilon}, m_1, m_2\right)\right) = -\frac{\epsilon}{4}\mathrm{Re}(\Delta(m_1, m_2))$. Let us consider two possibilities: $\mathrm{Re}(\Delta(m_1, m_2)) > 0$ and $\mathrm{Re}(\Delta(m_1, m_2)) < 0$.

- Case $\mathrm{Re}(\Delta(m_1, m_2)) > 0$: $\mathrm{Im}\left(A\left(-\infty, m_1, m_2\right)\right) > 0$ and $\mathrm{Im}\left(A\left(+\infty, m_1, m_2\right)\right) < 0$. To check if there are branch cut crossings in this situation, we can look at t_-. Specifically, the term $\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}$:

  $$\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} = \sqrt{\mathrm{Re}(\Delta(m_1, m_2)) + (\mathrm{Im}(m_1 - m_2))^2} > \mathrm{Im}(m_1) - \mathrm{Im}(m_2)\,, \tag{71}$$

  which implies that in this case $t_- < 0$ and so there are no branch cut crossings. So, we can have one of three possibilities for the position of x_-: $x_- < 0$, $0 < x_- < 1$, $x_- > 1$. For each one, we have:

  $$\begin{aligned} x_- < 0: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) < 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) < 0\,, \\ 0 < x_- < 1: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) > 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) < 0\,, \\ x_- > 1: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) > 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) > 0\,. \end{aligned}$$

  Importantly, note that we never have $\mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) < 0$ and $\mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) > 0$ in this case. Thus, Eq. (65) is simply given by

  $$B(k^2, M_1, M_2) = \sqrt{\pi}\left[G(1) - G(0)\right]\,. \tag{72}$$

- Case $\mathrm{Re}(\Delta(m_1, m_2)) < 0$: $\mathrm{Im}\left(A\left(-\infty, m_1, m_2\right)\right) < 0$ and $\mathrm{Im}\left(A\left(+\infty, m_1, m_2\right)\right) > 0$. To check if there are branch cut crossings in this situation, we can look at $t_- = \mathrm{Im}(m_1) - \mathrm{Im}(m_2) - \sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}$. Specifically, the term $\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))}$:

  $$\sqrt{\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2))} = \sqrt{\mathrm{Re}(\Delta(m_1, m_2)) + (\mathrm{Im}(m_1 - m_2))^2}\,, \tag{73}$$

  so, if $\mathrm{Re}(\Delta(m_1, m_2)) > 0$, then $t_- < 0$, which implies there are no branch cut crossings. On the other hand, if $\mathrm{Re}(\Delta(m_1, m_2)) < 0$, then $t_- > 0$, which implies there is a branch cut crossing, as long as $\Delta(\mathrm{Re}(m_1), \mathrm{Re}(m_2)) > 0$.

  So, we can have one of three possibilities for the position of x_-: $x_- < 0$, $0 < x_- < 1$, $x_- > 1$. For each one, we have:

  $$\begin{aligned} x_- < 0: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) > 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) > 0\,, \\ 0 < x_- < 1: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) < 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) > 0\,, \\ x_- > 1: &\quad \mathrm{Im}\left(A\left(0, m_1, m_2\right)\right) < 0 \text{ and } \mathrm{Im}\left(A\left(1, m_1, m_2\right)\right) < 0\,. \end{aligned}$$

  For $x_- < 0$ and $x_- > 1$, the branch cut crossing is in the integration region, so we need to take the crossing into account. In the two cases, it amounts to subtracting 2π from the expression inside the square brackets in Eq. (65). Therefore, for these situations, Eq. (65) is simply given by

  $$B(k^2, M_1, M_2) = \sqrt{\pi}\left[G(1) - G(0)\right]\,. \tag{74}$$

  For $0 < x_- < 1$, there is no branch cut crossing in the integration region, so Eq. (65) becomes

  $$B(k^2, M_1, M_2) = \sqrt{\pi}\left[G(1) - G(0) + 2\pi\right]\,. \tag{75}$$

What is remarkable is that, regardless of the relative signs of $\mathrm{Im}(m_1)$ and $\mathrm{Im}(m_2)$, and combining these results with Eq. (44), we always have that if $\mathrm{Im}(A(1, m_1, m_2)) > 0$ and $\mathrm{Im}(A(0, m_1, m_2)) < 0$, $B(k^2, M_1, M_2)$ is given by

$$B(k^2, M_1, M_2) = \frac{\sqrt{\pi}}{k}\left[i \log\left(2\sqrt{m_1} + i(m_1 - m_2 - 1)\right) - i \log\left(2\sqrt{m_2} + i(m_1 - m_2 + 1)\right) + 2\pi\right], \tag{76}$$

and in all other cases:

$$B(k^2, M_1, M_2) = \frac{\sqrt{\pi}}{k}\left[i \log\left(2\sqrt{m_1} + i(m_1 - m_2 - 1)\right) - i \log\left(2\sqrt{m_2} + i(m_1 - m_2 + 1)\right)\right]. \tag{77}$$

This interesting observation makes $B(k^2, M_1, M_2)$ extremely efficient to evaluate numerically.

3.2.3 Relating the integrals using contour integration

This last expression Eq. (77) hints, by its simplicity, at some closer relation between the case where the masses have the same sign of the imaginary part and where they have opposite signs. Indeed, we can prove using contour integration that the two integrals are closely related.

First, let us call x_1 and x_2 the roots of the second order polynomial $x(1-x) + m_1 x + m_2(1-x)$. Then, we can write Eq. (58) as

$$B(k^2, M_1, M_2) = \sqrt{\pi}\left(\int_0^{\frac{1}{2}-\frac{1}{\epsilon}} \frac{dx}{\sqrt{(x_1 - x)(x - x_2)}} + \int_{\frac{1}{2}+\frac{1}{\epsilon}}^{1} \frac{d\hat{x}}{\sqrt{(x_1 - x)(x - x_2)}} + \pi\right). \tag{78}$$

One can prove that, if $\mathrm{Im}(m_1) > 0$, $\mathrm{Im}(m_2) < 0$, and $\mathrm{Re}(m_i) > 0$, then $\mathrm{Im}(x_1) > 0$ and $\mathrm{Im}(x_2) > 0$. First, observe that $x_1 + x_2 = 1 + m_1 - m_2$ and that $x_1 x_2 = -m_2$. Taking the real and imaginary parts of each of those two equalities gives four equations with four unknows (the components of x_1 and x_2). One of them is $\mathrm{Im}(x_1 + x_2) = \mathrm{Im}(m_1 - m_2)$, which tells us that $\mathrm{Im}(x_1 + x_2) > 0$. From $\mathrm{Re}(x_1 + x_2) = 1 + \mathrm{Re}(m_1 - m_2)$ and $\mathrm{Im}(x_1 x_2) = -\mathrm{Im}(m_2)$, we get

$$\mathrm{Re}(x_1) = \frac{\mathrm{Im}(x_1)(1 + \mathrm{Re}(m_1 - m_2)) + \mathrm{Im}(m_2)}{\mathrm{Im}(x_1 - x_2)}, \tag{79}$$

$$\mathrm{Re}(x_2) = \frac{-\mathrm{Im}(x_2)(1 + \mathrm{Re}(m_1 - m_2)) - \mathrm{Im}(m_2)}{\mathrm{Im}(x_1 - x_2)}, \tag{80}$$

and, pluging in $\mathrm{Re}(x_1 x_2) = -\mathrm{Re}(m_2)$ and using $\mathrm{Im}(x_1 + x_2) = \mathrm{Im}(m_1 - m_2)$, we get

$$\begin{aligned} \mathrm{Im}(x_1)\,\mathrm{Im}(x_2)\left[(\mathrm{Im}(x_1 - x_2))^2 + (1 + \mathrm{Re}(m_1 - m_2))^2 + 4\,\mathrm{Re}(m_2)\right] = -\,\mathrm{Im}(m_1)\,\mathrm{Im}(m_2) + \\ + \mathrm{Re}(m_1)\,\mathrm{Im}(m_2)(\mathrm{Im}(m_2) - \mathrm{Im}(m_1)) + \mathrm{Re}(m_2)\,\mathrm{Im}(m_1)(\mathrm{Im}(m_1) - \mathrm{Im}(m_2)), \end{aligned} \tag{81}$$

so if $\mathrm{Re}(m_i) > 0$, then $\mathrm{Im}(x_1)\,\mathrm{Im}(x_2) > 0$. Combined with $\mathrm{Im}(x_1 + x_2) > 0$, it implies that $\mathrm{Im}(x_1) > 0$ and $\mathrm{Im}(x_2) > 0$.

Thus, we can separate the square roots in the two terms:

$$\sqrt{(x_1 - x)(x - x_2)} = \sqrt{x_1 - x}\sqrt{x - x_2}. \tag{82}$$

Let us now define $f(z) = \sqrt{x_1 - z}\sqrt{z - x_2}$ for complex z. The function $f(z)$ has two horizontal branch cuts in the upper imaginary plane, but in the lower imaginary plane f is analytic. Thus, by using a contour constituted of the real line and a lower semi-circle C, we find that

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}} + \int_0^{\pi} d\theta \frac{(-i)e^{-i\theta}}{\sqrt{-e^{-i\theta}}\sqrt{e^{-i\theta}}} = 0 \tag{83}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}} = \pi \tag{84}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}} = \pi + \int_0^{-\infty} \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}} + \int_{\infty}^{1} \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}}, \tag{85}$$

and therefore, when $\mathrm{Im}(m_1) > 0$, $\mathrm{Im}(m_2) < 0$, and $\mathrm{Re}(m_i) > 0$, we obtain:

$$B(k^2, M_1, M_2) = \sqrt{\pi} \int_0^1 \frac{dx}{\sqrt{x_1 - x}\sqrt{x - x_2}} = \sqrt{\pi} \int_0^1 \frac{dx}{\sqrt{(x_1 - x)(x - x_2)}}\,, \tag{86}$$

which is exactly Eq. (27). Note that the requirement $\mathrm{Re}(m_i) > 0$ is important in this case. In general, there could be a crossing of a branch cut between $x = 0$ and $x = 1$ if $\mathrm{Im}(m_1) > 0$ and $\mathrm{Im}(m_2) < 0$. So the two cases are indeed closely related. We will use similar argument to simplify the triangle integral

4 Triangle integral with Schwinger parameters

Let us compute the triangle integral,

$$T(k_1^2, k_2^2, M_1, M_2, M_3) = \int \frac{d^3q}{\pi^{3/2}} \frac{1}{((\vec{k}_1 - \vec{q})^2 + M_1)(q^2 + M_2)((\vec{k}_2 + \vec{q})^2 + M_3)} \tag{87}$$

in a similar way to the bubble integral.

4.1 Case where the imaginary part of the masses have the same sign

First let us consider the case of all three masses M_1, M_2 and M_3 having positive imaginary parts. Using Schwinger parameterization, the momentum integral becomes,

$$\begin{aligned} T(\dots) = \frac{i}{\pi^{3/2}} \int_0^{+\infty} \prod ds_i \int_q (1 + i\epsilon_1)(1 + i\epsilon_2)(1 + i\epsilon_3) \\ \exp\left[i \left((\vec{k}_1 - \vec{q})^2 + M_1 \right) (1 + i\epsilon_1)s_1 + i(q^2 + M_2)(1 + i\epsilon_2)s_2 + i \left((\vec{k}_2 + \vec{q})^2 + M_3 \right) (1 + i\epsilon_3)s_3 \right]. \end{aligned} \tag{88}$$

Expanding the exponent,

$$(1 + i\epsilon_1)s_1(k_1^2 + q^2 - 2k_1 \cdot q + M_1) + (1 + i\epsilon_2)s_2(q^2 + M_2) + (1 + i\epsilon_3)s_3(k_2^2 + q^2 + 2k_2 \cdot q + M_3) \tag{89}$$

$$\begin{aligned} &= q^2 \left[(1 + i\epsilon_1)s_1 + (1 + i\epsilon_2)s_2 + (1 + i\epsilon_3)s_3\right] - 2q \cdot \left[k_1(1 + i\epsilon_1)s_1 - k_2(1 + i\epsilon_3)s_3\right] + (1 + i\epsilon_1)s_1(k_1^2 + M_1) \\ &\quad + (1 + i\epsilon_2)s_2M_2 + (1 + i\epsilon_3)s_3(k_2^2 + M_3) \end{aligned} \tag{90}$$

$$= S \left(q^2 - 2q \cdot \frac{(1 + i\epsilon_1)s_1k_2 - (1 + i\epsilon_3)s_3k_2}{S} \right) + (1 + i\epsilon_1)s_1(k_1^2 + M_1) + (1 + i\epsilon_2)s_2M_2 + (1 + i\epsilon_3)s_3(k_2^2 + M_3) \tag{91}$$

$$= S \left(q - \frac{s_1k_1 - s_3k_2}{S} \right)^2 - \frac{(s_1k_1 - s_3k_2)^2}{S} + s_1k_1^2 + s_3k_2^2 + s_1M_1 + s_2M_2 + s_3M_3 \tag{92}$$

$$= S(\dots)^2 - \frac{s_1^2k_1^2 - 2s_1s_3k_1 \cdot k_2 + s_3^2k_2^2}{S} + (s_1 + s_2 + s_3)(s_1k_1^2 + s_3k_2^2)/S + s_1M_1 + s_2M_2 + s_3M_3 \tag{93}$$

$$= S(\dots)^2 + \frac{s_1s_3k_3^2 + s_1s_2k_1^2 + s_2s_3k_2^2}{s_1 + s_2 + s_3} + s_1M_1 + s_2M_2 + s_3M_3 \tag{94}$$

where we take the limit $\epsilon_i \to 0$ where no singularities appear and define $S = s_1 + s_2 + s_3 + i\tilde{\epsilon}$, and use $\vec{k}_1 + \vec{k}_3 + \vec{k}_3 = 0$. Doing the Gaussian integral in q, we obtain,

$$T = \frac{i}{(-i)^{3/2}} \int_0^{+\infty} ds_1 ds_2 ds_3 \frac{e^{iI}}{S^{3/2}} \tag{95}$$

where $I = (s_1s_2k_1^2 + s_3s_1k_3^2 + s_2s_3k_2^2)/S + s_1M_1 + s_2M_2 + s_3M_3$. Changing integration variables $s_1 = \tau x_1, s_2 = \tau x_2, s_3 = \tau(1 - x_1 - x_2) = \tau x_3$, where $\tau \in [0, +\infty[$, $x_1 \in [0, 1]$, $x_2 \in [0, 1]$, and $x_1 + x_2 < 1$, and observing that the Jacobian is τ^2, we obtain

$$T = -\frac{1}{(-i)^{1/2}} \int_0^1 dx_1 dx_2 \int_0^{+\infty} d\tau \tau^{1/2} e^{i\tau\tilde{I}}\,, \tag{96}$$

where we defined $\tilde{I} = x_1 x_2 k_1^2 + x_2 x_3 k_2^2 + x_1 x_3 k_3^2 + x_1 M_1 + x_2 M_2 + x_3 M_3$ Doing the τ integral,

$$T = \frac{\sqrt{\pi}}{2} \int dx_1 dx_2 \tilde{I}^{-3/2}. \tag{97}$$

Next, we make another change of variables $x_2 = (1-x)y$, $x_1 = x$, obtaining

$$T = \frac{\sqrt{\pi}}{2} \int_0^1 \frac{dxdy(1-x)}{\left(M_1 x + M_2 y(1-x) + M_3(1-y)(1-x) + k_1^2 xy(1-x) + k_2^2(1-x)^2 y(1-y) + k_3^2 x(1-x)(1-y)\right)^{3/2}} \tag{98}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^1 \frac{dxdy(1-x)}{(-ay^2 + by + c)^{3/2}}, \tag{99}$$

where $a = k_2^2(1-x)^2$, $b = (1-x)(k_2^2 + M_2 - M_3 + x(k_1^2 - k_2^2 - k_3^2))$, and $c = M_3(1-x) + k_3^2 x(1-x) + M_1 x$. Doing the indefinite integral in y, we get:

$$T(k_1^2, k_2^2, k_3^2, M_1, M_2, M_3) = \frac{\sqrt{\pi}}{2} \int_0^1 dx \left. \frac{2(1-x)(2ay - b)}{(b^2 + 4ac)\sqrt{-ay^2 + by + c}} \right|_{y=0}^{y=1}, \tag{100}$$

valid if $b^2 + 4ac \neq 0$. If $b^2 + 4ac = 0$, we obtain

$$T(k_1^2, k_2^2, k_3^2, M_1, M_2, M_3) = \frac{\sqrt{\pi}}{2} \int_0^1 dx \left. \frac{2(x-1)}{(b - 2ay)\sqrt{-\frac{(b-2ay)^2}{a}}} \right|_{y=0}^{y=1}. \tag{101}$$

Let us consider the case $b^2 + 4ac \neq 0$. Replacing the values of a, b, and c, and rearranging, we obtain:

$$T(x, y, k_1^2, k_2^2, k_3^2, M_1, M_2, M_3) = \frac{\sqrt{\pi}}{2} \int dx \left. \frac{N_1 x + N_0}{\sqrt{R_2 x^2 + R_1 x + R_0}\,(S_2 x^2 + S_1 x + S_0)} \right|_{y=0}^{y=1} \tag{102}$$

where $N_1, N_0, R_2, R_1, R_0, S_2, S_1, S_0$ are functions of y, independent of x, given by:

$$\begin{aligned}
N_1 &= -2k_1^2 + 2(1-2y)k_2^2 + 2k_3^2 \\
N_0 &= -2M_2 + 2M_3 + 2k_2^2(-1+2y) \\
R_2 &= k_3^2(-1+y) - k_1^2 y + k_2^2(1-y)y \\
&= k_3^2(-1+y) - k_1^2 y \\
R_1 &= k_1^2 y + 2k_2^2 y\,(y-1) + (1-y)k_3^2 + M_1 - M_2 y + M_3(y-1) \\
&= k_1^2 y + (1-y)k_3^2 + M_1 - M_2 y + M_3(y-1) \\
R_0 &= M_3(1-y) + k_2^2 y(1-y) + M_2 y \\
&= M_3(1-y) + M_2 y \\
S_2 &= k_1^4 - 2k_1^2(k_2^2 + k_3^2) + (k_2^2 - k_3^2)^2 \\
S_1 &= 2\left(k_1^2(k_2^2 + M_2 - M_3) - k_2^4 + k_2^2(k_3^2 + 2M_1 - M_2 - M_3) + k_3^2(M_3 - M_2)\right) \\
S_0 &= k_2^4 + 2k_2^2(M_2 + M_3) + (M_2 - M_3)^2,
\end{aligned}$$

where for the second equality in R_2, R_1, and R_0 we have used the fact that y is either 1 or 0.

Before integrating T we can put it in a simplified form, by factoring the second order polynomials:

$$T = \frac{\sqrt{\pi}}{2} \int_0^1 dx \, \frac{N_1 x + N_0}{\sqrt{R_2(x-y_+)(x-y_-)} S_2(x-x_+)(x-x_-)} \Bigg|_{y=0}^{y=1} \tag{103}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^1 dx \, \frac{1}{\sqrt{R_2(x-y_+)(x-y_-)}} \left(\frac{N_0 + N_1 x_+}{S_2(x-x_+)(x_+ - x_-)} - \frac{N_0 + N_1 x_-}{S_2(x-x_-)(x_+ - x_-)} \right) \Bigg|_{y=0}^{y=1} \tag{104}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^1 dx \, \frac{c_1}{\sqrt{R_2(x-y_+)(x-y_-)}(x-x_+)} + \frac{c_2}{\sqrt{R_2(x-y_+)(x-y_-)}(x-x_-)} \Bigg|_{y=0}^{y=1}, \tag{105}$$

where $c_1 = \frac{N_0 + N_1 x_+}{S_2(x_+ - x_-)}$ and $c_2 = -\frac{N_0 + N_1 x_-}{S_2(x_+ - x_-)}$. Defining $F_{\rm int}$ as

$$F_{\rm int}(R_2, y_+, y_-, x_0) = \frac{\sqrt{\pi}}{2} \int_0^1 dx \frac{1}{\sqrt{R_2(x-y_+)(x-y_-)}(x-x_0)}, \tag{106}$$

we can write T as

$$T = c_1 F_{\rm int}(R_2, y_+, y_-, x_+) + c_2 F_{\rm int}(R_2, y_+, y_-, x_-)|_{y=0}^{y=1}. \tag{107}$$

Note that this approach is only valid for $S_2 \neq 0$. The case $S_2 = 0$ corresponds to totally squeezed triangles satisfying $|k_3| = ||k_1| \pm |k_2||$ or $k_3 = 0$. The evaluation of $F_{\rm int}$ is discussed in Section 4.6.

4.2 Case where all masses have positive real part

Let us consider the case of $\{\mathrm{Re}(M_1), \mathrm{Re}(M_2), \mathrm{Re}(M_3)\} > 0$. In this case we can choose the Schwinger parameterization,

$$\frac{1}{A} = \int_0^\infty ds \exp(-As) \tag{108}$$

for $\mathrm{Re}(A) > 0$. In this case, the triangle integral can be written as,

$$T(k_1^2, k_2^2, M_1, M_2, M_3) = \int_q \int \prod ds_i \exp\left[-\left((\vec{k}_1 - \vec{q})^2 + M_1 \right) s_1 - (q^2 + M_2) s_2 - \left((\vec{k}_2 - \vec{q})^2 + M_3 \right) s_3 \right]. \tag{109}$$

Since if $\{\mathrm{Re}(M_1), \mathrm{Re}(M_2), \mathrm{Re}(M_3)\} > 0$, $\mathrm{Re}(A) > 0$. Expanding the exponential,

$$-(s_1 + s_2 + s_3) \left(q^2 - 2q \cdot \frac{s_1 k_1 - s_3 k_2}{s_1 + s_2 + s_3} \right) - s_1 k_1^2 - s_1 M_1 - s_2 M_2 - s_3 k_2^2 - s_3 M_3 \tag{110}$$

$$= -(s_1 + s_2 + s_3) \left(q - \frac{s_1 k_1 - s_3 k_2}{s_1 + s_2 + s_3} \right)^2 + \frac{(s_1 k_1 - s_3 k_2)^2}{s_1 + s_2 + s_3} - s_1 k_1^2 - s_3 k_2^2 - s_1 M_1 - s_2 M_2 - s_3 M_3 \tag{111}$$

$$= -(s_1 + s_2 + s_3) \left(q - \frac{s_1 k_1 - s_3 k_2}{s_1 + s_2 + s_3} \right)^2 - \frac{s_1 s_3 k_3^2 + s_1 s_2 k_1^2 + s_2 s_3 k_2^2}{s_1 + s_2 + s_3} - s_1 M_1 - s_2 M_2 - s_3 M_3 \tag{112}$$

We can perform the Gaussian integral as $s_1 + s_2 + s_3 > 0$. After the Gaussian integral in q, we make the change of variables $s_1 = \tau x_1, s_2 = \tau x_2, s_3 = \tau(1 - x_1 - x_2) = \tau x_3$,

$$\int dx_1 dx_2 \tau^2 \frac{\exp\left[-\tau \frac{x_1 x_3 k_3^2 + x_1 x_2 k_1^2 + x_2 x_3 k_2^2}{x_1 + x_2 + x_3} - \tau x_1 M_1 - \tau x_2 M_2 - \tau x_3 M_3 \right]}{(\tau x_1 + \tau x_2 + \tau x_3)^{3/2}} \tag{113}$$

$$= \frac{\sqrt{\pi}}{2} \int dx_1 dx_2 \frac{1}{\left(x_1(1 - x_1 - x_2) k_3^2 + x_1 x_2 k_1^2 + x_2(1 - x_1 - x_2) k_2^2 + x_1 M_1 + x_2 M_2 + (1 - x_1 - x_2) M_3 \right)^{3/2}} \tag{114}$$

where in the last line we performed the τ integration. We now arrive at the same integral as the case when M_1, M_2 and M_3 having positive imaginary parts.

4.3 Case where the imaginary part of the masses have the opposite sign

Let us consider the case of two masses M_1 and M_2 having positive imaginary parts and M_3 having a negative imaginary part.

$$\begin{aligned} T(\dots) = -i\int\prod ds_i \int_q (1+i\epsilon_1)(1+i\epsilon_2)(1-i\epsilon_3)\times \\ \exp\left[i\left((k_1-q)^2+M_1\right)(1+i\epsilon_1)s_1 + i\left(q^2+M_2\right)(1+i\epsilon_2)s_2 - i\left((k_2+q)^2+M_3\right)(1-i\epsilon_3)s_3\right] \end{aligned} \tag{115}$$

Expanding the exponent (factoring out an i),

$$(1+i\epsilon_1)s_1(k_1^2+q^2-2k_1\cdot q+M_1)+(1+i\epsilon_2)s_2(q^2+M_2)-(1-i\epsilon_3)s_3(k_2^2+q^2+2k_2\cdot q+M_3) \tag{116}$$

$$\begin{aligned} &= q^2\left[(1+i\epsilon_1)s_1+(1+i\epsilon_2)s_2-(1-i\epsilon_3)s_3\right]-2q\cdot\left[k_1(1+i\epsilon_1)s_1+k_2(1-i\epsilon_3)s_3\right]+(1+i\epsilon_1)s_1(k_1^2+M_1) \\ &\quad +(1+i\epsilon_2)s_2M_2-(1-i\epsilon_3)s_3(k_2^2+M_3) \end{aligned} \tag{117}$$

$$= S_-\left(q^2-2q\cdot\frac{(1+i\epsilon_1)s_1k_1+(1-i\epsilon_3)s_3k_2}{S_-}\right)+(1+i\epsilon_1)s_1(k_1^2+M_1)+(1+i\epsilon_2)s_2M_2-(1-i\epsilon_3)s_3(k_2^2+M_3) \tag{118}$$

$$= S_-\left(q-\frac{s_1k_1+s_3k_2}{S_-}\right)^2-\frac{\left[s_1k_1+s_3k_2\right]^2}{S_-}+s_1k_1^2-s_3k_2^2+s_1M_1+s_2M_2-s_3M_3 \tag{119}$$

$$= S_-(\dots)^2-\frac{s_1^2k_1^2+2s_1s_3k_1\cdot k_2+s_3^2k_2^2}{S_-}+(s_1+s_2-s_3)(s_1k_1^2-s_3k_2^2)/S_-+s_1M_1+s_2M_2-s_3M_3 \tag{120}$$

$$= S_-(\dots)^2-\frac{2s_1s_3k_1\cdot k_2+s_1s_3k_2^2-s_1s_2k_1^2+s_2s_3k_2^2+s_1s_3k_1^2}{S_-}+s_1M_1+s_2M_2-s_3M_3 \tag{121}$$

$$= S_-(\dots)^2+\frac{s_1s_2k_1^2-s_1s_3k_3^2-s_2s_3k_2^2}{S_-}+s_1M_1+s_2M_2-s_3M_3 \tag{122}$$

where we take the limit $\epsilon_i\to 0$ where no singularities appear and redefine $S_-=s_1+s_2-s_3+i\tilde{\epsilon}$. After the Gaussian integral we obtain,

$$T=\frac{-i}{(-i)^{3/2}}\int ds_1ds_2\frac{1}{S_-^{3/2}}e^{iI} \tag{123}$$

where $I=(s_1s_2k_1^2-s_3s_1k_3^2-s_2s_3k_2^2)/S_-+s_1M_1+s_2M_2-s_3M_3$. Changing variables to $s_1=\tau x_1, s_2=\tau x_2, s_3=\tau(1-x_1-x_2)=\tau x_3$, and noting that the Jacobian is τ^2, we get:

$$T=\frac{1}{(-i)^{1/2}}\int dx_1dx_2d\tau\frac{\tau^{1/2}}{(2(x_1+x_2)-1+i\tilde{\epsilon})^{3/2}}e^{i\tau\tilde{I}} \tag{124}$$

where $\tilde{I}=\frac{x_1x_2k_1^2-x_2x_3k_2^2-x_1x_3k_3^2}{2(x_1+x_2)-1+i\tilde{\epsilon}}+x_1M_1+x_2M_2-x_3M_3$. Doing the τ integral yields,

$$T=-\frac{\sqrt{\pi}}{2}\int_0^1 dx_1dx_2dx_3\delta(1-\sum x_i)\left(x_1+x_2-x_3+i\tilde{\epsilon}\right)^{-3/2}\tilde{I}^{-3/2} \tag{125}$$

Again evaluating this integral with the principal value,

$$T=-\frac{\sqrt{\pi}}{2}\int_0^1 dx_1dx_2dx_3\delta(1-\sum x_i)\left(P.V.\frac{1}{x_1+x_2-x_3}-\frac{i\pi}{2}\delta(x_2+x_1-\frac{1}{2})\right)(x_1+x_2-x_3+i\epsilon)^{-1/2}\tilde{I}^{-3/2}\,. \tag{126}$$

The term not involving the principal value can be easily evaluated:

$$T = -\frac{\sqrt{\pi}}{2}\int_0^1 dx_1dx_2dx_3\delta(1-\sum x_i)\left(-\frac{i\pi}{2}\delta(x_2+x_1-\frac{1}{2})\right)(x_1+x_2-x_3+i\epsilon)^{-1/2}\tilde{I}^{-3/2} \quad (127)$$

$$= -\frac{\sqrt{\pi}}{2}\int_0^1 dx_1dx_2\left(-\frac{i\pi}{2}\delta(x_2+x_1-\frac{1}{2})\right)(2(x_1+x_2)-1+i\epsilon)^{-1/2}\tilde{I}^{-3/2} \quad (128)$$

$$= -\frac{\sqrt{\pi}}{2}\int_0^1 dx_1\left(-\frac{i\pi}{2}\right)(i\epsilon)^{-1/2}\left(\frac{x_1x_2k_1^2-x_2x_3k_2^2-x_1x_3k_3^2}{i\epsilon}+x_1M_1+x_2M_2-x_3M_3\right)^{-3/2} \quad (129)$$

$$\underset{\epsilon\to 0}{=} o(\epsilon)\,, \quad (130)$$

which means it is just 0. So we are left with

$$T = -\frac{\sqrt{\pi}}{2}P.V.\int_0^1 dx_1dx_2dx_3\delta(1-\sum x_i)\frac{1}{x_1+x_2-x_3}\frac{1}{\sqrt{x_1+x_2-x_3+i\epsilon}}\tilde{I}^{-3/2}\,, \quad (131)$$

which in fact is a convergent integral (no need of the principal value). For later convenience, let us write explicitly the integration region in the plane x_1-x_2:

$$R = \int_0^1 dx_1dx_2dx_3\delta(1-\sum x_i) = \int_0^{\frac{1}{2}}dx_1\int_0^{\frac{1}{2}-x_1}dx_2 + \int_0^{\frac{1}{2}}dx_1\int_{\frac{1}{2}-x_1}^{1-x_1}dx_2 + \int_{\frac{1}{2}}^1 dx_1\int_0^{1-x_1}dx_2 \quad (132)$$

We can then make the following substitutions, $\tilde{x}_1 = \frac{x_1}{2(x_1+x_2)-1}, \tilde{x}_2 = \frac{x_2}{2(x_1+x_2)-1}$ (note that the Jacobian is $\frac{1}{(1-2(\tilde{x}_1+\tilde{x}_2))^3}$). The region of integration is $\{\tilde{x}_1\tilde{x}_2 > 0\}\setminus\{(\tilde{x}_1>0)\wedge(\tilde{x}_2>0)\wedge(\tilde{x}_1+\tilde{x}_2<1)\}$, that is the first and third quadrants except the triangle whose vertices are (0,0), (0,1) and (1,0) First, let us see how R is expressed:

$$\begin{aligned} R = &\int_{-\infty}^0 d\tilde{x}_1\int_{-\infty}^0\frac{d\tilde{x}_2}{(1-2(\tilde{x}_1+\tilde{x}_2))^3} + \int_0^{\frac{1}{2}}d\tilde{x}_1\int_{1-\tilde{x}_1}^{+\infty}\frac{d\tilde{x}_2}{(2(\tilde{x}_1+\tilde{x}_2)-1)^3} + \\ &+\int_{\frac{1}{2}}^{+\infty}d\tilde{x}_1\int_{\frac{1}{2}}^{+\infty}\frac{d\tilde{x}_2}{(2(\tilde{x}_1+\tilde{x}_2)-1)^3} + \int_{\frac{1}{2}}^1 d\tilde{x}_1\int_{1-\tilde{x}_1}^{\frac{1}{2}}\frac{d\tilde{x}_2}{(2(\tilde{x}_1+\tilde{x}_2)-1)^3} + \int_1^{+\infty}d\tilde{x}_1\int_0^{\frac{1}{2}}\frac{d\tilde{x}_2}{(2(\tilde{x}_1+\tilde{x}_2)-1)^3} \\ &= \int_{-\infty}^0 d\tilde{x}_1\int_{-\infty}^0\frac{d\tilde{x}_2}{(1-2(\tilde{x}_1+\tilde{x}_2))^3} + \left(\int_0^1 d\tilde{x}_1\int_{1-\tilde{x}_1}^{+\infty} + \int_1^{+\infty}d\tilde{x}_1\int_0^{+\infty}\right)\frac{d\tilde{x}_2}{(2(\tilde{x}_1+\tilde{x}_2)-1)^3}\,. \end{aligned} \quad (133)$$

Our new expression for T becomes:

$$T = -\frac{\sqrt{\pi}}{2}\int d\tilde{x}_1d\tilde{x}_2\left|\frac{1}{(1-2(\tilde{x}_1+\tilde{x}_2))^3}\right|\left(\frac{1}{2(\tilde{x}_1+\tilde{x}_2)-1}\right)^{-3/2}\left(\frac{\tilde{I}}{2(\tilde{x}_1+\tilde{x}_2)-1}\right)^{-3/2} \quad (134)$$

$$\begin{aligned} \Rightarrow T = -\frac{\sqrt{\pi}}{2}&\left(-\int_{-\infty}^0 d\tilde{x}_1\int_{-\infty}^0 d\tilde{x}_2 + \int_0^1 d\tilde{x}_1\int_{1-\tilde{x}_1}^{+\infty} + \int_1^{+\infty}d\tilde{x}_1\int_0^{+\infty}d\tilde{x}_2\right) \\ &\left(\frac{1}{2(\tilde{x}_1+\tilde{x}_2)-1}\right)^{3/2}\left(\frac{\tilde{I}}{2(\tilde{x}_1+\tilde{x}_2)-1}\right)^{-3/2}, \end{aligned} \quad (135)$$

$$\begin{aligned} \Rightarrow T = &\frac{\sqrt{\pi}}{2}\left(\int_{-\infty}^0 d\tilde{x}_1\int_{-\infty}^0 d\tilde{x}_2\right)\frac{(-i)}{(1-2(\tilde{x}_1+\tilde{x}_2))^{3/2}}\left(\frac{(-i)}{(1-2(\tilde{x}_1+\tilde{x}_2))^{3/2}}\right)^{-1}\tilde{I}^{-3/2} \\ &-\frac{\sqrt{\pi}}{2}\left(\int_0^1 d\tilde{x}_1\int_{1-\tilde{x}_1}^{+\infty} + \int_1^{+\infty}d\tilde{x}_1\int_0^{+\infty}d\tilde{x}_2\right)\frac{1}{(2(\tilde{x}_1+\tilde{x}_2)-1)^{3/2}}\left(\frac{1}{(2(\tilde{x}_1+\tilde{x}_2)-1)^{3/2}}\right)^{-1}\tilde{I}^{-3/2} \end{aligned} \quad (136)$$

$$\Rightarrow T = \frac{\sqrt{\pi}}{2}\left(\int_{-\infty}^0 d\tilde{x}_1\int_{-\infty}^0 d\tilde{x}_2 - \int_0^1 d\tilde{x}_1\int_{1-\tilde{x}_1}^{+\infty}d\tilde{x}_2 - \int_1^{+\infty}d\tilde{x}_1\int_0^{+\infty}d\tilde{x}_2\right)\tilde{I}^{-3/2}\,, \quad (137)$$

where we have redefined

$$\tilde{I} \equiv k_1^2\tilde{x}_1\tilde{x}_2 + k_2^2\tilde{x}_2(1-\tilde{x}_1-\tilde{x}_2) + k_3^2\tilde{x}_1(1-\tilde{x}_1-\tilde{x}_2) + \tilde{x}_1M_1 + \tilde{x}_2M_2 + M_3(1-\tilde{x}_1-\tilde{x}_2)\,, \quad (138)$$

and used the fact that the sign of $\mathrm{Im}(\tilde{I})$ is constant in each integration region.

We can go further and do another change of variables: $\tilde{x}_1 = x$, $\tilde{x}_2 = (1-x)y$, whose Jacobian is $1-x$. Let us separate the integration region in three parts, one in the third quadrant, and another two in the first quadrant. In the third quadrant, we have

$$\int_{-\infty}^{0} d\tilde{x}_1 \int_{-\infty}^{0} d\tilde{x}_2 = \int_{-\infty}^{0} dx(1-x) \int_{-\infty}^{0} dy\,, \tag{139}$$

and in the first quadrant we have

$$\int_{0}^{1} d\tilde{x}_1 \int_{1-\tilde{x}_1}^{+\infty} d\tilde{x}_2 = \int_{0}^{1} dx \int_{1}^{+\infty} dy(1-x)\,, \tag{140}$$

and

$$\int_{1}^{+\infty} d\tilde{x}_1 \int_{0}^{+\infty} d\tilde{x}_2 = \int_{1}^{+\infty} dx \int_{-\infty}^{0} dy|1-x| = \int_{1}^{+\infty} dx \int_{-\infty}^{0} dy(x-1)\,. \tag{141}$$

So, we get for T:

$$T = \frac{\sqrt{\pi}}{2}\left(\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy - \int_{0}^{1} dx \int_{1}^{+\infty} dy + \int_{1}^{+\infty} dx \int_{-\infty}^{0} dy\right)(1-x)\tilde{I}^{-3/2}\,, \tag{142}$$

where $\tilde{I}$ is given in terms of x and y by

$$\tilde{I} = M_1 x + M_2 y(1-x) + M_3(1-y)(1-x) + k_1^2 xy(1-x) + k_2^2(1-x)^2 y(1-y) + k_3^2 x(1-x)(1-y)\,. \tag{143}$$

We observe that $\tilde{I}$ is a second order polynomial in x and y. So we have

$$\tilde{I} = A_x(x-x_1)(x-x_2) = A_y(y-y_1)(y-y_2)\,, \tag{144}$$

where A_x and A_y are given by:

$$A_x = y(-k_1^2 + k_2^2 + k_3^2) - k_2^2 y^2 - k_3^2\,, \tag{145}$$

$$A_y = -k_2^2(x-1)^2\,. \tag{146}$$

It is interesting to notice that $A_x < 0$ (for k_1, k_2, and k_3 forming a triangle) and $A_y < 0$. Now, we can write the integrand as

$$\begin{aligned}\frac{1-x}{\tilde{I}^{3/2}} &= \frac{1-x}{(A_x(x-x_1)(x-x_2))^{3/2}} = \frac{1-x}{A_x(x-x_1)(x-x_2)}\frac{1}{\sqrt{A_x(x-x_1)(x-x_2)}}\\ &= \frac{1-x}{A_x(x-x_1)(x-x_2)}\frac{s(x_1-x, x-x_2)}{\sqrt{|A_x|}\sqrt{x_1-x}\sqrt{x-x_2}}\,,\end{aligned} \tag{147}$$

or as

$$\frac{1-x}{\tilde{I}^{3/2}} = \frac{1-x}{A_y(y-y_1)(y-y_2)}\frac{s(y_1-y, y-y_2)}{\sqrt{|A_y|}\sqrt{y_1-y}\sqrt{y-y_2}}\,. \tag{148}$$

As a side note, since $\tilde{I}$ crosses the real line when $y = \frac{x\,\mathrm{Im}(M_1-M_3)+\mathrm{Im}(M_3)}{(x-1)\,\mathrm{Im}(M_2-M_3)}$, then if $x<0$, the crossing happens at a positive y, which means there is no crossing in the third quadrant. Likewise, if $x>1$, the crossing happens at a positive y. Therefore, in the region $x<0$ and $y<0$, there are no crossings, which means that $s(x_1-x, x-x_2) = s(x_1, -x_2)$, with x_1 and x_2 taken at $y=0$. In the same way, in the region $x>1$ and $y<0$ there are no crossings, which means that $s(x_1-x, x-x_2) = s(x_1-1, 1-x_2)$, with x_1 and x_2 taken at $y=0$. For convenience, we write here the values for x_1 and x_2 when $y=0$:

$$x_1(y=0) = \frac{\sqrt{(k_3^2+M_1-M_3)^2+4k_3^2M_3}+k_3^2+M_1-M_3}{2k_3^2} \tag{149}$$

$$x_2(y=0) = \frac{-\sqrt{(k_3^2+M_1-M_3)^2+4k_3^2M_3}+k_3^2+M_1-M_3}{2k_3^2}\,. \tag{150}$$

Similar relations apply for y.

Now we are ready to proceed from Eq. (142). Let us analyze separately two of the three integration regions. We have,

$$\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy (1-x)\tilde{I}^{-3/2} = \int_{-\infty}^{0} dx \int_{-\infty}^{0} dy \frac{1-x}{A_x(x-x_1)(x-x_2)} \frac{s(x_1,-x_2)}{\sqrt{|A_x|}\sqrt{x_1-x}\sqrt{x-x_2}} \tag{151}$$

$$\int_{1}^{+\infty} dx \int_{-\infty}^{0} dy (1-x)\tilde{I}^{-3/2} = \int_{1}^{+\infty} dx \int_{-\infty}^{0} dy \frac{1-x}{A_x(x-x_1)(x-x_2)} \frac{s(x_1-1,1-x_2)}{\sqrt{|A_x|}\sqrt{x_1-x}\sqrt{x-x_2}} , \tag{152}$$

so that

$$\begin{aligned} &\left(\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy + \int_{1}^{+\infty} dx \int_{-\infty}^{0} dy\right)(1-x)\tilde{I}^{-3/2} \\ &= \int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \left(\int_{-\infty}^{0} dx + \int_{1}^{+\infty} dx \frac{s(x_1-1,1-x_2)}{s(x_1,-x_2)}\right) \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} , \end{aligned} \tag{153}$$

where x_1 and x_2 inside s are evaluated at $y=0$. We can then have 2 cases: either $s(x_1-1,1-x_2) = s(x_1,-x_2)$ or $s(x_1-1,1-x_2) = -s(x_1,-x_2)$.

4.3.1 Case where $s(x_1-1,1-x_2) = s(x_1,-x_2)$

For now, let us take the first case. Then, Eq. (153) becomes

$$\int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \left(\int_{-\infty}^{+\infty} dx - \int_{0}^{1} dx\right) \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} , \tag{154}$$

and it is straightforward to check that the first term $\int_{-\infty}^{+\infty} dx(\ldots) = 0$ (by computing the antiderivative eg in Mathematica). Plugging in Eq. (142) yields

$$T = \frac{\sqrt{\pi}}{2}\left(-\int_{0}^{1} dx \int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} - \int_{0}^{1} dx \int_{1}^{+\infty} dy (1-x)\tilde{I}^{-3/2}\right) , \tag{155}$$

and noting that, for $0 < x < 1$ and $y > 1$

$$\begin{aligned} (1-x)\tilde{I}^{-3/2} &= \frac{1-x}{A_x(x-x_1)(x-x_2)} \frac{s(x_1(y=1),-x_2(y=1))}{\sqrt{|A_x|}\sqrt{x_1-x}\sqrt{x-x_2}} = \frac{(1-x)s(x_1,-x_2)}{|A_x|^{3/2}(x_1-x)^{3/2}(x-x_2)^{3/2}} \\ &= \frac{(1-x)s(y_1-1,1-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} , \end{aligned} \tag{156}$$

where $s(y_1-1,1-y_2)$ is evaluated at $x=0$, and also

$$\frac{s(x_1,-x_2)}{|A_x|^{3/2}} \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} = \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} , \tag{157}$$

where $s(y_1,-y_2)$ is evaluated at $x=0$ as well. Let us consider the case where $s(y_1,-y_2) = s(y_1-1,1-y_2)$. Then Eq. (155) becomes

$$\begin{aligned} T &= -\frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \left(\int_{-\infty}^{0} dy + \int_{1}^{+\infty} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} \\ &= -\frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \left(\int_{-\infty}^{+\infty} dy - \int_{0}^{1} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} , \end{aligned} \tag{158}$$

where again we can easily check that $\int_{-\infty}^{+\infty} dy(\ldots)$ vanishes. Finally, we obtain:

$$T = \frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \int_{0}^{1} dy \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} , \tag{159}$$

which is the expression we were looking after! Notice the similarity with Eq. (99), with the difference that the denominator is already factorized. This is necessary in this case to avoid any branch cuts in the region of integration.

This can be simplified and written in terms of $F_{\rm int}$. Keep in mind that this is only valid if two conditions are satisfied: $s(x_1-1, 1-x_2) = s(x_1, -x_2)$, with x_1 and x_2 evaluated at $y=0$, and $s(y_1-1, 1-y_2) = s(y_1, -y_2)$, with y_1 and y_2 evaluated at $x=0$. For the masses used in our decomposition, those criteria are always satisfied.

We are now in position to evaluate Eq. (159). We check in Sec. 4.4 that if $s(y_1-1, 1-y_2) = s(y_1, -y_2)$ then they are equal to 1, and $\mathrm{Im}(y_1)\,\mathrm{Im}(y_2) > 0$. Integrating in y yields:

$$T = \sqrt{\pi}\int_0^1 dx\ \frac{(2y-(y_1+y_2))}{|k_2|^3(x-1)^2\sqrt{y_1-y}\sqrt{y-y_2}(y_1-y_2)^2}\Bigg|_{y=0}^{y=1}\,. \tag{160}$$

Since y_1 and y_2 are not polynomials in x, it is convenient to regroup this expression and rewrite it in terms of x. We obtain:

$$\begin{aligned}
T &= \frac{\sqrt{\pi}}{|k_2|}\int_0^1 dx\ \frac{s(y_1-y, y-y_2)(2y-(y_1+y_2))}{|A_y|\sqrt{(y_1-y)(y-y_2)}(y_1-y_2)^2}\Bigg|_{y=0}^{y=1} && (161)\\
&= \frac{\sqrt{\pi}}{|k_2|}\int_0^1 dx\ \frac{(2y-(y_1+y_2))}{\sqrt{|A_y|}\sqrt{A_y(y-y_1)(y-y_2)}(y_1-y_2)^2}\Bigg|_{y=0}^{y=1} && (162)\\
&= \frac{\sqrt{\pi}}{|k_2|}\int_0^1 dx\ \frac{(2y-(y_1+y_2))}{|k_2|(1-x)\sqrt{A_x(x-x_1)(x-x_2)}(y_1-y_2)^2}\Bigg|_{y=0}^{y=1} && (163)\\
&= \frac{\sqrt{\pi}}{2}\int dx\ \frac{N_1x+N_0}{\sqrt{R_2x^2+R_1x+R_0}\,(S_2x^2+S_1x+S_0)}\Bigg|_{y=0}^{y=1}\,, && (164)
\end{aligned}$$

which coincides with Eq. (102), with the same parameters. Note that this is only valid because $s(x_1-1, 1-x_2) = s(x_1, -x_2)$ and so there are no branch cuts in the integration region. We can also see this by calculating the values of x and y for which $\tilde{I}$ crosses the real axis (where we may encounter a branch cut of the square root). The pairs (x, y) form a curve given by

$$y = \frac{x\,\mathrm{Im}(M_1) + (1-x)\,\mathrm{Im}(M_3)}{(x-1)\,\mathrm{Im}(M_2-M_3)}\,. \tag{165}$$

If $x=0$, then $y_{\rm cross} = \frac{-\mathrm{Im}(M_3)}{\mathrm{Im}(M_2-M_3)}$, meaning that $0 < y_{\rm cross} < 1$. If $y=0$, then $x_{\rm cross} = \frac{-\mathrm{Im}(M_3)}{\mathrm{Im}(M_1-M_3)}$, also implying that $0 < x_{\rm cross} < 1$. Therefore, if $s(x_1-1, 1-x_2) = s(x_1, -x_2)$ at $y=0$ and $s(y_1-1, 1-y_2) = s(y_1, -y_2)$ at $x=0$, there is no branch cut crossing in the integration region. More about this later.

As a reminder, the parameters $N_1, N_0, R_2, R_1, R_0, S_2, S_1, S_0$ are functions of y, independent of x, given by:

$$\begin{aligned}
N_1 &= -2k_1^2 + 2(1-2y)k_2^2 + 2k_3^2\\
N_0 &= -2M_2 + 2M_3 + 2k_2^2(-1+2y)\\
R_2 &= k_3^2(-1+y) - k_1^2 y + k_2^2(1-y)y\\
&= k_3^2(-1+y) - k_1^2 y\\
R_1 &= k_1^2 y + 2k_2^2 y\,(y-1) + (1-y)k_3^2 + M_1 - M_2 y + M_3(y-1)\\
&= k_1^2 y + (1-y)k_3^2 + M_1 - M_2 y + M_3(y-1)\\
R_0 &= M_3(1-y) + k_2^2 y(1-y) + M_2 y\\
&= M_3(1-y) + M_2 y\\
S_2 &= k_1^4 - 2k_1^2(k_2^2+k_3^2) + (k_2^2-k_3^2)^2\\
S_1 &= 2\left(k_1^2(k_2^2+M_2-M_3) - k_2^4 + k_2^2(k_3^2+2M_1-M_2-M_3) + k_3^2(M_3-M_2)\right)\\
S_0 &= k_2^4 + 2k_2^2(M_2+M_3) + (M_2-M_3)^2\,,
\end{aligned}$$

where for the second equality in R_2, R_1, and R_0 we have used the fact that y is either 1 or 0. Thus, we recover Eq. (105).

4.3.2 Case where $s(x_1-1,1-x_2)=-s(x_1,-x_2)$

Now we return to Eq. (153) and consider the case $s(x_1-1,1-x_2)=-s(x_1,-x_2)$. In this case, Eq. (153) becomes,

$$\int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \left(\int_{-\infty}^{0} dx - \int_{1}^{\infty} dx\right) \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} \tag{166}$$

As the integral in x is now well defined over the real line, we can extend the region of integration so Eq. (153) becomes,

$$\int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \left(\int_{-\infty}^{0} dx - \int_{-\infty}^{\infty} dx + \int_{-\infty}^{1} dx\right) \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} \tag{167}$$

$$= \int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \left(\int_{-\infty}^{0} dx + \int_{-\infty}^{1} dx\right) \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} \tag{168}$$

$$= \int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \int_{0}^{1} \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} + 2\int_{-\infty}^{0} dy \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \int_{-\infty}^{0} dx \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}}. \tag{169}$$

Plugging back into Eq. (142),

$$T = \frac{\sqrt{\pi}}{2}\left(\left(\left(\int_{0}^{1} dx \int_{-\infty}^{0} dy + 2\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy\right) \frac{s(x_1,-x_2)}{|A_x|^{3/2}} \frac{1-x}{(x_1-x)^{3/2}(x-x_2)^{3/2}} - \int_{0}^{1} dx \int_{1}^{+\infty} dy (1-x)\tilde{I}^{-3/2}\right)\right. \tag{170}$$

$$= \frac{\sqrt{\pi}}{2}\left(\left(\left(\int_{0}^{1} dx \int_{-\infty}^{0} dy + 2\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} - \int_{0}^{1} dx \int_{1}^{+\infty} dy \frac{(1-x)s(y_1-1,1-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}}\right)\right. . \tag{171}$$

Again we can use our trick to evaluate $s(y_1,-y_2)=s(y_1+\infty,-\infty-y_2)$ and $s(y_1-1,1-y_2)=s(y_1-\infty,\infty-y_2)$ evaluated at $x=0$.

Consider the first case $\operatorname{sign}(\operatorname{Im} y_1)=\operatorname{sign}(\operatorname{Im} y_2)\rightarrow s(y_1,-y_2)=s(y_1-1,1-y_2)=1$, then the above equation becomes,

$$T = \frac{\sqrt{\pi}}{2}\left(2\int_{-\infty}^{1} dx \int_{-\infty}^{0} dy + \int_{0}^{1} dx \int_{0}^{1} dy\right) \frac{(1-x)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}}. \tag{172}$$

In the second case $\operatorname{sign}(\operatorname{Im} y_1)\neq\operatorname{sign}(\operatorname{Im} y_2)\rightarrow (s(y_1,-y_2)=-s(y_1-1,1-y_2))$, then we have,

$$T = \frac{\sqrt{\pi}}{2}\left(2\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy + \int_{0}^{1} dx \int_{0}^{1} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}}. \tag{173}$$

Now consider $\operatorname{sign}(\operatorname{Im} x_1)=\operatorname{sign}(\operatorname{Im} x_2)$ and $\operatorname{sign}(\operatorname{Im} y_1)\neq\operatorname{sign}(\operatorname{Im} y_2)$, then Eq. (155) becomes,

$$T = -\frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \left(2\int_{-\infty}^{0} dy + \int_{0}^{1} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} \tag{174}$$

Now consider $\operatorname{sign}(\operatorname{Im} x_1)=\operatorname{sign}(\operatorname{Im} x_2)$ and $\operatorname{sign}(\operatorname{Im} y_1)\neq\operatorname{sign}(\operatorname{Im} y_2)$, then Eq. (155) becomes,

$$T = -\frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \left(2\int_{-\infty}^{0} dy + \int_{0}^{1} dy\right) \frac{(1-x)s(y_1,-y_2)}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}}. \tag{175}$$

Notice that when $s(x_1-1,1-x_2)=-s(x_1,-x_2)$ and $s(y_1-1,1-y_2)=s(y_1,-y_2)$ we must add two regions of integration to our result, namely the region $x<0,y<0$ and the region $0<x<1,y<0$. Let us examine frist the integration in y,

$$T_y = \int dy \frac{1-x}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} = \frac{4y-2(y_1+y_2)}{k_2^3(x-1)^2\sqrt{y_1-y}\sqrt{y-y_2}(y_1-y_2)^2} \tag{176}$$

where we will subsequently take the limits of $\lim_{y\to 0} T_y(x) - \lim_{y\to -\infty} T_y(x)$,

$$\lim_{y\to 0} T_y(x) = \frac{-2(y_1+y_2)(1-x)}{k_2^2(x-1)^2(y_1-y_2)^2\sqrt{-k_2^2(x-1)^2 y_1 y_2}} \tag{177}$$

$$\lim_{y\to -\infty} T_y(x) = \frac{-4i}{k_2^3(x-1)^2(y_1-y_2)^2}. \tag{178}$$

Now, let us evaluate the region of integration $x<0, y<0$,

$$\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy \frac{1-x}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} = \int_{-\infty}^{0} dx \left(\lim_{y\to 0} T_y(x) - \lim_{y\to -\infty} T_y(x)\right) \tag{179}$$

$$\int_{-\infty}^{0} dx \lim_{y\to 0} T_y(x) = \int_{-\infty}^{0} dx \frac{c_1}{\sqrt{R_2(x-y_+)(x-y_-)}(x-x_+)} + \frac{c_2}{\sqrt{R_2(x-y_+)(x-y_-)}(x-x_-)}\Bigg|_{y=0} \tag{180}$$

$$\begin{aligned} &= c_1 F_{\text{int}}(0,y_+,y_-,x_+) + c_2 F_{\text{int}}(0,y_+,y_-,x_-) \\ &+ \lim_{x\to -\infty} \left(c_1 F_{\text{int}}(x,y_+,y_-,x_+) + c_2 F_{\text{int}}(x,y_+,y_-,x_-)\right)|_{y=0} \end{aligned} \tag{181}$$

$$\int_{-\infty}^{0} dx \lim_{y\to -\infty} T_y(x) = \int_{-\infty}^{0} dx \frac{-4i|k_2|}{S_2(x-x_+)(x-x_-)} \tag{182}$$

$$= \frac{-4i|k_2|}{S_2} \frac{\log(-x_+) - \log(-x_-)}{x_+ - x_-} \tag{183}$$

where A_y, y_1, y_2 are defined in Eq. (144), c_1, c_2, R_2, S_2 are given in section 4.3.1 evaluated at $y=0$. We also define the roots y_+, y_-, x_+, x_-,

$$y_+ = x_1 = \frac{1}{2}\left(1+m_1-m_3+\sqrt{(1+m_1-m_3)^2+4m_3}\right) \tag{184}$$

$$y_- = x_2 = \frac{1}{2}\left(1+m_1-m_3-\sqrt{(1+m_1-m_3)^2+4m_3}\right) \tag{185}$$

$$x_\pm = \frac{k_2^4 - (k_1-k_3)(k_1+k_3)(M_2-M_3) + k_2^2(-k_1^2-k_3^2-2M_1+M_2+M_3) \pm 2\sqrt{R}}{S_2} \tag{186}$$

$$S_2 = (k_1-k_2-k_3)(k_1+k_2-k_3)(k_1-k_2+k_3)(k_1+k_2+k_3) \tag{187}$$

$$\begin{aligned} R =& -k_2^2(k_2^4 M_1 + k_3^4 M_2 - k_2^2(k_3^2(M_1+M_2) + (M_1-M_2)(M_1-M_3)) \\ &+ k_3^2(M_1-M_2)(M_2-M_3) + k_1^4 M_3 - k_1^2((M_1-M_3)(M_2-M_3) + k_2^2(k_3^2+M_1+M_3) + k_3^2(M_2+M_3))) \end{aligned} \tag{188}$$

In the region $0<x<1, y>1$,

$$\begin{aligned} \int_0^1 dx \int_1^{+\infty} dy \frac{1-x}{|A_y|^{3/2}(y_1-y)^{3/2}(y-y_2)^{3/2}} =& -\left(c_1 F_{\text{int}}(x,y_+,y_-,x_+) + c_2 F_{\text{int}}(x,y_+,y_-,x_+)\right)|_{x=0}^{x=1} \\ &+ \frac{4i|k_2|}{S_2} \frac{\log(x-x_+) - \log(x-x_-)}{x_+ - x_-}|_{x=0}^{x=1} \end{aligned} \tag{189}$$

4.4 Note on $s(a,b)$

In the previous section, we evaluate expressions of the form $s(x_1-x, x-x_0)$. In the regions of interest, i.e. $x>0||x<0$, we may take $x\to -\infty||\infty$ to further simplify the expression. Consider,

$$\lim_{x\to\pm\infty} s(x_1-x, x-x_0) = \lim_{x\to\pm\infty} (-1)^{\theta(-\operatorname{Im}(x_1-x))\theta(-\operatorname{Im}(x-x_0))\theta(\operatorname{Im}((x_1-x)(x-x_0)))}(-1)^{\theta(\operatorname{Im}(x_1-x))\theta(\operatorname{Im}(x-x_0))\theta(-\operatorname{Im}((x_1-x)(x-x_0)))} \tag{190}$$

$$= \lim_{x\to\pm\infty} (-1)^{\theta(-\operatorname{Im} x_1)\theta(\operatorname{Im}(x_0))\theta(\operatorname{Im}(-x_1x_0+x_1x+xx_0-x^2))}(-1)^{\theta(\operatorname{Im} x_1)\theta(-\operatorname{Im} x_0)\theta(-\operatorname{Im}(-x_1x_0+x_1x+xx_0-x^2))} \tag{191}$$

$$= \lim_{x\to\pm\infty} (-1)^{\theta(-\operatorname{Im} x_1)\theta(x_0)\theta(x(\operatorname{Im} x_1+\operatorname{Im} x_0))}(-1)^{\theta(\operatorname{Im} x_1)\theta(-\operatorname{Im} x_0)\theta(-x(\operatorname{Im} x_1+\operatorname{Im} x_0))}. \tag{192}$$

The above expression takes on different values depending on $\operatorname{sign}(\operatorname{Im} x_1)$, $\operatorname{sign}(\operatorname{Im} x_0)$. There are three cases:

- Case 1: $\operatorname{sign}(\operatorname{Im} x_1) = \operatorname{sign}(\operatorname{Im} x_0) \Rightarrow \lim_{x\to\pm\infty} s(x_1 - x, x - x_0) = 1$
- Case 2: $\operatorname{sign}(\operatorname{Im} x_1) > 0, \operatorname{sign}(\operatorname{Im} x_0) < 0 \Rightarrow \lim_{x\to\pm\infty} s(x_1 - x, x - x_0) = (-1)^{\theta(\mp(\operatorname{Im} x_1 - |\operatorname{Im} x_0|))}$
- Case 3: $\operatorname{sign}(\operatorname{Im} x_1) < 0, \operatorname{sign}(\operatorname{Im} x_0) > 0 \Rightarrow \lim_{x\to\pm\infty} s(x_1 - x, x - x_0) = (-1)^{\theta(\pm(\operatorname{Im} x_0 - |\operatorname{Im} x_1|))}$

In Eq. (153), we investigate $s(x_1 - x, x - x_2)|_{x=1} \stackrel{?}{=} s(x_1 - x, x - x_2)|_{x=0}$. However, we can equivalently investigate $s(x_1 - x, x - x_2)|_{x=\infty} \stackrel{?}{=} s(x_1 - x, x - x_2)|_{x=-\infty}$ as $s(x_1 - 1, 1 - x_2) = s(x_1 - \infty, \infty - x_2)$ and $s(x_1, -x_2) = s(x_1 + \infty, -\infty - x_2)$. Using our above cases, we find,

$$s(x_1 - 1, 1 - x_2) = \begin{cases} s(x_1, -x_2) = 1, & \text{if } \operatorname{sign}(\operatorname{Im} x_1) = \operatorname{sign}(\operatorname{Im} x_2) \\ -s(x_1, -x_2), & \text{otherwise} \end{cases}$$

Hence, Eq. (154) is valid if $\operatorname{sign}(\operatorname{Im} x_1) = \operatorname{sign}(\operatorname{Im} x_2)$.

Let us now investigate the conditions under which $\operatorname{sign}(\operatorname{Im} x_1) \neq \operatorname{sign}(\operatorname{Im} x_2)$ such that $s(x_1 - 1, 1 - x_2) = -s(x_1, -x_2)$. $x_1(y = 0)$ and $x_2(y = 0)$ have the following form,

$$x_1 = \frac{1}{2}\left(1 + m_1 - m_3 + \sqrt{(1 + m_1 - m_3)^2 + 4m_3}\right) \tag{193}$$

$$x_2 = \frac{1}{2}\left(1 + m_1 - m_3 - \sqrt{(1 + m_1 - m_3)^2 + 4m_3}\right) \tag{194}$$

where $m_i = M_i/k_3^2$. Thus, the condition for the the imaginary part to change signs is,

$$\operatorname{Im} m_1 - \operatorname{Im} m_3 < |\operatorname{Im} \sqrt{(1 + m_1 - m_3)^2 + 4m_3}|. \tag{195}$$

Similarly, the condition for $\operatorname{sign}(\operatorname{Im} y_1) \neq \operatorname{sign}(\operatorname{Im} y_2)$ such that $s(y_1 - 1, 1 - x_2) = -s(y_1, -y_1)$ is,

$$\operatorname{Im} m_2 - \operatorname{Im} m_3 < |\operatorname{Im} \sqrt{(1 + m_2 - m_3)^2 + 4m_3}|. \tag{196}$$

where $m_i = M_i/k_2^2$.

4.5 Alternative way to calculate triangle integral with opposite imaginary masses

Recall from Eq. (142), in the case of M_3 having negative imaginary mass, the regions of integration are split into 3. We can compute these integrals taking their respective integration limits,

$$I_1 = \frac{\sqrt{\pi}}{2}\int_{-\infty}^{0} dx \int_{-\infty}^{0} dy (1 - x)\tilde{I}^{-3/2} \tag{197}$$

$$I_2 = \frac{\sqrt{\pi}}{2}\int_{1}^{+\infty} dx \int_{-\infty}^{0} dy (1 - x)\tilde{I}^{-3/2} \tag{198}$$

$$I_3 = \frac{\sqrt{\pi}}{2}\int_{0}^{1} dx \int_{1}^{+\infty} dy (1 - x)\tilde{I}^{-3/2}. \tag{199}$$

Let us now examine the first region of integration I_1,

$$I_1 = \frac{\sqrt{\pi}}{2}\int_{-\infty}^{0} dx \lim_{y\to 0} T_y(x) - \lim_{y\to-\infty} T_y(x) \tag{200}$$

$$= \frac{\sqrt{\pi}}{2}\left(c_1 F_{\text{int}}(x, y_+, y_-, x_+) + c_2 F_{\text{int}}(x, y_+, y_-, x_-)\right)|_{x=-\infty}^{x=0}|_{y=0} - \frac{\log(x_+) - \log(x_-)}{x_+ - x_-}\frac{2\sqrt{\pi}|k_2| i}{S_2} \tag{201}$$

where we have,

$$\lim_{x\to-\infty} F_{\text{int}}(x, y_+, y_-, x_\pm) = \frac{2\arctan\left[\frac{\sqrt{x_\pm - y_+}}{\sqrt{-x_\pm + y_-}}\right] i}{|k_2|\sqrt{y_- - x_\pm}\sqrt{x_\pm - y_+}}. \tag{202}$$

However, one needs to be careful of branch cut crossings in the region $x < 0$ and we will add the contribution of the branch cut the same way we do in the case of all imaginary masses having the same sign. We can change the form of the log term and will be relevant later for cancellation of log terms originating from the other two regions of integration,

$$\begin{aligned} \log(x) &= \log|x| + i\mathrm{Arg}(x) && (203)\\ \log(-x) &= \log|x| + i\mathrm{Arg}(-x) && (204)\\ \log(x) &= \log(-x) + i\mathrm{Arg}(x) - i\mathrm{Arg}(-x). && (205) \end{aligned}$$

This leads to the condition,

$$\log(x) = \begin{cases} \log(-x) + \pi i, & \text{if } \mathrm{Im}(x) > 0 \\ \log(-x) - \pi i, & \text{if } \mathrm{Im}(x) < 0 \end{cases}$$

and hence,

$$\log(x_+) - \log(x_-) = \begin{cases} \log(-x_+) - \log(-x_-), & \text{if } \mathrm{sign}(\mathrm{Im}\, x_+) = \mathrm{sign}(\mathrm{Im}\, x_-) \\ \log(-x_+) - \log(-x_-) + 2\mathrm{sign}(\mathrm{Im}\, x_+)\pi i, & \text{if } \mathrm{sign}(\mathrm{Im}\, x_+) \neq \mathrm{sign}(\mathrm{Im}\, x_-) \end{cases}$$

therefore, Eq. (200) becomes,

$$\frac{\sqrt{\pi}}{2}(c_1 F_{\mathrm{int}}(x, y_+, y_-, x_+) + c_2 F_{\mathrm{int}}(x, y_+, y_-, x_-))|_{x=-\infty}^{x=0}|_{y=0} - \frac{\log(-x_+) - \log(x_-) + 2\mathrm{sign}(\mathrm{Im}(x_+))\pi i \theta(-\mathrm{Im}(x_+)\,\mathrm{Im}(x_-))}{x_+ - x_-}\frac{2\sqrt{\pi}|k}{S_2} \quad (206)$$

The second region of integration I_2 is,

$$\begin{aligned} I_2 &= \frac{\sqrt{\pi}}{2}\int_1^{+\infty} dx \lim_{y\to 0} T_y(x) - \lim_{y\to -\infty} T_y(x) && (207)\\ &= \frac{\sqrt{\pi}}{2}\left(c_1 F_{\mathrm{int}}(x, y_+, y_-, x_+) + c_2 F_{\mathrm{int}}(x, y_+, y_-, x_-)\right)|_{x=1}^{x=+\infty}|_{y=0} + \frac{\log(1 - x_+) - \log(1 - x_-)}{x_+ - x_-}\frac{2\sqrt{\pi}|k_2|i}{S_2} && (208) \end{aligned}$$

where we have,

$$\lim_{x\to\infty} F_{\mathrm{int}}(x, y_+, y_-, x_\pm) = \frac{2\arctan\left[\frac{\sqrt{x_\pm - y_+}}{\sqrt{-x_\pm + y_-}}\right] i}{|k_2|\sqrt{-x_\pm + y_-}\sqrt{x_\pm - y_+}} \quad (209)$$

and again we evaluate the branch cut crossings in the region $x > 1$.

Lastly, the region I_3,

$$\begin{aligned} I_3 &= \frac{\sqrt{\pi}}{2}\int_0^1 dx \lim_{y\to +\infty} T_y(x) - \lim_{y\to 1} T_y(x) && (210)\\ &= \frac{\log(1 - x_+) - \log(1 - x_-) - \log(-x_+) + \log(-x_-)}{x_+ - x_-}\frac{2\sqrt{\pi}|k_2|i}{S_2} - \frac{\sqrt{\pi}}{2}\left(c_1 F_{\mathrm{int}}(x, y_+, y_-, x_+) + c_2 F_{\mathrm{int}}(x, y_+, y_-, x_-)\right)|_{x=0}^{x=1}|_{y=1} && (211) \end{aligned}$$

and we evaluate the branch cut crossing in the region $0 < x < 1$.

Thus, the triangle integral is the,

$$
\begin{aligned}
T &= I_1 + I_2 - I_3 && (212)\\
&= \frac{\sqrt{\pi}}{2}\left(c_1 F_{\text{int}}(x, y_+, y_-, x_+) + c_2 F_{\text{int}}(x, y_+, y_-, x_+)\right)|_{x=0,y=1}^{x=1} \\
&- \frac{\sqrt{\pi}}{2}\left(c_1 F_{\text{int}}(x, y_+, y_-, x_+) + c_2 F_{\text{int}}(x, y_+, y_-, x_+)\right)|_{x=0,y=0}^{x=1} + \frac{4\text{sign}(\operatorname{Im} x_+)\pi\sqrt{\pi}|k_2|\theta(-\operatorname{Im} x_+ \operatorname{Im} x_-)}{S_2(x_+ - x_-)} && (213)\\
&= \frac{\sqrt{\pi}}{2}\Big(P(a, y_+, y_-)(c_1 \tilde{F}_{\text{int}}(x, y_+, y_-, x_+) + c_2 \tilde{F}_{\text{int}}(x, y_+, y_-, x_-))\Big)|_{x=0,y=1}^{x=1} \\
&- \frac{\sqrt{\pi}}{2}\Big(s(y_+ - 1, 1 - y_+)P(a, y_+, y_-)(c_1 \tilde{F}_{\text{int}}(1, y_+, y_-, x_+) + c_2 \tilde{F}_{\text{int}}(1, y_+, y_-, x_-)) \\
&- P(a, y_+, y_-)(c_1 \tilde{F}_{\text{int}}(0, y_+, y_-, x_+) + c_2 \tilde{F}_{\text{int}}(0, y_+, y_-, x_-))\Big)|_{y=0} + \frac{4\text{sign}(\operatorname{Im} x_+)\pi\sqrt{\pi}|k_2|\theta(-\operatorname{Im} x_+ \operatorname{Im} x_-)}{S_2(x_+ - x_-)} && (214)
\end{aligned}
$$

where one should also include contributions from branch cut crossings in $-\infty < x < \infty$.

4.6 Derivation of F_{int}

In this section, we derive a closed form expression for the function F_{int} defined in Eq. (106), that we rewrite here for convenience:

$$F_{\text{int}}(R_2, y_+, y_-, x_0) = \frac{\sqrt{\pi}}{2}\int_0^1 dx \frac{1}{\sqrt{R_2(x - y_+)(x - y_-)(x - x_0)}}\,, \tag{215}$$

where R_2 is a negative real, y_+, y_-, and x_0 are in general complex numbers.

This integral only makes sense if the square root in the integrand does not cross any branch cut. Thus, we will separate the square root using our formula.

$$F_{\text{int}}(R_2, y_+, y_-, x_0) = \frac{\sqrt{\pi}}{2}\int_0^1 dx \frac{s(y_+ - x, x - y_-)}{\sqrt{|R_2|}\sqrt{(y_+ - x)}\sqrt{(x - y_-)(x - x_0)}}\,. \tag{216}$$

Since there are no branch cut crossings by hypothesis, $s(y_+ - x, x - y_-)$ is constant which means we can take $s(y_+ - x, x - y_-) = s(y_+, -y_-)$. Integrating yields:

$$F_{\text{int}}(R_2, y_+, y_-, x_0) = s(y_+, -y_-)\frac{\sqrt{\pi}}{\sqrt{|R_2|}} \left.\frac{\arctan\left(\frac{\sqrt{y_+ - x}\sqrt{x_0 - y_-}}{\sqrt{x_0 - y_+}\sqrt{x - y_-}}\right)}{\sqrt{x_0 - y_+}\sqrt{x_0 - y_-}}\right|_{x=0}^{x=1}. \tag{217}$$

This would be the final result if the function arctan did not have any branch cuts. However, it has two, both in the imaginary axis. The first goes from i to $+i\infty$ and the second goes from $-i$ to $-i\infty$. The jump works as follows:

$$
\begin{aligned}
\lim_{\epsilon\to 0} \arctan(x\,i) - \arctan(x\,i - \epsilon) &= \pi\,,\ |x| > 1\,,\\
\lim_{\epsilon\to 0} \arctan(x\,i + \epsilon) - \arctan(x\,i - \epsilon) &= \frac{\pi}{2}\,,\ |x| = 1\,.
\end{aligned} \tag{218}
$$

So we need to find when the argument of the arctan intersects a branch cut. Let us define, for complex z,

$$A(z, y_+, y_-, x_0) \equiv \frac{\sqrt{y_+ - z}\sqrt{x_0 - y_-}}{\sqrt{x_0 - y_+}\sqrt{z - y_-}}\,. \tag{219}$$

If there is a branch cut crossing, then we have $A^2 \leq -1$, which means $\arg(A^2) = \pi$ and $|A^2| \geq 1$.

Let us first focus on $\arg(A^2) = \pi$ and $|A^2| > 1$.

These two conditions reduce to

$$\arg\frac{(y_+ - z)(x_0 - y_-)}{(x_0 - y_+)(z - y_-)} = \pi\ [2\pi] \tag{220}$$

$$\Rightarrow \arg\left(\frac{z - y_+}{z - y_-}\right) = \arg\left(\frac{x_0 - y_-}{x_0 - y_+}\right)\,, \tag{221}$$

and

$$\left|\frac{(y_+ - z)(x_0 - y_-)}{(x_0 - y_+)(z - y_-)}\right| > 1 \tag{222}$$

$$\Rightarrow \frac{|z - y_+|}{|z - y_-|} > \frac{|x_0 - y_+|}{|x_0 - y_-|} \,. \tag{223}$$

The first condition describes in the complex plane an arc of the circle defined by the points y_+, y_-, and x_0. The ends of the arc are y_+ and y_-, and the arc includes x_0. The second condition describes a region delimited by the circle described by $\frac{|z-y_+|}{|z-y_-|} = \frac{|x_0-y_+|}{|x_0-y_-|}$, which is the Apollonius definition of a circle, with foci y_+ and y_-, that passes by x_0. So by finding the intersection of the circle defined by y_+, y_-, and x_0 with the line segment [0,1] in the real line (and guaranteeing that the second condition is satisfied), we get the branch cut crossings.

A simpler way would be to define $B \equiv A^2 + 1$, and note that the conditions for the branch cut crossings are equivalent to $B < 0 \Leftrightarrow \arg(B) = \pi$. This way, we only have one condition to check, which is numerically more efficient. B is given by

$$B = \frac{(x_0 - z)(y_+ - y_-)}{(x_0 - y_+)(z - y_-)} \,, \tag{224}$$

so that

$$\arg(B) = \pi \Leftrightarrow \arg\left(\frac{z - x_0}{z - y_-}\right) = \pi + \arg\left(\frac{y_+ - x_0}{y_+ - y_-}\right) \; [2\pi] \,, \tag{225}$$

which describes the arc of the circle defined by y_+, y_-, and x_0, that ends in x_0 and y_- and does NOT include y_+.

So, to calculate the possible intersection points with the [0,1] segment, we obtain the equation of the circle, using the following expression

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ |y_+|^2 & \operatorname{Re} y_+ & \operatorname{Im} y_+ & 1 \\ |y_-|^2 & \operatorname{Re} y_- & \operatorname{Im} y_- & 1 \\ |x_0|^2 & \operatorname{Re} x_0 & \operatorname{Im} x_0 & 1 \end{vmatrix} = 0 \,. \tag{226}$$

Then, we set $y = 0$ and get a quadratic equation for x, $ax^2 + bx + c = 0$, where

$$a = \operatorname{Im}(y_-)(\operatorname{Re}(y_+) - \operatorname{Re}(x_0)) + \operatorname{Im}(y_+)(\operatorname{Re}(x_0) - \operatorname{Re}(y_-)) + \operatorname{Im}(x_0)(\operatorname{Re}(y_-) - \operatorname{Re}(y_+)) \tag{227}$$

$$b = \operatorname{Im}(x_0)\left(\operatorname{Im}(y_+)^2 - \operatorname{Im}(y_-)^2 + \operatorname{Re}(y_+)^2 - \operatorname{Re}(y_-)^2\right) + \tag{228}$$

$$\operatorname{Im}(y_-)\left(\operatorname{Re}(x_0)^2 - \operatorname{Re}(y_+)^2\right) + \operatorname{Im}(y_+)\left(\operatorname{Im}(y_-)^2 - \operatorname{Re}(x_0)^2 + \operatorname{Re}(y_-)^2\right) + \operatorname{Im}(x_0)^2(\operatorname{Im}(y_-) \tag{229}$$

$$- \operatorname{Im}(y_+)) - \operatorname{Im}(y_+)^2 \operatorname{Im}(y_-) \tag{230}$$

$$c = \operatorname{Im}(y_+)^2(\operatorname{Im}(y_-)\operatorname{Re}(x_0) - \operatorname{Im}(x_0)\operatorname{Re}(y_-)) + \operatorname{Im}(y_+)\left(\operatorname{Re}(y_-)\left(\operatorname{Im}(x_0)^2 - \operatorname{Re}(x_0)\operatorname{Re}(y_-) + \operatorname{Re}(x_0)^2\right) - \right. \tag{231}$$

$$\left.\operatorname{Im}(y_-)^2\operatorname{Re}(x_0)\right) + \operatorname{Re}(y_+)(\operatorname{Im}(x_0)(\operatorname{Im}(y_-)^2 - \operatorname{Re}(y_+)\operatorname{Re}(y_-) \tag{232}$$

$$+ \operatorname{Re}(y_-)^2) + \operatorname{Im}(y_-)\operatorname{Re}(x_0)(\operatorname{Re}(y_+) - \operatorname{Re}(x_0)) - \operatorname{Im}(x_0)^2\operatorname{Im}(y_-)) \,. \tag{233}$$

From then, we calculate the discriminant $\Delta = b^2 - 4ac$. If $\Delta \leq 0$, there are no branch cut crossings for real x. If $\Delta > 0$, there are two crossings x_1 and x_2 between the full circle and the real line. Then, we check if x_1 and x_2 are between 0 and 1. If x_1 and/or x_2 lie in [0,1], then we check Eq. (225) to verify that the point belongs to the correct arc. Now, after this we can have 3 cases:

- No branch cut crossings: then we just use Eq. (217) to evaluate F_{int}.
- 1 branch cut crossing: we add/subtract π to the arctan in Eq. (217), if the sign of $\operatorname{Re}\frac{dA}{dx}$ at the crossing is negative/positive, respectively.
- 2 branch cut crossings: in this case, we always return to the original branch of the arctan, so there is no need to add anything to Eq. (217).

Finally, there is still the case $B = 0$, which holds only for $z = x_0$. This means that if $0 < x_0 < 1$, then we have an extra branch cut crossing (that only adds/subtracts $\pi/2$ instead of π, depending on the sign of $\operatorname{Re}\frac{dA}{dx}$).

4.7 $s(y_+, -y_-)$ function in $F_{\rm int}$

In the expression for $F_{\rm int}$ in Eq. (217), there is the factor $s(y_+, -y_-)$. This factor is undefined if either y_+ and/or y_- vanish. In that case, we use Eq. (216) to set the value of s, taking any $x > 0$. For example, when $y_+ = 0$, and $y_- \neq 0$, we get $s(-x, -y_-)$. When $y_+ = y_- = 0$, we obtain $s(-x, x) = 1$.

4.8 Limiting values for arctan

Now that P is well defined, let us look at $\tilde{F}_{\rm int} \equiv \frac{2\arctan\left(\frac{\sqrt{x-y_1}\sqrt{x_0-y_2}}{\sqrt{x-y_2}\sqrt{y_1-x_0}}\right)}{\sqrt{y_1-x_0}\sqrt{x_0-y_2}}$. The issues that we are focusing on are:

1. When the result is indeterminate;
2. When we cross a branch cut.

The result can be indeterminate in 2 situations: when $x = y_2$ and when $x_0 = y_2$. Regarding the first case, we know that x runs from 0 to 1, and we know that, for a general complex z:

$$\lim_{X\to 0^+} \arctan\left(\frac{z}{X}\right) = \frac{\sqrt{z^2}\pi}{2z} \tag{234}$$

$$\lim_{X\to 0^+} \arctan\left(\frac{z}{iX}\right) = i\frac{\sqrt{-z^2}\pi}{2z}\,. \tag{235}$$

The problematic values of y_2 are 0 and 1 because they are the values at which we calculate the antiderivative.

When $y_2 = 0$, we have for a general function f, $\lim_{x\to 0^+} f(\sqrt{x-y_2}) = \lim_{X\to 0^+} f(X)$. When $y_2 = 1$, we have $\lim_{x\to 1^-} f(\sqrt{x-y_2}) = \lim_{X\to 0^+} f(iX)$. Therefore when $y_2 = 0$ we use Eq. (234) with $z = \frac{\sqrt{-y_1}\sqrt{x_0}}{\sqrt{y_1-x_0}}$, giving

$$\tilde{F}_{\rm int} = \sqrt{\left(\frac{\sqrt{-y_1}\sqrt{x_0}}{\sqrt{y_1-x_0}}\right)^2}\frac{\pi}{2}\frac{\sqrt{y_1-x_0}}{\sqrt{-y_1}\sqrt{x_0}} = \sqrt{\frac{-y_1\,x_0}{y_1-x_0}}\frac{\pi}{2}\frac{\sqrt{y_1-x_0}}{\sqrt{-y_1}\sqrt{x_0}}\,, \tag{236}$$

and this last expression can be simplified using the generalized product rule for square roots.

and when $y_2 = 1$ we use Eq. (235), with the same z.

When $x_0 = y_2$ we just use the Taylor approximation for arctan for a small argument z: $\arctan(z) \approx z$.