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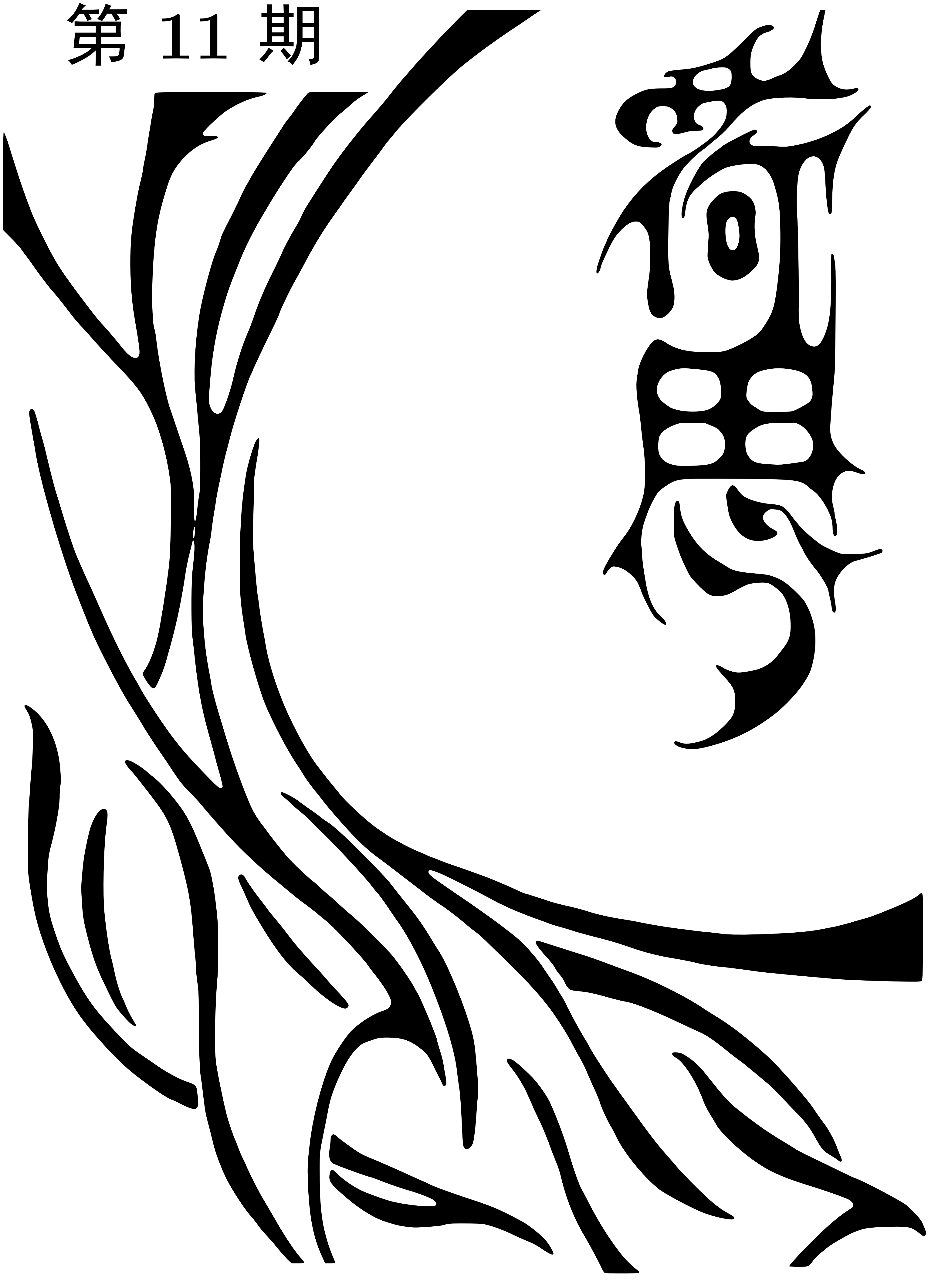

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研究探讨

专题介绍

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Study Note of "Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables"

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1 Chapter 1

The first part of my studying note was written about Chapter 2.1 of A.Majda's book "Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables", focusing on the local existence of smooth solutions for the general system of conservation laws, $\frac{\partial u}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = S(u, x, t)$. I set $S = 0$ here for simplicity in exposition, since all the proofs given below remain valid in the general case.

1.1 Main Theorem

Theorem 1.1. *Assume* $u_0 \in H^s, s > N/2 + 1$ *and* $u_0(x) \in G$, $\overline{G_1} \subset\subset G$, *Then there is a time interval* $[0, T]$ *with* $T > 0$, *so that*

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = 0, \tag{1.1}$$

$$u(x, 0) = u_0(x) \tag{1.2}$$

have a unique classical solution $u(x,t) \in C^1(R^N \times [0,T])$. *Furthermore,*$u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *and* T *depends on* $\| u_0 \|_s$ *and* G_1, *i.e.* $T(\| u_0 \|_s, G_1)$

We now separate the proof of Theorem 1.1 into two parts, Theorem 1.2 and Theorem 1.3.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, there is a unique classical solution* $u \in C^1([0,T] \times R^N)$*to the equations in (1.1) with* $u \in L^\infty([0.T], H^s) \cap C_w([0,T], H^s) \cap Lip([0,T], H^{s-1})$

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Theorem 1.3. *Any classical solution of (1.1) with $u(x,t) \in \overline{G_2} \subset\subset G$ satisfying the regularity stated in the conclusion of Theorem 1.2 on some interval $[0,T]$ satisfies the additional regularity $u \in C([0,T],H^s) \cap C^1([0,T],H^{s-1})$*

We first begin the proof of Theorem 1.2

1.2 Proof of Theorem 1.2

First, we smooth the initial data to avoid technical difficulties regarding the smoothness of the coefficients in the associated linearized problems for this iteration scheme.

We choose $j(x) \in C_0^\infty(R^N)$, $suppj \subseteq \{x| \mid x \mid\leq 1\}$,$j \geq 0$, $\int j = 1$, set $j_\epsilon = \epsilon^{-N} j(x/\epsilon)$ we define $J_\epsilon u \in C^\infty(R^N) \cap H^s(R^N)$ by

$$J_\epsilon u(x) = \int_{R^N} j_\epsilon(x-y)u(y)dy \tag{1.3}$$

we have properties:

$$\text{For } u \in H^s \parallel J_\epsilon u - u \parallel_s \to 0 \qquad \text{as } \epsilon \to 0. \tag{1.4}$$

$$\text{For } u \in H^1 \text{ and } \epsilon \leq \epsilon_0, \parallel J_\epsilon u - u \parallel_0 \leq C\epsilon \parallel u \parallel \tag{1.5}$$

set $\epsilon_k = 2^{-k}\epsilon_0$, $k = 0,1,2,3,\ldots$ and set $u_0^k(x) = J_\epsilon u_0$, $k = 0,1,2,\ldots$, where $\epsilon_0 > 0$ will be chosen later.

Differentiate the nonlinear terms in (1.2)and get

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \tilde{A}_j(u)\frac{\partial u}{\partial x_j} = 0 \tag{1.6}$$

apply the symmetrizing matrix $A_0(u)$, which satisfies

$$CI \leq A_0(u) \leq C^{-1}I \tag{1.7}$$
$$A_0(u) = A_0(u)^* \tag{1.8}$$

with a constant C uniform for $u \in G_1$ with $G_1 \subset\subset G$

$$A_0\tilde{A}_j = A_j \text{ with } A_j = A_j^*, \qquad j = 1,2,\ldots,N. \tag{1.9}$$

Now it is sufficient to prove Theorem 1.2 for the quasi-linear symmetric system

$$A_0(u)\frac{\partial u}{\partial t} + \sum_{i=1}^{N} A_j(u)\frac{\partial u}{\partial x_j} = 0 \tag{1.10}$$
$$u(x,0) = u_0(x) \tag{1.11}$$

we set

$$u^0(x,t) = u_0^0(x) \tag{1.12}$$

then for $k = 0, 1, 2, 3, \ldots$, we define $u^{k+1}(x,t)$ inductively as the solution of

$$A_0(u^k)\frac{\partial u^{k+1}}{\partial t} + \sum_{i=1}^{N} A_j(u^k)\frac{\partial u^{k+1}}{\partial x_j} = 0 \tag{1.13}$$

$$u^{k+1}(x,0) = u_0^{x,0}(x) \tag{1.14}$$

However, we need to prove this iterates in (1.13) is well defined. That is, $\exists G_2$ with $\overline{G_2} \subset\subset G$ s.t.$u^k \in G_2$ for $k = 1, 2, \ldots$. From (1.4),we know that when $\epsilon \to 0$, then$\| u_0 - u_0^0 \|_s \to 0$. We suppose G_2 to be an open set satisfying$G_1 \subset G_2$,$\overline{G}_2 \subset\subset G$, since $u_0 \in G_1$, so there exists $R > 0$ s.t. $\| u - u_0^0 \|_{L^\infty} < C_s R$ implies that $u \in \overline{G_2}$, where C_s is the constant in the Sobolev embedding estimate $\| v \|_{L^\infty} \leq \| v \|_s$. So we get:

$$\| u - u_0^0 \|_s \leq R \text{ implies that } u(x) \in \overline{G_2} \tag{1.15}$$

Also when ϵ_0 is small enough, we have:

$$\| u_0 - u_0^k \|_s \leq CR/4, C \text{ is the constant in (1.6)} \tag{1.16}$$

Then at interval $[0, T_k]$, where T_k satisfies $\| u^k - u_0^0 \|_{s,T_k} \leq R$, from the discussion above we know the eqnarray of u_{k+1} is well defined on $[0, T_k]$. Actually, because u^0 is smooth, and by induction hypotheses we suppose u^k is smooth and is already known, then (1.13) is a linear equation of u^{k+1} with smooth coefficients and initial data, so we know $u^{k+1} \in C^\infty([0, T_k] \times R^N)$. It will be known from the lemma below that T_k has a positive low bound, i.e $\exists T_* > 0$ s.t $T_k \geq 0$, $k = 1, 2, \ldots$, Then if the lemma hold, we will get a series of smooth solutions $\{u^k\}_{k=0}^\infty$ on $[0, T_*) \times R^N$, which later will be proven to be convergent to u, the solution of Equation(1.10).

Lemma 1.1. *(Boundness in the High Norm) There are constants $L > 0$, $T_* > 0$, so that the solutions $u^{k+1}(x,t)$ defined in (1.10) for $k = 0, 1, 2, \ldots$. satisfy:*

$$\| u^{k+1} - u_0^0 \|_{s,T_*} \leq R \tag{1.17}$$

$$\| \frac{\partial u^{k+1}}{\partial t} \|_{s-1,T_*} \leq R \tag{1.18}$$

Proof of Lemma 1.1: we will prove this lemma inductively we set $v^{k+1} = u^{k+1} - u_0^0$ and compute v^{k+1} satisfies

$$A_0(u^k)\frac{\partial v^{k+1}}{\partial t} + \sum_{i=1}^{N} A_j(u^k)\frac{\partial v^{k+1}}{\partial x_j} = H^k \tag{1.19}$$

$$v^{k+1}(x,0) = u_0^{k+1}(x) - u_0^0(x) \tag{1.20}$$

Where

$$H^k = -\sum_{j=1}^{N} A_j(u^k)\frac{\partial U_0^0}{\partial x_j} \tag{1.21}$$

by induction hypothesis, $u^k(x,t) \in \overline{G^2}$,$(x,t) \in R^N \times [0,T_*]$ where T_* will be chosen later so as to be not related with k. For similarity we ignore the superscripts and consider $u \in C^\infty$,$v \in C^\infty$,$u(x,t) \in \overline{G_2}$,$(x,t) \in R^N \times [0,T_*]$ satisfying

$$A_0(u)\frac{\partial v}{\partial t} + \sum_{i=1}^{N} A_j(u)\frac{\partial v}{\partial x_j} = H \tag{1.22}$$

$$v(x,0) = v_0(x) \tag{1.23}$$

Denote $v_\alpha = D^\alpha v$for $\mid \alpha \mid < s$. Divide the equation by $A_0(u)$ on both hands and differentiate with respect to x αtimes.

$$D^\alpha(\frac{\partial v}{\partial t}) + \sum_{i=1}^{N} D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial v}{\partial x_j}) = D^\alpha(A_0^{-1}(u)H) \tag{1.24}$$

$$\begin{aligned}\frac{\partial v_\alpha}{\partial t} + \sum_{i=1}^{N} A_0^{-1}(u)A_j(u)\frac{\partial v_\alpha}{\partial x_j} + \sum_{i=1}^{N} D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial v}{\partial x_j}) -\\ -\sum_{i=1}^{N} A_0^{-1}(u)A_j(u)\frac{\partial v_\alpha}{\partial x_j} = D^\alpha(A_0^{-1}(u)H)\end{aligned} \tag{1.25}$$

$$A_0(u)\frac{\partial v_\alpha}{\partial t} + \sum_{i=1}^{N} A_j(u)\frac{\partial v_\alpha}{\partial x_j} = A_0(u)D^\alpha(A_0^{-1}(u)H) + F\alpha \tag{1.26}$$

$$F_\alpha = \sum_{i=1}^{N} A_0(u)[A_0^{-1}(u)A_j(u)\frac{\partial v_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial v}{\partial x_j})] \tag{1.27}$$

We assume that we can find a constant $\overline{C}$, depending only on $\overline{G_2}$, $\| u \|_s$, R, s, s.t

$$(\sum_{1\leq|\alpha|\leq s} \| F_\alpha \|_0^2)+(\sum_{|\alpha|\leq s} \| A_0 D^\alpha (A_0^{-1}H) \|_0^2)\leq \overline{C}(\overline{G_2}, \| u \|_s, R, s)(\| v \|_s^2+1), \tag{1.28}$$

Where $\| v \|_0=(\int_{R^N} | v |^2)^{\frac{1}{2}}$. Now we have control the nonlinear terms on the right hand side of (1.22), so now we can use Energy method to estimate (1.22). Denote $E_\alpha(t)=(A_0 v_\alpha, v_\alpha)=\int_{R^N} v^T A_0 v$

$$\begin{aligned}
&\frac{\partial}{\partial t}E_\alpha(t)\\
&=(A_{0t}v_\alpha, v_\alpha)+2(A_0 v_{\alpha t}, v_\alpha)\\
&=(A_{0t}v_\alpha, v_\alpha)+2(A_0(u)D^\alpha(A_0^{-1}(u)H)+F_\alpha-\sum_{j=1}^N A_j v_{\alpha x_j}, v_\alpha)\\
&=(A_{0t}v_\alpha, v_\alpha)+2(A_0(u)D^\alpha(A_0^{-1}(u)H)+F_\alpha, v_\alpha)-\sum_{j=1}^N (A_j v_\alpha, v_\alpha)_{x_j}+\sum_{j=1}^N (A_{jx_j} v_\alpha, v_\alpha)\\
&=((divA)v_\alpha, v_\alpha)+2(A_0(u)D^\alpha(A_0^{-1}(u)H)+F_\alpha, v_\alpha)\\
&\leq \| divA \|_{L^\infty} \| v_\alpha \|_0^2+((\| F_\alpha \|_0^2)+(\| A_0 D^\alpha (A_0^{-1}H) \|_0^2)) \| v_\alpha \|_0^2
\end{aligned} \tag{1.29}$$

Sum over $| \alpha |\leq s$ and consider the bound of the nonlinear part, we get

$$\frac{\partial}{\partial t}C \| v \|_s^2 \leq (\| divA \|_{L^\infty}+\overline{C}+1) \| v \|_s^2+\overline{C} \tag{1.30}$$

By Gronwall Inequality,we get

$$\| v(T) \|_s^2\leq exp(C^{-1}(\| divA \|_{L^\infty}+\overline{C}+1)T)(\| u_0^{k+1}-u_0^0 \|_s+C^{-1}\overline{C}T) \tag{1.31}$$

So

$$\begin{aligned}
&\| v \|_{s,T}=\max_{0\leq t\leq T} \| v(t) \|_s^2\\
&\leq exp(C^{-1}(\| divA \|_{L^\infty}+\overline{C}+1)T)(\| u_0^{k+1}-u_0^0 \|_s+C^{-1}\overline{C}T)
\end{aligned} \tag{1.32}$$

Choose ϵ_0 small enough, then it follows that $\| u_0^{k+1}-u_0^0 \|_s\leq R/2$. So $\| u^{k+1}-u_0^0 \|_{s,T}\leq exp(C^{-1}(\| divA \|_{L^\infty}+\overline{C}+1)T)(R/2+C^{-1}\overline{C}T)$

Let T_* satisfies $exp(C^{-1}(\| divA \|_{L^\infty}+\overline{C}+1)T)(R/2+C^{-1}\overline{C}T)<R$, then the first part of lemma 1 proved.

For the second part, on $[0,T_*]$we have

$$\begin{aligned}
&\| \frac{\partial u^{k+1}}{\partial t} \|_{s-1,T_*} \\
&\leq C\sum_{j=1}^{N} \| A_j(u^k)\frac{\partial u^{k+1}}{\partial x_j} \|_{s-1,T_*} \\
&\leq C \| u^{k+1} \|_{s,T_*} \max_{1\leq j\leq N} \| A_j \|_{L^\infty,\overline{G_2}} \\
&\leq C(\| u_0^0 \|_s +R) \max_{1\leq j\leq N} \| A_j \|_{L^\infty,\overline{G_2}}
\end{aligned} \tag{1.33}$$

So L is independent of T_*.

Lemma 1.1 proved

We also need another lemma

Lemma 1.2. *$\exists T_{**}\leq T_*$, $\alpha<1$, and $\{\beta\}_{k=1}^{\infty}$with $\sum\mid\beta\mid<\infty$,s.t. u^{k+1}satisfy $\| u^{k+1}-u^k \|^{0,T_{**}}\leq\| u^k-u^{k-1} \|^{0,T_{**}} +\beta_k$.*

Proof of Lemma 1.2:

First we get

$$A_0(u^k)\frac{\partial(u^{k+1}-u^k)}{\partial t}+\sum_{i=1}^{N}A_j(u^k)\frac{\partial(u^{k+1}-u^k)}{\partial x_j}=F^k \tag{1.34}$$

$$v^{k+1}(x,0)=u_0^{k+1}(x)-u_0^0(x) \tag{1.35}$$

where

$$F^k=-\sum_{j=1}^{N}(A_j(u^k)-A_j(u^{k-1}))\frac{\partial u^k}{\partial x_j} \tag{1.36}$$

For the same reason when we use energy estimates we can get

$$\| u^{k+1}-u^k \|_{o,T}\leq Ce^{CT}(\| u_0^{k+1}-u_0^k \|_0 +T \| F_k \|_{0,T}) \tag{1.37}$$

From lemma 1.1 and Taylor expansion we know

$$\begin{aligned}
&\| F_k \|_0 \\
&\leq \sum_{i=1}^{N}\max_{1\leq j\leq N} \| \frac{\partial}{\partial t}A_j \|_{L^\infty,\overline{G}_2} \frac{\partial u^k}{\partial x_j} \| u^k-u^{k-1} \|_0 \\
&\leq \max_{1\leq j\leq N} \| \frac{\partial}{\partial t}A_j \|_{L^\infty,\overline{G}_2}\| u_k \|_1\| u^k-u^{k-1} \|_0 \\
&\leq C \| u^k-u^{k-1} \|_0
\end{aligned} \tag{1.38}$$

$$C=\max_{1\leq j\leq N} \| \frac{\partial}{\partial t}A_j \|_{L^\infty,\overline{G}_2} (R+\| u_0^0 \|_{s,T_*}) \tag{1.39}$$

$\forall T < T_*$, we have

$$\| F_k \|_{0,T} \leq C \| u^k - u^{k-1} \|_{0,T} \tag{1.40}$$

Then we choose ϵ_0 small enough so that we can let $\| u^{k+1} - u^k \|_0 \leq C2^{-k}$. So by (1.37) and (1.38) we can choose proper $T = T_{**}$ and $\beta_k = C2^{-k}$, we get lemma 1.2 proved.

Now we get back to the proof of Theorem 1.2:

First from the lemma 1.2, by inductive process it easily follows that $\sum_{k=0}^{\infty} \| u^{k+1} - u^k \|_{0,T_{**}} < \infty$

When we consider u^k, its initial data u_0^k is in $H^s(R^N) \cap C^\infty(R^N)$. So by Energy Estimate, we can easily know that $\forall t < T_{**}$, $u^k(\bullet, t) \in H^s(R^N)$. Also from lemma 1.1 we know that $\| \frac{\partial u^k}{\partial t} \| \leq L$. So $\forall t_1 < t_2 < T_{**}$,

$$\begin{aligned}
& \| u^k(x,t_1) - u^k(x,t_2) \|^2_{H^{s-1}(R^N)} \\
&= \sum_{|\alpha| \leq s} \int_{R^N} | D^\alpha u^k(x,t_1) - D^\alpha u^k(x,t_2) |^2 \, dx \\
&= \sum_{|\alpha| \leq s} \int_{R^N} | \frac{\partial D^\alpha u^k(x,t_*)}{\partial t} |^2 \, dx (t_1 - t_2)^2 \\
&= \| \frac{\partial u^k}{\partial t} \|^2_{H^{s-1}} (t_1 - t_2)^2 \\
&\leq L^2 (t_1 - t_2)^2 \\
&\Rightarrow \| u^k(x,t_1) - u^k(x,t_2) \|_{H^{s-1}(R^N)} \leq L \, | t_1 - t_2 | \\
&\Rightarrow u^k \in Lip([0, T_{**}], H^{s-1}(R^N))
\end{aligned} \tag{1.41}$$

Especially, we get $u^k \in C([0, T_{**}], L^2(R^N))$.
By lemma 1.2 we know $\sum_{k=1}^{\infty} \| u^{k+1} - u^k \|_{0,T_{**}} < \infty$,
So $\exists M$, s.t $\forall k$, $\| u^k \|_{0,T_{**}} < M$. i.e $\forall 0 < t < T_{**}$, $\| u^k \|_0 < M$.
Also, $\forall 0 < t_1 < t_2 < T_{**}$, $\forall k$, $\| u^k(x,t_1) - u^k(x,t_2) \|_0 \leq L \, | t_1 - t_2 |$.

So by Arezola-Ascoli Theorem,$\{u^k\}$ has a subsequence convergence to some $u \in C([0, T_{**}], L^2(R^N)])$. And since u^k is a cauchy sequence in the space, so $\lim_{k \to \infty} u^k = u$ in $C([0, T_{**}], L^2(R^N)])$, i.e $\lim_{k \to \infty} \| u^k - u \|_{0,T_{**}} = 0$.

By lemma 1.1 we can assume that

$$\| u^k \|_{s,T_{**}} + \| \frac{\partial u^k}{\partial t} \|_{s-1,T_{**}} \leq C \tag{1.42}$$

$$u^k(x,t) \in \overline{G_2} \subset\subset G, (x,t) \in R^N \times [0, T_{**}] \tag{1.43}$$

and by Sobolev space interpolation inequalities

$$\| v \|_{s'} \leq C_s \| v \|_0^{1-s'/s} \| v \|_s^{s'/s} \tag{1.44}$$

Then $\| u^k - u^l \|_{s',T_{**}} \leq C(\| u^k - u^l \|_{0,T_{**}})^{1-s'/s}$.
So

$$\lim_{t\to\infty} \| u^k - u^l \|_{s',T_{**}} = 0, \forall s' < s \tag{1.45}$$

So if $s' > N/2+1$(since $s > N/2+1$,so such s' exists)
Then $\| v \|_{C^1(R^N)} \leq C \| v \|_{H^{s'}(R^N)}$.
Then $\| v \|_{C^1(R^N),T_{**}} \leq C \| v \|_{H^{s'}(R^N),T_{**}}$. Because $u^k \to u$ in $C([0,T_{**}], H^{s'}(R^N))$, then from the norm control showed above we know $u^k \to u$ in $C([0,T_{**}], C^1(R^N))$. And then, by considering

$$\frac{\partial u^{k+1}}{\partial t} = -A_0^{-1}(u^k)\sum_{j=1}^{N} A_j(u^k)\frac{\partial u^{k+1}}{\partial t} \tag{1.46}$$

We get $\frac{\partial u^k}{\partial t} \in C([0,T_{**}, C(R^N)])$ and $\frac{\partial u^k}{\partial t} \to \frac{\partial u}{\partial t}$ in$C([0,T_{**}], C(R^N))$
so $u \in C^1([0,T_{**}] \times R^N)$ is the classical solution of (1.10).

What left to be proved is $u \in C_w([0,T_{**}], H^s) \cap Lip([0,T_{**}], H^{s-1})$

The proof of the first part:
Since $u^k \to u$ in $C([0,T], H^{s'})$ for$s' < s$, $[\phi, u^k(t)] \to [\phi, u(t)]$ for any $\phi \in H^{-s'}$ uniformly on$[0,T_*]$. And since $\| u^k \|_{s,T_{**}} \leq R+ \| u_0^0 \|_s$ and $H^{-s'}$is dense in H^{-s}, $\forall \overline{\phi} \in H^{-s}$ $\exists \phi$ s.t $| \phi - \overline{\phi} |_{H^{-s}} < \frac{\epsilon}{3(R+\|u_0^0\|_s)}$
$\Rightarrow | (\overline{\phi}, u-u^k) | = | (\phi, u-u^k) + (\phi - \overline{\phi}, u-u^k) | \leq | (\phi, u-u^k) | + | (\phi-\overline{\phi})u | + | ((\phi-\overline{\phi}, u^k)) < \epsilon$ when k is big enough
$\Rightarrow [\overline{\phi}, u^k(t)] \to [\overline{\phi}, u(t)]$ uniformly on$[0,T_{**}]$ for any$\overline{\phi} \in H^{-s}$ since (ϕ, u^K) is continuous, so $u \in C_w([0,T_{**}], H^s)$

The proof of the second part: From Equation(1.41) we know $\| u^k(x,t_1) - u^k(x,t_2) \|_{H^{s-1}(R^N)} \leq L | t_1 - t_2 |$, and since $s-1 < s$, by Equation (45) we know $u^k \to u$ in $C([0,T_{**}], H^{s-1})$. So $\| u(x,t_1) - u(x,t_2) \|_{H^{s-1}(R^N)} \leq L | t_1 - t_2 |$. Then $u \in Lip([0,T_{**}], H^{s-1})$
So Theorem 1.2 get proved.
Now we define the norm $\| v \|_{s,A_0(t)}^2$ by $\| v \|_{s,A_0(t)}^2 = \sum_{|\alpha|\leq s}\int_{R^N}(D^\alpha v, A_0(t)D^\alpha v)dx$ for $0 \leq t \leq T_{**}$, $A_0(t)$ is the short-hand notation for $A_0(u(x,t))$. As a consequence of (1.42),(1.43)and(1.45), we can conclude that

$$A_0(u(x,t)) \in C([0,T_*], C(R^N)) \tag{1.47}$$

$$CI \leq A_0(U(x,t)) \leq C^{-1}I, (x,t) \in R^N \times [0, T_{**}] \tag{1.48}$$

From (1.47),(1.48),we can get two facts:

$$C \parallel v \parallel_s^2 \leq \parallel v \parallel_{s,A_0(t)}^2 \leq C^{-1} \parallel v \parallel_s^2 \tag{1.49}$$

$$\overline{\lim_{t\downarrow 0}} \parallel v \parallel_{s,A_0(t)}^2 = \overline{\lim_{t\downarrow 0}} \parallel v \parallel_{s,A_0(t)}^2 \tag{1.50}$$

for any $v(t) \in C_w([0, T_{**}], H^s)$
Now we start the proof of Theorem 1.3

1.3 Proof of Theorem 1.3

First, it is sufficient to prove that $u \in C([0,T], H^s)$ Since we soon concludes from (2.18) that $u \in C^1([0,T], H^{s-1})$.

Furthermore, we only need to prove the strong right continuity of u at t=0, since by translation we can use the same method to prove the strong continuity property of each point. Similarly we know when we take the inverse direction of time we can prove the left continuity property.

Because $CI \leq A_0(u) \leq C^{-1}I$, so $\parallel \bullet \parallel_{s,A_0(t)}$ and $\parallel \bullet \parallel_s$ are equivalent.

Moreover, considering the existing conclusion of uniform convexity in Functional Analysis, if $w_n \to w$ weakly in Hilbert space, then $w_n \to w$ strongly is equivalent to $\parallel w \parallel \geq \overline{\lim}_{n\to\infty} \parallel w_n \parallel$

So by (1.49)(1.50), we will prove the right continuity at 0 in H^s if we prove $\parallel u_0 \parallel_{s,A_0(0)}^2 \geq \overline{\lim}_{t\downarrow 0} \parallel u(t) \parallel_{s,A_0(0)}^2 \geq \overline{\lim}_{t\downarrow 0} \parallel u(t) \parallel_{s,A_0(t)}^2$

In this way we unify the norm by using (1.49)(1.50).

We need the following proposition:

Proposition 1.1. *(Moser-type Calculus Inequalities)*
(A) For $f, g \in H^s \cap L^\infty$ and $\mid \alpha \mid \leq s$

$$\parallel D^\alpha (fg) \parallel_0 \leq C_s(\mid f \mid_{L^\infty} \parallel D^s g \parallel_0 + \mid g \mid_{L^\infty} \parallel D^s f \parallel_0) \tag{1.51}$$

(B)For $f \in H^s$, $Df \in L^\infty$, $g \in H^{s-1} \cap L^\infty$ and $\mid \alpha \mid \leq s$

$$\parallel D^\alpha (fg) - f D^\alpha g \parallel_0 \leq C_s(\mid Df \mid_{L^\infty} \parallel D^{s-1} g \parallel_0 + \mid g \mid_{L^\infty} \parallel D^s f \parallel_0) \tag{1.52}$$

(C) Assume $g(u)$ is a smooth vector-valued function on G, $u(x)$ is a continuous function with $u(x) \in G_1$, $\overline{G_1} \subset\subset G$, and $u(x) \in H^s \cap L^\infty$, then for $s \leq 1$,

$$\| D^s g(u) \|_0 \leq \left| \frac{\partial g}{\partial u} \right|_{s-1,\overline{G_1}} | u |_{L^\infty}^{s-1} \| D^s u \|_0 \tag{1.53}$$

Also we need the following lemma:

Lemma 1.3. *If u is the local solution of Theorem 2 on some interval*$[0, T_{**}]$, *then is* $f(s) \in L^1[0, T_{**}]$ *s.t*

$$\begin{aligned}
&\| u(t) \|_{s,A_0(t)}^2 \\
&= \sum_{|\alpha|\leq s} \int_{R^N} (D^\alpha u, A(u(t)) D^\alpha u) dx \\
&\leq \sum_{|\alpha|\leq s} \int_{R^N} (D^\alpha u_0, A(u_0) D^\alpha u_0) dx + \int_0^t | f(s) | ds \\
&= \| u_0 \|_{s,A_0(0)}^2 + \int_0^t | f(s) | ds
\end{aligned} \tag{1.54}$$

Proof: because u^k are in $C^\infty \cap H^s$,(by Energy estimates), and by energy estimates ,for $0 \leq t \leq T_{**}$,

$$\begin{aligned}
&\frac{\partial}{\partial t} \sum_{|\alpha|\leq s} \int_{R^N} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) dx \\
&= \int_{R^N} (div A(u^k) u^{k+1}, u^{k+1}) + 2 \int_{R^N} (F_s^{k+1}, u^{k+1})
\end{aligned} \tag{1.55}$$

where

$$u_\alpha^{k+1} = D^\alpha u^{k+1}, \tag{1.56}$$

$$\begin{aligned}
&F_s^{k+1} \\
&= \int_{1\leq|\alpha|\leq s, 1\leq j\leq N} A_0(u^k)[A_0^{-1}(u^k) A_j(u^k) \frac{\partial u_\alpha^{k+1}}{\partial x_j} - D^\alpha (A_0^{-1} A_j(u^k) \frac{\partial u^{k+1}}{\partial x_j})]
\end{aligned} \tag{1.57}$$

From the high norm estimate of u^k in Lemma 1.1,and by Proposition 1.1, we know that the right hand side of (1.56) can be controlled by an integrable function$f(x)$. In fact $f(s) \in L^\infty([0, T_{**}])$. Integrating the both sides of (1.56) with respect to t, then we get
$\forall 0 \leq t \leq T_{**}$,

$$\begin{aligned}
&\sum_{|\alpha|\leq s} \int_{R^N} (D^\alpha u^{k+1}(t), A_0(u^k(t)) D^\alpha u^{k+1}(t)) \\
&\leq \sum_{|\alpha|\leq s} \int R^N \int_{R^N} (D^\alpha u_0^{k+1}, A_0(u_0^k) D^\alpha u_0^{k+1}) + \int_0^t | f(s) | ds
\end{aligned} \tag{1.58}$$

From (1.3) we know

$$\lim_{k\to\infty}\sum_{|\alpha|\le s}\int R^N\int_{R^N}(D^\alpha u_0^{k+1}, A_0(u_0^k)D^\alpha u_0^{k+1}) = \| u_0 \|^2_{s,A_0(0)} \tag{1.59}$$

Also when $k\to\infty$ from (1.45) and Sobolev embedding estimate we know

$$\max_{0\le t\le T_*} \mid A_0(u^k(t)) - A_0(u(t)) \mid_{L^\infty} \to 0 \tag{1.60}$$

Moreover from the weak convergence when $k\to\infty$, for fixed $t>0$ it follows that

$$\begin{aligned}&\overline{\lim_{k\to\infty}}(u^{k+1}(t), u^{k+1}(t))_{s,A_0(t)} \le (u(t), u(t))_{s,A_0(t)}\\ &\to \| u(t) \|^2_{s,A(t)} \le \overline{\lim_{k\to\infty}}\sum_{|\alpha|\le s}\int_{R^N}(D^\alpha u^{k+1}(t), A_0(u^k(t))D^\alpha u^{k+1}(t))\end{aligned} \tag{1.61}$$

Then (1.54) follows from (1.59) and the equation above, this complete the prove of lemma 1.3.

Now Theorem 1.3 is an immediate result of lemma 1.3. So Theorem 1.3 proved.

Now we have proved Theorem 1.2 and Theorem 1.3, so Theorem 1.1 proved.

2 Chapter 2

This part is my studying note on Chapter 2.2 , focusing on a continuation principle for smooth solutions of the hyperbolic system:

$$A_0(x)u_t + \sum_{j=1}^{N} A_j(u)u_{x_j} = 0, u(x,0) = u_0(x) \tag{2.1}$$

where

$$CI \le A_0(u(x,t)) \le C^{-1}I \tag{2.2}$$

2.1 Main Theorem

Theorem 2.1. *Assume that $u_0 \in H^s$ for some $s > \frac{N}{2}+1$. Let $T>0$ be some given time. Assume that there are fixed constants M_1, M_2 and a fixed open set G_1 with $\overline{G}_1 \subset\subset G$ (all independent of T_*) so that for any interval* *of classical existence* $[0, T_*]$, $T_* \leq T$ *for* $u(t)$ *from Theorem 1.1 in Chapter 1, the following a priori estimates are satisfied:*

$$| div\overrightarrow{A} |_{L^\infty} \leq M_1, 0 \leq t \leq T_* \tag{2.3}$$

$$| Du |_{L^\infty} \leq M_2, 0 \leq t \leq T_* \tag{2.4}$$

$$u(x,t) \in \overline{G}_1 \subset\subset G, (x,t) \in R^N \times [0, T_*] \tag{2.5}$$

Then the classical solution $u(t)$ *exists on the interval* $[0, T]$, *with* $u(t)$ *in* $C([0,T], H^s) \cap C^1([0,T], H^{s-1})$. *Furthermore,* $u(t)$ *satisfies the a priori estimate*

$$\| u \|_{s,T_*} \leq Cexp((M_1 + M_2)CT_*) \| u_0 \|_s \tag{2.6}$$

for T_* *with* $0 \leq T_* \leq T$ *and the two constants* C *in (2.6) depend only on* s *and* $\overline{G}_1$, *i.e.,* $C(s, \overline{G}_1)$.

2.2 Proof of Theorem 2.1

: In order to prove this theorem, we first prove the a priori estimates (2.6). We need the following proposition:

Proposition 2.1. *(Moser-type Calculus Inequalities)*
(A) For $f, g \in H^s \cap L^\infty$ *and* $| \alpha | \leq s$

$$\| D^\alpha (fg) \|_0 \leq C_s(| f |_{L^\infty} \| D^s g \|_0 + | g |_{L^\infty} \| D^s f \|_0) \tag{2.7}$$

(B)For $f \in H^s$, $Df \in L^\infty$, $g \in H^{s-1} \cap L^\infty$ *and* $| \alpha | \leq s$

$$\| D^\alpha (fg) - f D^\alpha g \|_0 \leq C_s(| Df |_{L^\infty} \| D^{s-1} g \|_0 + | g |_{L^\infty} \| D^s f \|_0) \tag{2.8}$$

(C) Assume $g(u)$ *is a smooth vector-valued function on* G, $u(x)$ *is a continuous function with* $u(x) \in G_1$, $\overline{G_1} \subset\subset G$, *and* $u(x) \in H^s \cap L^\infty$, *then for* $s \leq 1$,

$$\| D^s g(u) \|_0 \leq | \frac{\partial g}{\partial u} |_{s-1, \overline{G_1}} | u |^{s-1}_{L^\infty} \| D^s u \|_0 \tag{2.9}$$

2.2.1 Energy Estimate

We first take α-times derivative of both side of Equation(2.1), then we get the following equations:

$$A_0(u) \frac{\partial u_\alpha}{\partial t} + \sum_{j=1}^{N} \frac{\partial u_\alpha}{\partial x_j} = F_\alpha \tag{2.10}$$

Where

$$F_\alpha = \sum_{i=1}^{N} A_0(u)[A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j})] \tag{2.11}$$

$$\begin{aligned}
&\sum_{1\le|\alpha|\le s} \| F_\alpha \|_0 \\
&= \sum_{1\le|\alpha|\le s, 1\le j\le N} \| A_0(u)[A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j})] \|_0 \\
&\le \sum_{1\le|\alpha|\le s, 1\le j\le N} C^{-1} \| A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j}) \|_0 \\
&\le \sum_{1\le|\alpha|\le s, 1\le j\le N} C^{-1}C_s(| D(A_0^{-1}A_j) |_{L^\infty} \| D^{s-1}\frac{\partial u}{\partial x_j} \|_0 + | \frac{\partial u}{\partial x_j} |_{L^\infty} \| D^s(A_0^{-1}A_j) \|_0
\end{aligned} \tag{2.12}$$

The first "$\le$" holds because of (2.9), and the second "$\le$" holds because of Proposition 2.1(B).

By Proposition 2.1(C), we have

$$\begin{aligned}
&\| D^s(A_0^{-1}A_j(u)) \|_0 \\
&\le C_s \, | \frac{\partial(A_0^{-1}A_j(u))}{\partial u} |_{s-1,\overline{G}_1} | u |_{L^\infty}^{s-1} \| D^s u \|_0 \\
&\le \tilde{C}(s, \overline{G}_1) \| D_s u \|_0
\end{aligned} \tag{2.13}$$

We assume a priori estimates $| \frac{\partial u}{\partial x_j} |_{L^\infty} \le M_2$, then we have So

$$\sum_{1\le|\alpha|\le s} \| \partial F_\alpha \|_0 \le \tilde{C}(s, \overline{G}_1) M_2 \| D^s u \|_0 \tag{2.14}$$

Denote $E(t) = \sum_{o\le|\alpha|\le s}(A_0 v_\alpha, v_\alpha)$ So from the equation we know that

$$\frac{\partial E}{\partial t} = \sum_{o\le|\alpha|\le s} (div\overrightarrow{A} u_\alpha, u_\alpha) + 2(F_\alpha, u_\alpha) \tag{2.15}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}E(t) \\
&= \sum_{o\le|\alpha|\le s} (div\overrightarrow{A}u_\alpha, u_\alpha) + 2(F_\alpha, u_\alpha) \\
&\le M_1 \sum_{o\le|\alpha|\le s} (u_\alpha, u_\alpha) + 2(F_\alpha, u_\alpha) \\
&\le C^{-1}M_1 \sum_{o\le|\alpha|\le s} (A_0u_\alpha, u_\alpha) + 2 \parallel F_\alpha \|_0 \| u_\alpha \|_0 \\
&\le C^{-1}M_1E + 2\tilde{C}M_2 \parallel u_\alpha \|_0^2 \\
&\le C^{-1}(M_1 + 2\tilde{C})E
\end{aligned}
\tag{2.16}
$$

So by Gronwall Inequality,

$$
E(t) \le e^{C^{-1}(M_1+2\tilde{C}M_2)t} \parallel u_0 \|_0 \tag{2.17}
$$

So

$$
\sum_{o\le|\alpha|\le s} \parallel u_\alpha(t) \|^2 \le C^{-1}E(t) \le C^{-1}e^{C^{-1}(M_1+2\tilde{C}M_2)t} \parallel u_0 \|_0 \tag{2.18}
$$

So priori estimate (2.6) gets proved.

Then if the H^s classical solution u only exists on $[0, T_*)$ for some $T_* \le T$, then we apply the a priori estimate (2.6), and use the local existence theorem in Chapter 1, beginning at $T - \epsilon$, where $\epsilon > 0$ is appropriately small, then we can continue this H^s classical solution beyond T_*, contradiction. So now we get $u(t) \in C([0,T], H^s)$, which follows $Du(t) \in C([0,T], H^{s-1})$. Then use the Equations(1):$\frac{\partial u}{\partial t} = -A_0^{-1}\sum_{j=1}^N A_j \frac{\partial u}{\partial x_j}$, we can easily know that $u_t \in C([0,T], H^{s-1})$ too. So $u(t) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$

So Theorem 2.1 get proved

Theorem 2.1 has two corollaries. Here is the first one:

Corollary 2.1. *Assume $u_0 \in H^s$ for some $s > \frac{N}{2}+1$. Assume that $u(x,t)$ is a classical solution of (2.1) on some interval $[0,T]$ with $u \in C^1([0,T] \times R^N)$, then on the same time interval $[0,T]$ necessarily $u \in C([0,T], H^s)$. In particular, if $u_0 \in \cap_s H^s$, on any interval $[0,T]$, where u belongs to $C([0,T], H^{s_0})$ for some s_0, $s_0 > \frac{N}{2} + 1$, automatically u is a function in $C^\infty([0,T] \times R^N)$.*

2.3 Proof of Corollary 2.1

If $u \in C^1([0,T] \times R^N)$, then the a priori estimates (2.3),(2.4),(2.5) satisfies automatically. Also since $u_0 \in H^s$, and a priori estimate (2.6), we know $u \in C([0,T], H^s)$. Especially when the latter situation also holds, then on interval $[0,T]$ we have $u_0 \in \cap_s H_s$ hold. Then by the a priori estimate (2.6) we have $u \in \cap_s C([0,T], H^s)$. So $u \in C([0,T], C^\infty(R^N))$ by the Sobolev

Imbedding theorem. Also the equation $\frac{\partial u}{\partial t} = -A_0^{-1}\sum_{j=1}^{N} A_j \frac{\partial u}{\partial x_j}$ implies that $u_t \in C^\infty$, so we get $u \in C^\infty([0,T] \times R^N)$.

Here is the second one:

Corollary 2.2. *Assume $u_0 \in H^s$ for some $s > \frac{N}{2} + 1$. Then $[0,T)$ with $T < \infty$ is a maximal interval of H^s existence if and only if either*

$$\begin{aligned} &\mid u_t \mid_{L^\infty} + \mid Du \mid_{L^\infty} \to \infty \text{ as } t \uparrow T, \\ &or \qquad (2.19) \\ &\text{as } t \uparrow T, u(x,t) \text{ escape every compact subset } K \subset\subset G. \qquad (2.20) \end{aligned}$$

2.4 Proof of Corollary 2.2

If we choose some $T_* < T$, then neither of situation(2.19),(2.20) holds for T_*. Then it is obvious that a priori estimates (2.4),(2.5) hold for T_*, we only need to check the a priori estimate (2.3). We assume that there exists a constant M s.t $\forall 0 \le j \le N$, $\mid \frac{\partial A_j(u)}{\partial u} \mid_{L^\infty} < M$, and a constant p s.t. $\mid u_t \mid_{L^\infty} + \mid Du \mid_{L^\infty} < P$, $\forall t \in [0, T_*]$. Then we have the following calculation:

$$(A_0)_t = \frac{\partial A_0(u)}{\partial u}\frac{\partial u}{\partial t} \tag{2.21}$$

$$(A_j)_{x_j} = \frac{\partial A_j(u)}{\partial u}\frac{\partial u}{\partial x_j} \tag{2.22}$$

Then

$$\mid div\overrightarrow{A} \mid_{L^\infty} \le MP, 0 \le t \le T_* \tag{2.23}$$

So now a priori estimate (2.3) still holds. Then by theorem 2.1 we know the H^s classical solution exists on $[0, T_*]$. So the H^s classical solution exists on the interval $[0,T)$. Also, we suppose $[0, T_0)$ is the maximum interval of H^s local existence, then either of situation (2.19),(2.20) implies $T_0 \le T$. So $[0,T)$ is the maximum interval of H^s local existence.

3 Chapter 3

This part focus on Chapter 2.3 of A.Majda's book "Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables". In Chapter 1 and Chapter 2 we have given a complete treatment of the H^s classical existence theory under the assumption that $u_0(x) \in H^s(R^N)$. However, such conditions on the initial data are not always natural since for example the initial density might be a smooth plane wave $\rho(x \bullet w)$ with $\lim_{\tilde{x} \to +\infty} \rho_0(\tilde{x}) = \rho_+$ and $\lim_{\tilde{x} \to -\infty} \rho_0(\tilde{x}) = \rho_-$ with $\rho_+, \rho_- > 0$ and $\rho_- \ne \rho_+$. The proofs in the previous two notes were based upon the global energy principle for symmetric hyperbolic system. However, it is well known that solutions of hyperbolic equations have finite propagation speed and obey a local energy principle . To take advantage of the above local energy principle and also to allow for initial data like the density in the previous paragraph, we now focus on the uniformly local Sobolev space H^s_{ul} introduced by Kato.

In this note we still focus on the hyperbolic system

$$A_0(u)u_t + \sum_{j=1}^{N} A_j(u)u_{x_j} = 0,$$
$$\text{where } u(x,t) \in G_1, \overline{G}_1 \subset\subset G, u(x,0) = u_0(x) \tag{3.1}$$

$$CI \leq A_0(u) \leq C^{-1}I \text{ for } u \in \overline{G}_1 \tag{3.2}$$

First we want to introduce the uniformly local Sobolev spaces H^s_{ul}. These spaces defined in the following fashion: let $\theta \in C^\infty_0(R^N)$ be a function so that $\theta \geq 0$ and $\theta(x) = 1$, if $\mid x \mid \leq \frac{1}{2}$; $\theta(x) = 0$, if $\mid x \mid > 1$ and define $\theta_{d,y}(x) = \theta(\frac{x-y}{d})$.

Definition 3.1. *The function u belongs to the uniformly local Sobolev spaces $u \in H^s_{ul}$, provided that there is some $d > 0$ so that*

$$\sup_{y\in R^N} \parallel \theta_{d,y}u \parallel_s = \parallel u \parallel_{s,d} < \infty \tag{3.3}$$

Proposition 3.1. *The norms $\parallel . \parallel_{s,d}$ are all equivalent norms on H^s_{ul} as d varies; in particular,*

$$\parallel u \parallel_{s,d_1} \leq C \parallel u \parallel_{s,d_2} \tag{3.4}$$

Proof For $d_1 \leq d_2$, it is obvious that $\parallel u \parallel_{s,d_1} \leq \parallel u \parallel_{s,d_2}$. What left to be proved is that there exists a constant C which only related with s, d_1 and d_2, s.t. $\parallel u \parallel_{s,d_2} \leq C(s, d_1, d_2) \parallel u \parallel_{s,d_1}$. To prove this, we only need to proved the situation when $s = 0$, i.e $\parallel u \parallel_{s,d_2} \leq C(d_1, d_2) \parallel u \parallel_{s,d_1}$. When $s > 0$ the proof is similar.

We define $Cov(d, D)$=minimum number of $\frac{d}{2}$-balls it takes to cover D-balls. We can see that $\forall d, D > 0$, $Cov(d, D) < \infty$, and is only related with d, D. Then if $\parallel u \parallel_{0,d_2} = M \geq 0$, then $\sup_{y\in R^N} \parallel \theta_{d_2,y}u \parallel_0 = M$, so $\forall \epsilon > 0$, there exists $y_0 \in R^N$ s.t

$$\parallel \theta_{d_2,y_0}u \parallel_0 > M - \epsilon \tag{3.5}$$

Take $t = Cov(d_1, d_2)$ points $z_1, \dots, z_t$, s.t. t $\frac{d_1}{2}$-balls $B(z_1, \frac{d_1}{2}), \dots, B(z_t, \frac{d_1}{2})$ cover $B(y_0, d_2)$. Then we have

$$\begin{aligned} t \parallel u \parallel_{0,d_1} &= \sum_{i=1}^{t} \parallel u \parallel_{0,d_1} \geq \sum_{i=1}^{t} \parallel \theta_{d_1,z_i} u \parallel_0 \\ &\geq \sum_{i=1}^{t} \parallel 1_{B(z_t, \frac{d_1}{2})} u \parallel_0 \geq \parallel 1_{\cap_{i=1}^{t} B(z_t, \frac{d_1}{2})} u \parallel_0 \\ &\geq \parallel 1_{B(y_0,d_2)} u \parallel_0 \geq \parallel u \parallel_{0,d_2} - \epsilon \end{aligned} \tag{3.6}$$

So let $\epsilon \to 0$, we get

$$\parallel u \parallel_{0,d_2} \leq Cov(d_1, d_2) \parallel u \parallel_{0,d_1} . \tag{3.7}$$

Proved.

Next, we formulate the local energy principle for the linear symmetric hyperbolic equation

3.1 Energy Estimate

$$A_0(u)v_t + \sum_{j=1}^{N} A_j(u)v_{x_j} = F, v(x,0) = v_0(x) \tag{3.8}$$

where $u(x,t) \in G_1, \overline{G}_1 \subset\subset G$. We recall that

$$CI \leq A_0(u) \leq C^{-1} I \tag{3.9}$$

and also, there is a number $D > 0$ so that

$$\max_{|w|=1, u \in G_1} \mid (\sum_{j=1}^{N} A_j(u) w_j v, v) \mid \leq D \mid v \mid^2 \tag{3.10}$$

Then we define the number R by $R = \frac{D}{C}$.

Now we try to make out an energy estimates by using Green's Formula. First we denote

$$\Omega(y,T) = \{(t,x) \in R^{N+1} \mid\mid x - y \mid \leq d + T - t, 0 < t < T\} \tag{3.11}$$

From the equation we get

$$(A_0 v_t, v) + (A_j v_{x_j}, v) = (F, v) \tag{3.12}$$

So integrate over $\Omega(y,T)$, and use Green's Identity, we get:

$$
\begin{aligned}
&2\int_{\Omega(y,T)}(F,v)dxdt\\
&=2\int_{\Omega(y,T)}(A_0v_t,v)+(A_jv_{x_j},v)dxdt\\
&=\int_{\Omega(y,T)}(A_0v,v)_t-(A_{0t}v,v)+(A_jv,v)_{x_j}-(A_{jx_j}v,v)dxdt\\
&=-\int_{\Omega(y,T)}(div\overrightarrow{A}v,v)dxdt+\int_{\Omega(y,T)}(A_0v,v)_t+(A_jv,v)_{x_j}dxdt\\
&=-\int_{\Omega(y,T)}(div\overrightarrow{A}v,v)dxdt+\int_{|x-y|\le d}(A_0(u)v,v)(T)dx-\int_{|x-y|\le d+RT}(A_0(u)v_0,v_0)dx\\
&+\int_0^T\int_{|x-y|=d+R(T-s)}((A_0v,v),(A_1v,v),\ldots,(A_Nv,v))\bullet\frac{1}{\sqrt{1+R^2}}(R,\frac{x_1}{\mid x\mid},\ldots,\frac{x_n}{\mid x\mid})
\end{aligned}
\tag{3.13}
$$

Here $\frac{1}{\sqrt{1+R^2}}(R,\frac{x_1}{|x|},\ldots,\frac{x_n}{|x|})$ is the unit normal vector on $S=\{(x,t)\mid\mid x-y\mid= d+R(T-t),0<t<T\}$ pointing outwards.
Then we use the bounds (3.9) and (3.10),we have

$$
\begin{aligned}
&((A_0v,v),(A_1v,v),\ldots,(A_Nv,v))\bullet(R,\frac{x_1}{\mid x\mid},\ldots,\frac{x_n}{\mid x\mid})\\
&\qquad\ge RC\mid v\mid^2-D\mid v\mid^2=0
\end{aligned}
\tag{3.14}
$$

So

$$
\begin{aligned}
&\int_{|x-y|\le d}(A_0(u)v,v)(T)dx\\
&\le\int_{|x-y|\le d+RT}(A_0(u)v_0,v_0)dx+2\int_{\Omega(y,T)}(F,v)dxdt\\
&+\int_{\Omega(y,T)}(div\overrightarrow{A}v,v)dxdt\\
&\le\int_{|x-y|\le d+RT}(A_0(u)v_0,v_0)dx\\
&+\int_0^T\int_{|x-y|\le d+R(T-s)}2(F,v)+(div\overrightarrow{A}v,v)dxdt
\end{aligned}
\tag{3.15}
$$

Now we have this energy estimates,we get back to the original equation:

$$A_0(u)u_t+\sum_{j=1}^{N}A_j(u)u_{x_j}=0,u(x,0)=u_0(x)\tag{3.16}$$

We need the following proposition:

Proposition 3.2. *(Calculus Inequalities forH^s_{ul})*
(A) For $f, g \in H^s_{ul} \cap L^\infty$ and $|\alpha| \leq s$

$$\| D^\alpha(fg) \|_{0,d} \leq C_{s,d}(| f |_{L^\infty} \| g \|_{s,2d} + | g |_{L^\infty} \| f \|_{s,2d}) \tag{3.17}$$

(B)For $f \in H^s_{ul} \cap C^1$, $g \in H^{s-1}_{ul} \cap L^\infty$ and $|\alpha| \leq s$

$$\| D^\alpha(fg) - fD^\alpha g \|_{0,d} \leq C_{s,d}(| f |_{C^1} \| g \|_{s-1,2d} + | g |_{L^\infty} \| f \|_{s,2d}) \tag{3.18}$$

(C) Assume $g(u)$ is smooth on G, $u(x)$ continuous with $u(x) \in G_1$, $\overline{G_1} \subset\subset G$, and $u(x) \in H^s_{ul}$, then

$$\| g(u) \|_{s,d} \leq C(\overline{G}_1, s, d)(1 + \| u \|_{s,2d}) \tag{3.19}$$

We first take α-times derivative of both side of Equation(3.1), then we get the following equations:

$$A_0(u)\frac{\partial u_\alpha}{\partial t} + \sum_{j=1}^{N} \frac{\partial u_\alpha}{\partial x_j} = F_\alpha \tag{3.20}$$

Where

$$F_\alpha = \sum_{i=1}^{N} A_0(u)[A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j})] \tag{3.21}$$

$$\begin{aligned}
&\sum_{1\leq|\alpha|\leq s} \| F_\alpha \|_{0,d} \\
&= \sum_{1\leq|\alpha|\leq s, 1\leq j\leq N} \| A_0(u)[A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j})] \|_{0,d} \\
&\leq \sum_{1\leq|\alpha|\leq s, 1\leq j\leq N} C^{-1} \| A_0^{-1}(u)A_j(u)\frac{\partial u_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}(u)A_j(u)\frac{\partial u}{\partial x_j}) \|_{0,d} \\
&\leq \sum_{1\leq|\alpha|\leq s, 1\leq j\leq N} C^{-1}C_{s,d}(| A_0^{-1}A_j |_{C^1} \| \frac{\partial u}{\partial x_j} \|_{s-1,2d} + | \frac{\partial u}{\partial x_j} |_{L^\infty} \| A_0^{-1}A_j \|_{s,2d})
\end{aligned} \tag{3.22}$$

The first "$\leq$" holds because of (3.9), and the second "$\leq$" holds because of Proposition 3.2(B).

By Proposition 3.2(C), we have

$$\| A_0^{-1}A_j(u) \|_{s,2d} \leq C(\overline{G}_1, s, d)(1 + \| u \|_{s,4d}) \tag{3.23}$$

We assume a priori estimates $\mid \frac{\partial u}{\partial x_j} \mid_{L^\infty} \leq M_2$, then we have

$$\sum_{1\leq|\alpha|\leq s} \parallel F_\alpha \parallel_{0,d} \leq \tilde{C}(\overline{G}_1, s, d) M_2 (1 + \parallel u \parallel_{s,4d}) \tag{3.24}$$

Because $\parallel \bullet \parallel_{s,4d}$ is equivalent to $\parallel \bullet \parallel_{s,d}$ by Proposition 1, so actually we get the following:

$$\sum_{1\leq|\alpha|\leq s} \parallel F_\alpha \parallel_{0,d} \leq \tilde{C}(\overline{G}_1, s, d) M_2 (1 + \parallel u \parallel_{s,d}) \tag{3.25}$$

Now we back to the energy estimate (2.13), we assume a priori estimate $\mid div\overrightarrow{A} \mid \leq M_1$, then from above we have we have:

$$\begin{aligned}
&\int_{|x-y|\leq d} (A_0(u)u_\alpha, v)(T)dx \\
&\leq \int_{|x-y|\leq d+RT} (A_0(u)u_{0\alpha}, u_{0\alpha})dx + \int_0^T \int_{|x-y|\leq d+R(T-s)} 2(F_\alpha, u_\alpha) + (div\overrightarrow{A}u_\alpha, u_\alpha) \\
&\leq \int_{|x-y|\leq d+RT} C^{-1} \mid u_{0,\alpha} \mid^2 dx + \int_0^T \int_{|x-y|\leq d+R(T-s)} \mid F_\alpha \mid^2 + (1+M_1) \mid u_\alpha \mid^2
\end{aligned} \tag{3.26}$$

So

$$\begin{aligned}
&C \sum_{|\alpha|\leq s} \int_{|x-y|\leq d} (u_\alpha, u_\alpha)(T)dx \\
&\leq \sum_{|\alpha|\leq s} \int_{|x-y|\leq d} (A_0(u)u_\alpha, v)(T)dx \\
&\leq \sum_{|\alpha|\leq s} \int_{|x-y|\leq d+RT} C^{-1} \mid u_{0,\alpha} \mid^2 dx + \int_0^T \sum_{|\alpha|\leq s} \int_{|x-y|\leq d+R(T-s)} \mid F_\alpha \mid^2 + (1+M_1) \mid u_\alpha \mid^2 \\
&\leq \sum_{|\alpha|\leq s} \int_{|x-y|\leq d+RT} C^{-1} \parallel u_{0,\alpha} \parallel^2 dx + \int_0^T \sum_{|\alpha|\leq s} \int_{|x-y|\leq d+R(T-s)} \mid F_\alpha \mid^2 + (1+M_1) \mid u_\alpha \mid^2 \\
&\leq C^{-1} \parallel u_0 \parallel_{s,d+RT} + \int_0^T \sum_{|\alpha|\leq s} \parallel F_\alpha \parallel_{0,d+R(T-s)} + (1+M_1) \int_0^T \parallel u \parallel_{s,d+R(T-s)}
\end{aligned} \tag{3.27}$$

By Proposition 2.1, we know

$$\begin{aligned}
&\sum_{|\alpha|\leq s} \parallel F_\alpha \parallel_{0,d+R(T-s)} \\
&\leq C(d, d+RT) \sum_{|\alpha|\leq s} \parallel F_\alpha \parallel_{0,d} \\
&\leq C(\overline{G}_1, s, d, RT) M_2 (1 + \parallel u \parallel_{s,d})
\end{aligned} \tag{3.28}$$

Take supreme of $y \in R^N$ in $\sum_{|\alpha|\leq s}\int_{|x-y|\leq d}(u_\alpha, u_\alpha)(T)dx$, we get

$$
\begin{aligned}
&C \parallel u(T) \parallel_{s,d} \\
&\leq C^{-1} \parallel u_0 \parallel_{s,d+RT} + C(\overline{G}_1, s, d, RT)M_2 \int_0^T (1 + \parallel u \parallel_{s,d}) \\
&+ (1+M_1)\int_0^T \parallel u(t) \parallel_{s,d+R(T-s)} \qquad (3.29)
\end{aligned}
$$

i.e.

$$\parallel u(T) \parallel_{s,d} \leq \tilde{C}(C, d, RT, \overline{G}_1, s)(1 + \parallel u_0 \parallel_{s,d+RT} + (M_1+M_2)\int_0^T \parallel u(t) \parallel_{s,d}) \qquad (3.30)$$

Define $E(t) = \parallel u(t) \parallel_{s,d}$, then

$$E(T) \leq \tilde{C}(1 + E(0) + M\int_0^T E(t)dt) \qquad (3.31)$$

where $M = M_1 + M_2$. So by Gronwall Inequality, we have:

$$E(T) \leq \tilde{C}(E(0)+1)e^{\tilde{C}MT} \qquad (3.32)$$

i.e.

$$\parallel u(t) \parallel_{s,d} \leq \tilde{C}(\parallel u_0 \parallel_{s,d} + 1)e^{\tilde{C}MT} \qquad (3.33)$$

Thus we get the core energy estimate. Then there is nothing different left to prove the extension of Theorem 3.1 and Theorem 3.2. We state these results:

3.2 Main Theorem

Theorem 3.1. *(Annex) Assume $u_0 \in H^s_{ul}$, $s > \frac{N}{2}+1$ and $u_0(x) \in G_1$, $\overline{G}_1 \subset\subset G$. Then there is a time interval $[0,T]$ with $T > 0$ so that the equation in (1) have a unique classical solution $u \in C^1([0,T] \times R^N)$ with $u(x,t) \in G_2, \overline{G}_2 \subset\subset G$ for $(x,t) \in R^N \times [0,T]$. Furthermore, $u(t) \in C([0,T], H^s_{loc}) \cap C^1([0,T], H^{s-1}_{loc})$ and $u(t) \in L^\infty([0,T], H^s_{ul})$; also T depends only on s and $\parallel u_0 \parallel_{s,d}$.*

Theorem 3.2. *(Annex) If $u_0 \in H^s_{ul}, s > \frac{N}{2}+1$, and for any time interval $[0,T_*]$ of local H^s_{ul} existence with $T_* \leq T$, the following three a priori estimates are satisfied:*

$$| div\overrightarrow{A} |_{L^\infty} \leq M_1, 0 \leq t \leq T_* \qquad (3.34)$$

$$| Du |_{L^\infty} \leq M_2, 0 \leq t \leq T_* \qquad (3.35)$$

$$u(x,t) \in \overline{G}_1 \subset\subset G, (x,t) \in R^N \times [0,T_*] \qquad (3.36)$$

Then the classical H^s_{ul} solution of (1) exists on the interval $[0, T]$ *with* $u(t) \in C([0,T], H^s_{loc}) \cap C^1([0,T],\, H^s_{loc}) \cap L^\infty([0,T], H^s_{ul})$. *Also, a similar a priori estimate like the one in the notes before is also valid in* H^s_{ul} *norms:*

$$\| u(t) \|_{s,d,T_*} \leq \tilde{C}(\| u_0 \|_{s,d} + 1) e^{\tilde{C}MT} \tag{3·37}$$

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一个优美恒等式的证明

郑可迪*

来看一个优美的恒等式：

$$\int_0^1 \frac{1}{x^x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n} \tag{1}$$

据说这个等式是约翰-伯努利发现的，下面用微积分的方法给出一个证明。实际上，左边的 $\frac{1}{x^x} = e^{-xlnx}$。由泰勒展开：

$$e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

马上可以得到：

$$\int_0^1 \frac{1}{x^x} dx = \int_0^1 e^{-xlnx} dx = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{(xlnx)^k}{k!} dx \tag{2}$$

设$f(x) = xlnx$，对它求导，得到：$f'(x) = lnx + 1$。易知$x = e^{-1}$ 时，$f(x)$ 取到最小值$-\frac{1}{e}$，而我们知道

$$\lim_{x \to 0^+} f(x) = f(1) = 0$$

在区间 (0,1] 上，$|f(x)|$ 的最大值为e^{-1}。设$u_k(x) = (-1)^k \frac{(xlnx)^k}{k!}$。在区间 (0,1] 上，

$$|u_k(x)| \leqslant \frac{1}{k!e^k}$$

我们找到了一个强级数。由魏尔斯特拉斯比较判别法知，函数级数$\sum\limits_{k=0}^{\infty} u_k(x)$ 在 (0,1] 上一致收敛。于是 (2) 式右端的积分可以变为级数求和：

$$\int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{(xlnx)^k}{k!} dx = \sum_{k=0}^{\infty} \int_0^1 (-1)^k \frac{(xlnx)^k}{k!} dx \tag{3}$$

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由分部积分进行递推（其实这里有点像 Wallis 公式的推导）：

$$\int_0^1 (xlnx)^k dx = \frac{1}{k+1}\int_0^1 ln^k x dx^{k+1} = \frac{1}{k+1}(x^{k+1}ln^k x)\Big|_0^1 - \frac{k}{k+1}\int_0^1 x^k ln^{k-1} x dx$$
$$= -\frac{k}{k+1}\int_0^1 x^k ln^{k-1} x dx = ... = (-1)^k \frac{k!}{(k+1)^k}\int_0^1 x^k dx = (-1)^k \frac{k!}{(k+1)^{k+1}}$$

把这个结果代入到 (3) 中，马上得到：

$$\sum_{k=0}^{\infty}\int_0^1 (-1)^k \frac{(xlnx)^k}{k!} dx = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n} \tag{4}$$

结合 (2)(3)(4) 马上可以得到 (1) 式成立。

解析流形上的李群作用

郑志伟*

摘要

该理论的局部部分由 Sophus Lie 本人发展，标志着李群李代数整套理论的开始。该理论的整体部分是很新的结果，本文会做简单的介绍。

关键词：李群和李代数，无穷小作用，局部，整体。

1　引言

我们关心的是李群 G 在解析流形 M（以下只考虑解析的流形）上的作用的无穷小刻画。记 $\mathfrak{g}$ 为 G 的李代数，$J_a(M)$ 为流形 M 上的解析向量场构成的李代数。如果 G 在 M 上作用，则 $t \longmapsto \exp(-tX) \longmapsto \exp(-tX)\cdot x$ 给出了 M 上的一族曲线，由此给出 M 上的一个向量场。记这样的映射为 $\tau : \mathfrak{g} \longrightarrow J_a(M)$。李的第一个基本结果是说这样得到的 τ 是一个李代数同态。

再考虑 M 上所有的微分同胚构成的群 $Diff(M)$，则李群作用给出 G 到 $Diff(M)$ 的一个群同态，记为 α。若把 $Diff(M)$ 近似看做一个李群，则其李代数为 $J_a(M)$，可将 τ 看作 α 的微分。所以自然地，称 τ 被 G 在 M 上的作用决定，称每一个 $\mathfrak{g}$ 到 $J_a(M)$ 的李代数同态是 G 在 M 上的无穷小作用。李的第二个基本结果是说对每个 G 在 M 上的无穷小作用，都可以局部地（某种意义下唯一的）构造 G 在 M 上的作用，恰好决定该无穷小作用。

李本人并没有考虑整体的情况。事实上，并不是对每个无穷小作用，都可以找到一个整体作用与之对应。这里就可以举一个简单的观点：设 τ 是 G 在 M 上的无穷小作用，并设其由一个 G 在 M 上的整体作用决定。取 M' 为 M 的开子流形，令 $\tau'(X) = \tau(X)|M'(X \in \mathfrak{g})$，则 τ' 无法由一个整体作用决定。

最后，文章会简单介绍该理论整体意义下的结果。

*基数 01

2 基本定义

下面一直令 $n = dim(G), m = dim(M)$.

定义 2.1. D 为 $G \times M$ 中 $1 \times M$ 的开领域，φ 为从 D 到 M 的解析映射，满足

1. $\varphi(1, x) = x$
2. $E = \{(g, h, x) \in G \times G \times M | (h, x), (gh, x), (g, \varphi(h, x)) \in D$ 且 $\varphi(gh, x) = \varphi(g, \varphi(h, x))\}$ 是 $1 \times M$ 的开领域。

则称 φ 为 G 在 M 上的一个局部作用。

如果上述定义中 $D = G \times M$ 并且 $E = G \times G \times M$，则称 φ 为 G 在 M 上的一个整体作用。

将 $\varphi(g, x)$ 简记为 $g \cdot x$。

定义 2.2. 称每一个 $\mathfrak{g}$ 到 $J_a(M)$ 的李代数同态是 G 在 M 上的无穷小作用。

定义 2.3. G 上所有左不变微分算子构成的代数称为 G 的包络代数，记为 O。$\forall g \in G$，可将 O 与 $T_g^\infty(G)$ 等同。

记 $\varphi_x : g \longmapsto g^{-1} \cdot x$ 在 G 的单位元附近定义且解析，$\forall x$。记 $\tau_\varphi^\infty(a)_x = (d\varphi_x)_1^\infty(a), \forall a \in O$，这是 M 上 x 附近的一个微分算子。如果 $a = X_1 X_2 \dots X_r$，则 $\tau_\varphi^\infty(a)_x$ 由下式决定：

$$\tau_\varphi^\infty(a)_x(f) = (\frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_r} f(exp(-t_r X_r) \dots exp(-t_1 X_1) \cdot x))_0 \tag{1}$$

引理 2.1. $\forall a \in O$，$\tau_\varphi^\infty(a)(x \longmapsto \tau_\varphi^\infty(a)_x)$ 是 M 上的一个解析的微分算子。并且 $\tau_\varphi^\infty(a \longmapsto \tau_\varphi^\infty(a))$ 是从 O 到 M 上微分算子代数的同态。

引理的前半部分是显然的。我们只需要验证 τ_φ^∞ 是一个代数同态, 为此利用 (1) 进行计算即可。具体证明省略。

推论 2.1. $\tau_\varphi = \tau_\varphi^\infty|\mathfrak{g}$ 是从 $\mathfrak{g}$ 到 $J_a(M)$ 的李代数同态。

于是，τ_φ 是 G 在 M 上的一个无穷小作用，它被 τ 决定。

3 作用的图像与唯一性

令 $\tau(X \longmapsto \tau(X))$ 是 G 在 M 上的无穷小作用。相应的有 $\bar{\tau}(X) : (g, x) \longmapsto (X_g, \tau(X)_x)$，这是流形 $G \times M$ 上的解析向量场。易见，$\bar{\tau}$ 是从 $\mathfrak{g}$ 到 $J_a(G \times M)$ 的嵌入李代数同态。令

$$L_{(g,x)}^\tau := \{\bar{\tau}(X)_{(g,x)} | X \in \mathfrak{g}\}$$

于是有

$$L^\tau : (g,x) \longmapsto L^\tau_{(g,x)}$$

是 $G \times M$ 上一个相容的切向量系统, 维数等于 $dim(G) = n$。用 Frobenius 定理，对 $(g,x) \in G \times M$，可记 $S_{(g,x)}$ 为 L^τ 的经过 (g,x) 的极大积分子流形。

引理 3.1. *1.* $\forall h \in G$，定义

$$\lambda_h : (g,x) \longmapsto (hg,x)$$

为 $G \times M$ 上的一个解析同胚。则

$$\lambda_h(S_{(g,x)}) = S_{(hg,x)}$$

2. 如果 φ 是 G 在 M 上的整体作用使得 $\tau = \tau_\varphi$，则 $\forall x \in M$，映射：

$$\alpha_{\varphi,x} : g \longmapsto (g, g^{-1} \cdot x)$$

是从 G 到 $S_{(1,x)}$ 的解析同胚。

3. 假设 φ 是 G 在 M 上的局部作用使得 $\tau = \tau_\varphi$。给定 $x_0 \in M$，则存在 G 中 *1* 的连通开领域 $V(V = V^{-1})$，M 中 x_0 的开领域 U，满足：$\forall x \in U, \alpha_{\varphi,x}$ 将 V 解析同胚到 $S_{(1,x)}$ 的一个经过 $(1,x)$ 的开子流形上。

因为 L^τ 在 λ_h 诱导的切映射下不动，$\forall h \in G$，所以 $\lambda_h(S_{(g,x)})$ 是经过 (hg,x) 的积分子流形，反向分析马上证得引理第一部分。

设 φ 是 G 在 M 上的整体作用使得 $\tau = \tau_\varphi$，则 $\forall X \in \mathfrak{g}, x \in M$ 有：

$$(d\alpha_{\varphi,x})_g(X_g) = (X_g, \tau(X)_{g^{-1}\cdot x}) = \bar{\tau}(X)_{\alpha_{\varphi,x}(g)}$$

记 $A_x = \alpha_{\varphi,x}(G)$ 并给其从 $\alpha_{\varphi,x}$ 得到的解析结构。由之前分析知道 A_x 是 $S_{(1,x)}$ 的开子流形。作为 $G \times M$ 中的图像（graph），A_x 是闭的，从而等于 $S_{(1,x)}$。我们证明了引理的第二部分。

引理的第三部分是第二部分的局部形式，证明类似。

推论 3.1. 令 τ 为 G 在 M 上的一个无穷小作用。如果 φ_1 和 φ_2 均是 G 在 M 上的整体作用，并且 $\tau_{\varphi_1} = \tau_{\varphi_2} = \tau$，则 $\varphi_1 = \varphi_2$。如果 φ_1 和 φ_2 均是 G 在 M 上的局部作用，并且 $\tau_{\varphi_1} = \tau_{\varphi_2} = \tau$，则在 $1 \times M$ 的某个领域中 $\varphi_1 = \varphi_2$。

先考虑整体的情况。记 p_G 为 $G \times M$ 到 G 的投影，p_M 为 $G \times M$ 到 M 的投影，$p_{G,x} = p_G|S_{(1,x)}$。由上面引理知道 $p_{G,x}$ 是从 $S_{(1,x)}$ 的解析同胚，且 $\varphi_1(g,x) = p_M \circ p_{G,x}^{-1}(g^{-1}) = \varphi_2(g,x)$，所以 $\varphi_1 = \varphi_2$。

如果 φ_1 和 φ_2 仅仅是局部的作用。固定 $x_0 \in M$，如之前引理取 U, V。固定 $x \in U$，记 $A_{x,i} = \alpha_{\varphi_i,x}(V)(i = 1, 2)$。则 $A_{x,1}$ 和 $A_{x,2}$ 均是 $S_{1,x}$ 的经过 $(1, x)$ 的开子流形，从而 $A_{x,1} \cap A_{x,2}$ 同时是 $A_{x,1}$ 和 $A_{x,2}$ 的开子流形。考虑集合

$$W_x = \{g|g \in V, \varphi_1(g, x) = \varphi_2(g, x)\}$$

它对应与 $A_{x,1} \cap A_{x,2}$ 从而是 V 中开的，显然它也是闭的，所以 $W_x = V$。于是在 $V \times U$ 上 $\varphi_1 = \varphi_2$，由于 x_0 的任意性，必有在 $1 \times M$ 的某个开领域上 $\varphi_1 = \varphi_2$。

上述推论给出了唯一性。

4 李的基本定理

首先我们给出一个引理：

引理 4.1. *令 τ 是 G 在 M 上的一个无穷小作用，L^τ 是如之前一样定义的切向量系统。给定 $x_0 \in M$，则存在 G 中 1 的连通开领域 $V(V = V^{-1})$，M 中 x_0 的开领域 U，以及从 $V \times U$ 到 M 的解析映射 ψ，使得：*

1. *$\forall x \in M, \psi(1, x) = x$。*

2. *$(g, x) \longmapsto (g, \psi(g^{-1}, x))$ 是从 $V \times U$ 到 $(1, x_0)$ 的一个开领域的解析同胚。*

3. *$\forall x \in U$，映射*

$$g \longmapsto (g, \psi(g^{-1}, x))$$

是 V 到 $S_{(1,x)}$ 的一个包含 $(1, x)$ 的开子流形的解析同胚。

引理的证明略琐碎，以后有空我再码上。现在我们可以陈述和证明李的第一个基本定理：

定理 4.1. *令 τ 是 G 在 M 上的一个无穷小作用，给定 $x_0 \in M$，则存在 M 中 x_0 的开领域 U，以及 G 在 U 上的局部作用 φ，使得 $\forall X \in \mathfrak{g}, \tau_\varphi(X) = \tau(X)|U$。*

固定 $x_0 \in M$，取上述引理中的 V, U, ψ。令 $D = \{(g, x) \in V \times U|\psi(g, x) \in U\}$。易知 D 是 $1 \times U$ 在 $G \times U$ 中的一个开领域。令 $\varphi = \psi|D$，我们证明 φ 是 G 在 U 上的局部作用，使得 $\forall X \in \mathfrak{g}, \tau_\varphi(X) = \tau(X)|U$。

固定 $x \in U$。由定义，$\varphi_x : g \longmapsto (g, g^{-1} \cdot x)$ 是 V 到 $S_{(1,x)}$ 的开子流形的解析同胚。于是，任意 $X \in \mathfrak{g}$，$(d\varphi_x)_1(X_1) = \tau(X)_x$。我们剩下只需要验证的是 φ 在 U 上作用的局部结合律。为此，取 G 中单位元的连通开领域 $V_1 = V_1^{-1}, V_2 = V_2^{-1}$，M 中 x 的

连通开领域 U_1, U_2 满足：$V_2^2 \subset V_1, V_1^2 \subset V, \psi(V_2 \times U_2) \subset U_1, \psi(V_1 \times U_1) \subset U$ 易见，对 $(g,h,y) \in V_2 \times V_2 \times U_2, h \cdot y, gh \cdot y, g \cdot (h \cdot y)$ 有定义。欲证明：$gh \cdot y = g \cdot (h \cdot y), \forall (g,h,y) \in V_2 \times V_2 \times U_2, i.e.$ 这与 $\psi(h^{-1}g^{-1}, y) = \psi(h^{-1}, \psi(g^{-1}, y)), \forall (g,h,y) \in V_2 \times V_2 \times U_2$ 一样。固定 $g \in V_2, y \in U_2$，令 $z = \psi(g^{-1}, y)$，定义

$$\alpha(h) = (gh, \psi(h^{-1}g^{-1}, y)), \beta(h) = (gh, \psi(h^{-1}, z)), h \in V_2$$

由之前引理，α(相应的 ,β) 是从 V_2 到 $S_{(1,y)}$(相应的，$S_{(g,x)}$) 的开子流形 A（相应的，B）的解析同胚。A 和 B 均为 L^τ 的积分子流形，$(g,z) \in A \cap B$，所以 $A \cap B$ 是 A 和 B 的非空开子流形。类似上一节推论中的论述，可以证明在 V_2 上 $\alpha = \beta$，从而 $\psi(h^{-1}g^{-1}, y) = \psi(h^{-1}, z)$，由 $x \in U$ 的任意性，我们完成了证明。

5 向整体推广的第一步

上一节中得到的李的基本定理，从 G 和 M 的变量来看都是局部的。很自然的，我们问能否把定理推广到整体的情况。下面，我们首先考虑对 M 这边的推广。我们需要一个引理：

引理 5.1. *令 $\{U_i : i \in I\}$ 为 M 的一个局部有限开覆盖。任意的 $(i,j) \in I \times I$ 和任意的 $x \in U_i \cap U_j$，令 U_{ijx} 为 x 的包含于 $U_i \cap U_j$ 的一个开领域。则任意的 $x \in M$，可以选择 x 的开领域 U_x 使得：*

1. *若 $x \in U_i \cap U_j$，则 $U_x \subset U_{ijx}$*
2. *若 $U_x \cap U_j \neq \varnothing$，则存在 $i \in I$ 使得 $U_x \cup U_y \subset U_i$*

引理的证明完全是点集拓扑的，关键在于对每个 x 都设法取其足够小的领域。

利用这个引理我们有了第一步的整体推广。

定理 5.1. *令 τ 是 G 在 M 上的一个无穷小作用，则存在 G 在 M 上的局部作用 φ，使得 $\tau_\varphi = \tau$。*

根据上节定理，我们可以选取 M 的一个局部有限开覆盖 $\{U_i | i \in I\}$，以及 φ_i 为 G 在 U_i 上的局部作用，满足：$\tau_{\varphi_i}(X) = \tau(X)|U_i, (X \in \mathfrak{g}, i \in I)$。对每个 $(i,j) \in I \times I$，每个 $x \in U_i \cap U_j$，可以选取 x 的开领域 $U_{ijk} \subset U_i \cap U_j$，使得 φ_1, φ_2 在 $1 \times U_{ijk}$ 的某个领域上定义且相等。由上面的引理构作 U_x。对每个 $x \in M$，令 $I(x) = \{i \in I | x \in U_i\}$，令 W_x 是使得所有 $\varphi_i(i \in I(x))$ 可定义且取相同的值的 $(g,y) \in G \times U_x$ 的集合。$I(x)$ 有限，故 W_x 是 $1 \times U_x$ 的开领域。令 $\varphi_x = \varphi_i | W_x, i \in I(x)$。设有 $x, y \in M, s.t., W_x \cap W_y \neq \varnothing$，于

是 $U_x \cap U_y \neq \varnothing$，所以存在 $i \in I(x)$，使得 $U_x \cup U_y \subset U_i$，这意味着 $i \in I(x) \cap I(y)$，所以 $\varphi_x | W_x \cap W_y = \varphi_y | W_x \cap W_y = \varphi_i | W_x \cap W_y$，从而可以把全部 φ_x 拼成 φ，它就是我们要的局部作用！

6 G 上变量的整体推广

这一节我们不加证明地给出关于 G 上变量的整体推广的结论。这一部分较之前更加有趣，但是更难。首先，我们有定理 5.1 的推论如下：

定理 6.1. *设 M 紧，G 单连通。则任意 τ 是 G 在 M 上的一个无穷小作用，都存在 G 在 M 上的整体作用 φ，使得 $\tau_\varphi = \tau$。*

如果 M 不是紧的，则不一定可以找到一个整体作用决定我们给的无穷小作用，即使 G 是单连通的。但是我们现在有很好的结果：

定理 6.2. *令 τ 是 G 在 M 上的一个无穷小作用。如果：*

1. *G 单连通*
2. *$\forall X \in \mathfrak{g}, \tau(X)$ 是 M 上的整体向量场*

则存在唯一的 G 在 M 上的整体作用 φ，使得 $\tau_\varphi = \tau$.

事实上，这个定理还可以更强。可以把第二个条件减弱为：对 $\mathfrak{g}$ 的一个代数生成集中的任意向量场 X，$\tau(X)$ 是整体的。

参考文献

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Introduction to Heegner Points

Qiu Congling*

The theory of Heegner points started with Heegners paper in, and then Birch first undertook a systematic study in the 70's. Then, Gross, Zagier, Kolyvagin etc. It played an important role in the number theory of the next three decades, for example, the proof of Gauss class number 1 theorem. Now, it has been studied deeply, but is still not fully understood by us. Someone, like Zhang Shouwu, hopes to use this theory to prove the BSD conjecture.

The original application the theory of Heegner points is to find nontrivial rational points on elliptic curves. But in the later 80's, the Gross-Zagier formula linked it to L-functions and Kolyvagin's Euler system showed the functorial properties of Heegner points, just like Weber's theory of CM.

In this paper, we will establish some basic properties of Heegner points, the existence and some actions on them. Also, we introduce the work of Gross-Zagier and Kolyvagin. But we can not fully described the theory of L-functions and elliptic curves. So, some important results will be omitted, for example, Kolyvagin's theorem also shows the system of Heegner points controls the size of Selmer group.

1 Preliminary knowledge

Some necessary preliminary knowledge in number theory is given below. Also some about elliptic curves. Let K be an imaginary quadratic field, $\mathcal{O}_K$ the integer group. An order $\mathcal{O}$ in K is a subring of $\mathcal{O}_K$, at the same time a $\mathbb{Z}$-module of rank $[K:\mathbb{Q}]$. The conductor of $\mathcal{O}$ is

$$\mathfrak{c} = \{x \in \mathcal{O}_K | \mathcal{O}_K \subset \mathcal{O}\},$$

which is an integral $\mathcal{O}_K$-ideal. Notice that $\mathfrak{c} \subset \mathcal{O}$ and is the largest $\mathcal{O}$-ideal which is also an $\mathcal{O}_K$-ideal. A fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is a finitely generated $\mathcal{O}$-submodule in K. $\mathfrak{a}$ is called invertible if there is some fractional ideal $\mathfrak{b}$ such that

$$\mathfrak{a}\mathfrak{b} = \mathcal{O}.$$

These invertible ideals form an abelian group and

$$\mathfrak{a}^{-1} = \{x \in K : x\mathfrak{a} \subset \mathcal{O}\},$$

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which is the biggest fractional ideal such that

$$\mathfrak{a}\mathfrak{a}^{-1} \subset \mathcal{O}.$$

The importance of conductor is that it gives a criterion whether an integral ideal $\mathfrak{a}$ is invertible, saying

$$(\mathfrak{a}, \mathfrak{c}) = \mathcal{O} \Longleftrightarrow \mathfrak{a} \text{ is invertibe.}$$

When K is an quadratic field $\mathbb{Q}(\sqrt{d})$, where d is a square free integer, define the discriminant

$$d_K = \begin{cases} d & d \equiv 1(\mod 4) \\ 4d & otherwise \end{cases}$$

then

$$\mathcal{O}_K = \mathbb{Z}[w_K], \text{ where } w_K = \frac{d_K + \sqrt{d_K}}{2}.$$

An order of K must be of form

$$\mathcal{O} = \mathbb{Z}[cw_K], \text{ where } c = [\mathcal{O}_K : \mathcal{O}]$$

and $\mathfrak{c} = c\mathcal{O}_K$ is the conductor of $\mathcal{O}$. The discriminant of $\mathcal{O}$ is defined as $D = c^2 d_K$. Notice that $\mathcal{O}$ is uniquely determined by c or D. In fact, the definition of discriminant comes from the discriminant of a $\mathbb{Z}$-basis [?], in this way, we can define discriminant generally.

We also recall the ramification theory in quadratic fields, a prime number p is said

$$\text{split if } p = \mathfrak{p}_1\mathfrak{p}_2, \text{ then } \mathcal{O}/\mathfrak{p}_i \cong \mathbb{Z}/p\mathbb{Z}, \; i = 1, 2;$$
$$\text{ramified if } p = \mathfrak{p}_1^2, \text{ then } \mathcal{O}/\mathfrak{p}_1 \cong \mathbb{Z}/p\mathbb{Z};$$
$$\text{inert if } p \text{ is still prime, then } \mathcal{O}/p \cong \mathbb{F}_{p^2}.$$

The importance of discriminant is that the ramified prime numbers are precisely those dividing the discriminant d_K (this is valid for all number field). So when $p \neq 2$, $p||D$ (exactly divides) is equivalent to that p is ramified in K. 2 ramifies if and only if $d \not\equiv 1(\mod 4)$.

Class field theory and CM theory. The ring class field of an order $\mathcal{O}$ is defined by class field theory, and could be generated by value of $j(\mathfrak{a})$ over K, i.e.

$$H(\mathcal{O}) = K(j(\mathfrak{a}))$$

here $\mathfrak{a}$ is any ideal, regarded as a lattice, j is Klein j-invariant. In fact, two ideal in a same class have same j-value, and j-value between different classes are conjugate algebraic integers. In fact, they are roots of the so called Hilbert polynomial,[?] . There is an isomorphism (known as Artin map) between the class group Pic($\mathcal{O}$) of $\mathcal{O}$ and the galois group $G_{H/K}$. Denote $\sigma(\mathfrak{d})$ the Artin symbol (the image of artin of Artin map) $\mathfrak{d}$, then the main theorem of CM say

$$\mathfrak{d}(j(\mathfrak{a})) = j(\mathfrak{a}\mathfrak{d}^{-1}).$$

Elliptic curves [?] Ch3, Ch8. The invariant differential of an elliptic curve defined over k with Weierstarss equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in k$$

is defined as

$$\omega = \frac{d_x}{2y + a_1x + a_3}$$

which is invariant under translation, i.e. if $P \in E$, $T_P(Q) = P + Q$ is a translation, then

$$T_P^*(\omega) = \omega.$$

Check directly that it has no poles and zeros, i.e. $div(\omega = 0)$. If

$$\phi : E_1 \longrightarrow E_2$$

is an isogeny, and ω_1, ω_2 are the corresponding invariant differential, then

$$\phi^*\omega_2 = c\omega_1, c \in \bar{k}(x).$$

But since

$$div(c) = div(\phi^*\omega_2) - div(\omega_2) = \phi^*div(\omega_2) - div(\omega_2) = 0 - 0 = 0,$$

$c \in \bar{k}$. When E is defined over $\mathbb{C}$ and given by a Weierstarss uniformization $f : \mathbb{C}/L \to E$, then

$$f * \omega = cdz, c \in \mathbb{C}$$

.

The canonical height on E is a real quadratic form $\langle \cdot, \cdot \rangle$ on E, $\langle P, P \rangle = 0$ if and only if P is torsion. And it can be extended to a Hermitian form on $E(k) \otimes \mathbb{C}$.

The L-series of an elliptic curves defined over $\mathbb{Q}$, it's imaginary quadratic twist, and it's L-series over imaginary quadratic field can't be stated in a few lines, so we omit them.

2 CM elliptic curves

The classical correspondence

$$\{\text{Rank} - 2 \text{ lattice in } \mathbb{C}\}/\text{homothety} \longleftrightarrow \{\text{elliptic curves over } \mathbb{C}\},$$

which realizes complex tori as elliptic curves by the Weierstarss uniformizations. A isogeny between two complex tori is a holomorphic homomorphism keep group structure, must be multiplication by a complex number. Up to homothety, a degree-N cyclic isogeny (i.e. the kernel is $\mathbb{Z}/N\mathbb{Z}$) could be given by

$$\mathbb{C}/Z[\tau] \to \mathbb{C}/Z[1/N, \tau], \quad z \mapsto z. \tag{1}$$

It can also be defined in a totally algebraic way. Use Tate-module and some algebraic classification theroem, we can proof that $\mathrm{End}(E)$ is $\mathbb{Z}$ or an order in some imaginary quadratic field K when E is defined over some field F with characteristic 0[**?**]. In the latter case, we say E has CM (complex multiplication). When $F = \mathbb{C}$, the proof is much easier[**?**]. We sketch the proof. Let $L = \mathbb{Z}[\tau]$ be a lattice, $\forall \alpha \in \mathrm{End}(E_L) = \{x \in K : xL \subset L\}$, we have

$$\alpha = a + b\tau, \alpha\tau = c + d\tau$$

for some integer a, b, c, d, which gives

$$\alpha^2 - (a+d)\alpha + ad - bc = 0.$$

So $\mathrm{End}(E_L)$ is $\mathbb{Z}$-integral. If $\mathbb{Z} \subsetneqq \mathrm{End}(E_L)$, suppose the above $\alpha \notin \mathbb{Z}$, then $b \neq 0$. We have

$$b\tau^2 - (a-d)\tau - c = 0.$$

So, $\mathcal{O} = \mathrm{End}(E_L)$ is an order in K. By definition, L is a fractional ideal of $\mathcal{O} = \mathrm{End}(E)$. Moreover L is invertible. Indeed, It ca be shown by a more careful discussion. As in [**?**], suppose τ satisfies

$$Ax^2 + B^2 + C = 0, A, B, C \in \mathbb{Z}, \gcd(A, B, C) = 1$$

then

$$\mathcal{O}_\tau = \mathcal{O}_D = \mathbb{Z}[\frac{D + \sqrt{D}}{2}] \text{ and } L\overline{L} = \frac{1}{A}\mathcal{O}$$

i.e. $L^{-1} = A\overline{L}$. One has to realize that it doesn't mean that any fractional ideal is invertible, it depends on the order (one fractional ideal of $\mathcal{O}$ is fractional ideal of any order contained in $\mathcal{O}$. Further more, [**?**] shows $\mathrm{disc}(\mathcal{O}) = D = B^2 - 4AC$. We define $disc(L)$ to be the discriminant of the orders of L.

3 Moduli curve $X_0(N)$

$Y = Y_0(N)$ is the open modular curve of ordered pairs (E, E') with an degree-N cyclic isogeny

$$\phi : E \to E', \ker\phi \cong \mathbb{Z}/N\mathbb{Z}.$$

As in (1), it may be given by

$$(L, L')_\tau, L = \mathbb{Z}[\tau], L' = \mathbb{Z}[1/N, \tau] = \mathbb{Z}[N\tau]$$

Let

$$\Gamma_0(N) = \{\in SL_2(\mathbb{Z}) : N|c\}.$$

We have an isomorphism

$$Y_0(N) \cong \Gamma_0(N) \setminus \mathbb{H},\ (L, L')_\tau \mapsto \tau.$$

$X_0(N)$ is the natural compactification of $Y_0(N)$, classification the so-called generalized elliptic curves. let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q}$,

$$X_0(N) \cong \Gamma_0(N) \setminus \mathbb{H}^*.$$

The function field of $X_0(N)(\mathbb{C})$ is $\mathbb{C}(j(\tau), j(N\tau))$, and in fact this two functions satisfy a polynomial defined over $\mathbb{Q}$, called the N-th modular equation. Thus, $X_0(N)$ is defined over $\mathbb{Q}$, have a planar model realized by

$$\Gamma_0(N) \setminus \mathbb{H}^* \longrightarrow X_0(N),\ \tau \longmapsto (j(\tau), j(N\tau)).$$

4 Heegner points: existence

A Heegner on $Y_0(N)$ is a pair (E, E') with the same endomorphism ring $\mathcal{O}$. Choose $L = \mathfrak{a}, L' = \mathfrak{b}$ as fractional ideals of $\mathcal{O}$ with $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{b}/\mathfrak{a} \cong \mathbb{Z}/N\mathbb{Z}$. Then $\mathfrak{n} = \mathfrak{a}\mathfrak{b}^{-1}$ is an invertible integral ideal, with $\mathcal{O}/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. If we have such an $\mathfrak{n}$, choose an invertible fractional ideal $\mathfrak{a}$, (in fact an ideal class $[\mathfrak{a}]$), let $\mathfrak{b} = \mathfrak{a}\mathfrak{n}^{-1}$. Then $E_\mathfrak{a}, E_\mathfrak{b}$ are two elliptic curves with obvious isogeny. Denote this Heegner point as $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$.

Use the results and symbols in 1, we can easily conclude that the above happens, i.e. $\tau' = N\tau$ (after a homothety) and $D' = D$ if and only if

$$A = NA', C' = NC, B = B, \gcd(A', B', C') = \gcd(A, B, C) = 1.$$

Thus the existence of Heegner points is equivalent to that $D = B^2 - 4NC$ (replace NAC by NC) has an integer solution with $(N, B, C) = 1$.

I also give another description of the existence of Heegner points. It can be derived from the previous one by analyze the equation, but we can see it from another viewpoint.

Let $I_K(c)$ be the monoid of integral $\mathcal{O}_K$-ideals prime to c, $I_\mathcal{O}(c)$ the monoid of integral $\mathcal{O}$-ideals prime to c, i.e. invertible integral $\mathcal{O}$-ideals. Then there is a multiplicative bijection (isomorphism) between this two monoids given by

$$\mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}, \mathfrak{a} \in\ I_K(c);\ \mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_K, \mathfrak{a} \in\ I_\mathcal{O}(c) \tag{2}$$

From now on, for the discussion about Heegner point, we always suppose that N prime to c. So, we always have $N\mathcal{O}$ is an invertible ideal, so is every factor of N. Now, for $\mathfrak{a} \in\ I_K(c)$, consider a homomorphism

$$\mathcal{O}/\mathfrak{a} \cap \mathcal{O} \to \mathcal{O}_K/\mathfrak{a},$$

which is injective. Since c is prime to $\mathfrak{a}$, multiplication by c induces an isomorphsim of $\mathcal{O}_K/\mathfrak{a}$. But $c\mathcal{O}_K \subset \mathcal{O}$, the homomorphism is surjective. So, we conclude that the existence of an integral $\mathcal{O}$-ideal $\mathfrak{n}$ with

$$\mathcal{O}/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$$

is equivalent to the existence of an integral $\mathcal{O}_K$-ideal $\mathfrak{m}$ with $\mathcal{O}/\mathfrak{m} \cong \mathbb{Z}/N\mathbb{Z}$. Further more, it's equivalent to that every prime number p dividing N is split or ramified in K (if p is inert, their will appear $\mathbb{F}_{p^2}$ in the quotient), p with p^2 dividing N is split in K (if it's ramified, $p||D$, but it's shown in 3 that we must have $D \equiv B^2(\mod 4N)$, so $p|B$, and $P^2|D$, a contradiction). In the split case, p has only one prime $\mathfrak{p}$ of $\mathcal{O}$ dividing n over it, otherwise, there will be a subgroup $(\mathbb{Z}/p\mathbb{Z})^2$ in $\mathcal{O}/\mathfrak{n}$. Since we assumed $(c,N)=1$, $p|\gcd(D,N)$ means $p|d_K$ is ramified, so $p||N$. Moreover, if $p\neq 2$, $p||d_K$, if $p=2$, $4|d_K$. Usually, we accept the Heegner hypothesis assuming $\gcd(d_K,N)=1$, or equivalently, all $p|N$ are split. Then, there exist Heegner points for each c coprime to N.

We can count the number of Heegner points.Suppose the prime factorization of$N$ is

$$N=\prod_{i=1}^{r}p_i^{a_i}\prod_{j=1}^{s}q_j^{b_j},\ \ p_i|d_K, q_j\nmid d_K$$

(As we have shown, in fact $a_i=1$, and if assuming the Heegner hypothesis, $r=0$) and $h(\mathcal{O})$ be the class number. Then

$$\#\{\text{heegner points of level } N \text{ with order } \mathcal{O}\}=2^s h(\mathcal{O})$$

5 Galois action on Heegner points

Consider the action of $\mathrm{Aut}(\mathbb{C})$ on $X(\mathbb{C})$, which is given by

$$E: y^2=x^3+ax+b\longmapsto E^\sigma: y^2=x^3+\sigma a x+\sigma b,\ \ \sigma\in \mathrm{Aut}(\mathbb{C}).$$

Notice $\mathrm{Aut}(\mathbb{C})$ is a semi product of it's normal subgroup $\mathrm{Aut}_K(\mathbb{C})$ and the order-2 group generated by the complex conjugation. For the complex conjugation, we have

$$\overline{(\mathcal{O},\mathfrak{n},[\mathfrak{a}])}=(\mathcal{O},\bar{\mathfrak{n}},[\bar{\mathfrak{a}}]). \tag{3}$$

For $\sigma\in\mathrm{Aut}_K(\mathbb{C})$, since the j-invariants $j(\mathfrak{a})$ of all ideal classes of an order are conjugate algebraic integers, So the order of $(\mathcal{O},\mathfrak{n},[\mathfrak{a}])^\sigma$ is $\mathcal{O}$ too . In a special case, this point could be given by a explicit formula by the CM theory. Denote $\sigma(\mathfrak{d})$ the Artin symbol of $\mathfrak{d}$, then $\mathfrak{d}(j(\mathfrak{a}))=j(\mathfrak{a}\mathfrak{d}^{-1})$. Since $\mathfrak{a}\mathfrak{d}^{-1}(\mathfrak{b}\mathfrak{d}^{-1})^{-1}=\mathfrak{a}\mathfrak{b}^{-1}$, we have

$$(\mathcal{O},\mathfrak{n},[\mathfrak{a}])^\sigma=(\mathcal{O},\mathfrak{n},[\mathfrak{a}^\sigma])=(\mathcal{O},\mathfrak{n},[\mathfrak{a}\mathfrak{b}^{-1}]). \tag{4}$$

In particular, we know that $(\mathcal{O},\mathfrak{n},[\mathfrak{a}])$ are over $K(j(\mathcal{O}))$, the ring class field of $\mathcal{O}$).

6 Hecke operators and Atkin-Lehner involutions

For any congruence subgroup Γ, there ia a natural correspondence between wight-k invariant functions respect to Γ and degree-k homogeneous functions defined on $Y(\Gamma)$. When $\Gamma=\Gamma_0(N)$, it's given by

$$f(\tau)=F(\mathbb{C}/\mathbb{Z}[\tau],\mathbb{C}/\mathbb{Z}[1/N,\tau])$$

where F satisfies for all $m\in\mathbb{C}^*$,

$$F(\mathbb{C}/(m\mathbb{Z}[\tau]),\mathbb{C}/(m\mathbb{Z}[1/N,\tau]))=m^{-k}F(\mathbb{C}/\mathbb{Z}[\tau],\mathbb{C}/\mathbb{Z}[1/N,\tau]).$$

Extend F additively to a function defined on $\mathrm{Div}(X_0(N))$, then an operator T from $\mathrm{Div}X_0(N))$ to $\mathrm{Div}X_0(N))$ acts on F by $TF(D) = F(TD)$, so on f. Thus actions on modular space induce actions on modular forms. But historically, always the latter is discovered earlier.

The actions we discuss here are the Hecke operators (or Hecke correspondences) and the Atkin-Lehner involutions. We give their their actions on $X_0(N)$ at first, since it make more sense. Then we compute their actions on modular forms. And we shall prove that the Hecke operators and the Atkin-Lehner involutions on $X_0(N)$ map Heegner points to Heegner points of K.

Let $x = (E, E', \phi)$ be a point in $X_0(N)$, where ϕ is a given isogeny. Every m-cyclic subgroup C of E, with $C \cap \ker\phi = \emptyset$ gives another point $(E/C, E'/\phi(C), \phi/C) \in X_0(N)$, where ϕ/C means the induced isogeny. Let $E(m)$ be the set of cyclic subgroups of order m,

$$T_m(x) = \sum_{\substack{C\in E(m)\\ C\cup\ker\phi=\emptyset}} (E/C, E'/\phi(C), \phi/C).$$

We mainly focus on m with $\gcd(m, N) = 1$, then the classical product formula follows quickly. The point is tha when $\gcd(m, n) = 1$, taking m-cyclic and a n-cyclic gives a nm-cyclic, but taking apply T_p and T_{p^k} may gives a $(\mathbb{Z}/p\mathbb{Z})^2 \oplus (\mathbb{Z}/p^{k-1}\mathbb{Z})$, the quotient by it is the same with the quotient by a $\mathbb{Z}/p^{k-1}\mathbb{Z}$.

As a correspondence $X_0(N) \rightsquigarrow X_0(N)$, Hecke operator T_m, is defined by the following diagram

$$X_0(N) \xleftarrow{\alpha} X_0(mN) \xrightarrow{\beta} X_0(N)$$

Since $(m, N) = 1$ $(E, G) \in X_0(mN)$ must be of form $G = C \oplus D$, where C, D are cyclic of order N, m. Then α forgets D, β quotients by D:

$$\begin{aligned}\alpha &: (E, G) \mapsto (E, C)\\ \beta &: (E, G) \mapsto (E/D, (C \oplus D)/D)\end{aligned}$$

In particular, T_p is the composition

$$(E, C) \overset{\alpha^*}{\mapsto} \sum_{D\in E(p)} (E, (C \oplus D) \overset{\beta_*}{\mapsto} \sum_{D\in E(p)} (E/D, (C \oplus D)/D$$

By a bookkeeping computation,

$$T_p(\mathbb{Z}[\tau], \mathbb{Z}[1/N, \tau]) = \sum_{j=0}^{p} (\mathbb{Z}[\gamma_j\tau], \mathbb{Z}[1/N, \gamma_j\tau])$$

where $\gamma_j = \begin{bmatrix} 1 & j\\ 0 & p\end{bmatrix}$ for $j \neq p$, which is given by $< \frac{j+\tau}{p} >$ and $\gamma_p = \begin{bmatrix} p & 0\\ 0 & 1\end{bmatrix}$ which is given by $< \frac{1}{p} >$.

To get it's action on $S_k(\Gamma_0(N))$, only need to notice that

$$\Gamma_0(pN) = \Gamma_0(N) \cap \begin{bmatrix} 1 & 0\\ 0 & p\end{bmatrix}^{-1} \Gamma_0(N) \begin{bmatrix} 1 & 0\\ 0 & p\end{bmatrix}$$

and The following technical lemma [**?**]

Lemma 6.1. *Let $\alpha \in GL_2(\mathbb{R})^+$, and for a congruence subgroup Γ,*

$$\Gamma = \amalg(\Gamma \cap \alpha^{-1}\Gamma\alpha)\beta_i$$

then

$$\Gamma\alpha\Gamma = \amalg(\Gamma\alpha_i), \ \alpha_i = \alpha\beta_i.$$

Thus we get the usual double coset operators, which forms an abstract Hecke ring. And the actions on $S_k(\Gamma_0(N))$ is given by

$$T_k(p)f = p^{k/2-1}\sum_i f|_{[\alpha_i]_k}.$$

If $(\mathbb{Z}[\tau], \mathbb{Z}[1/N, \tau])$ is a heegner point of K, proceed as in section 2 and 4, we can show that $disc(\mathbb{Z}[\gamma_j\tau])$ and $disc(\mathbb{Z}[1/N, \gamma_j\tau])$ have the same discriminant dividing Dp^2 (recall as we defined in section 2, it's equal to the discriminant of minimal polynomial of $\gamma_j\tau$ and $N\gamma_j\tau$). So, they are of the same order, and $(\mathbb{Z}[\gamma_j\tau], \mathbb{Z}[1/N, \gamma_j\tau])$ is a Heegner point. But we usually can't say much about this order in general without explicit computation. Thus, we have a formula

$$T_p(\mathcal{O}, \mathfrak{n}, \mathfrak{a}) = \sum_{[\mathfrak{a}:\mathfrak{b}]=p} (\mathrm{End}(\mathfrak{b}), \mathfrak{n}_{\mathfrak{b}}, \mathfrak{b}). \tag{5}$$

[?] claim that $\mathfrak{n}_{\mathfrak{b}} = \mathfrak{n}\mathrm{End}(\mathfrak{b}) \cap \mathrm{End}(\mathfrak{b})$.

Moreover, $disc(\mathbb{Z}[\gamma_j\tau])|p^2D$ $(D = disc(\mathbb{Z}[\tau]))$. So, the conductor of (M, M') is prime to N, provided so is (L, L').

The Atkin-Lehner involutions are from taking dual isogeny. We define $w_N(E, E', \phi) = (E', E, \widehat{\phi})$, where $\widehat{\phi}$ is the dual isogeny of ϕ. Generally, let $Q|N$, $\gcd(Q, N/Q) = 1$, $G = \ker\phi$, $G' = \ker\phi'$, $G[Q], G'[Q]$ the (unique) order-Q subgroup part of G, G'. w_Q is the composition

$$E/G[Q] \to E/G \cong E' \to E'/G'[Q]$$

It's visible that

Observation. *w_Q is a intermediate process of $w_{Q'}$ if $Q|Q'$, when $Q = N$, we just take dual isogeny. It's clear $w_{Q_1}w_{Q_2} = w_{Q_1Q_2}$ for $\gcd(Q_1, Q_2) = 1$. Since $E/E[Q] \cong E$, we also have $w_Q^2 = 1$. Thus we find that all the w_Q's forms a group isomorphic to $(Z/2\mathbb{Z})^r$, denoted as W, where r is the number of factors of N.*

For the computation of the matrices of the actions on $S_k(\Gamma_0(N))$, as usual, take $E = \mathbb{C}/\mathbb{Z}[\tau]$, $G = \langle 1/N \rangle$, so $G[Q] = \langle 1/Q \rangle$. It's direct

$$w_p(E, G) = (\mathbb{C}/\mathbb{Z}\left[\frac{1}{Q}, \tau\right], \left\langle \frac{\tau}{Q} + \frac{1}{N} \right\rangle).$$

Only need to find a $\mathbb{Z}$-basis (ω_1, ω_2) of $\mathbb{Z}[\frac{1}{Q}, \tau]$, such that

$$\frac{\tau}{Q} + \frac{1}{N} = \omega_1/N,$$

then $w_Q(E,G) = (\mathbb{Z}[\omega_2/\omega_1], 1/N)$. Any choice of integers x, y such that $xQ - y(N/Q) = 1$ produce the desired $\omega_2 = x\tau + y/Q$. Thus

$$\omega_2/\omega_1 = \begin{bmatrix} Qx & y \\ N & Q \end{bmatrix} \tau,$$

denote this matrix as W_Q, In [?], it's given by

$$W_Q = \begin{bmatrix} Qx & y \\ Nz & Qw \end{bmatrix}$$

. Also, a direct computation shows W_Q is in the normalizer of $\Gamma_0(N)$ and the action doesn't depend on the choice of a, b. We have now obtain the Atkin-Lehner involution on $S_k(\Gamma_0(N))$. Compute directly, we can verify the observation in the end of the last paragraph in true when regarded as actions on $S_k(\Gamma_0(N))$.

Proceed as previous, by comparing discriminant, we can show that $w_Q(\mathcal{O}, \mathfrak{n}, \mathfrak{a})$ is of the same order with $(\mathcal{O}, \mathfrak{n}, \mathfrak{a})$. [?] gave a precise formula. Let $w_p = w_{p^{ord_p(N)}}$. We showed in section 4, p has only one prime $\mathfrak{p}$ of $\mathcal{O}$ dividing n over it so

$$w_p((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])) = (\mathcal{O}, [\mathfrak{n}\mathfrak{p}^{-k}\overline{\mathfrak{p}}^k, \mathfrak{a}\mathfrak{p}^{-k}]). \tag{6}$$

This coincides with our previous observation. Moreover $w_N = \prod_{p|N} w_p$ given by

$$w_N((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])) = (\mathcal{O}, \overline{\mathfrak{n}}, [\mathfrak{a}\mathfrak{n}^{-1}]) \tag{7}$$

is the dual isogeny, and if $\mathfrak{a} = \mathbb{Z}[\frac{1}{-N\tau}]$ (up to homothety), then

$$\mathfrak{a}\mathfrak{n}^{-1} = \mathbb{Z}\left[\frac{1}{-N\tau}\right].$$

Combining the actions of W and $G = G_{H/K}$, we see that $W \times G$ acts transitively on the set of Heegner points, and $W_p \in W$ with $p|d_K$ acts trivially. Thus, $W/\langle W_p : p|d_K\rangle \times G$ acts transitively and freely on the set of Heegner points.

7 Hecke algebra of Γ_0 and Atkin-Lehner theory in general

Define three commutative algebras

$$\mathbb{T}^0 = \mathbb{Z}[T_p, p \text{ is prime and } \gcd(p, N) = 1\ ],$$

$$\mathbb{T} = \mathbb{Z}[T_p, p \text{ is prime and } \gcd(p, N) = 1; T_q, q \text{ is prime and } q|N\ ],$$

(recall the definition, as in section 6, we can similarly show T_q is defined as sum of $\gamma_j,\ j \neq p$, satisfying $T_qT_n = T_{qn}$.)

$$\mathbb{T}_w = \mathbb{Z}[T_p, p \text{ is prime and } \gcd(p, N) = 1; w_q, q \text{ is prime and } q|N\ ].$$

Recall the action of w_N is

$$w_N(f) = \frac{-1}{N\tau^2} f(\frac{-1}{N\tau^2})$$

Notice T_q and w_q are not commutative. For the recursive relation of $\mathbb{T}$, we have a formula by formal L-series

$$\sum_{n=1}^{\infty} T_n n^s = \prod_{p\nmid N} (1 - T_p p^{-s} + p^{1-2s})^{-1} \prod_{p|N} (1 - T_p p^{-s})^{-1}.$$

Then Under the Petersson inner product for $S_k(N)$, both of T_p and w_q are hermitian, but T_q not. Thus there exists a basis f_i consisting of normalized eigenforms of all elements in $\mathbb{T}_0$, and all eigenvalues are real. When f is normalized, i.e. $a_0(f) = 1$, we have $T_p(f) = a_p(f)f$. There exists a so-called old subspace, i.e. space formed by $f(e\tau)$ with $f(\tau) \in S_k(M)$ and $eM|N$. Its orthogonal complement is called new space, which is stable under $\mathbb{T}$ and w_q.

The eigenspaces are either old or new, moreover, each new space is spanned by a single normalized form, called a new form. The action of T_q, w_q on new form is as follows:

$$w_q(f) = \pm f;\ U_q(f) = a_q(f)f,\ \text{if } q||N, U_q(f) = -q^{k-1}w_f,\ \text{if } q^2|N, a_q = 0,$$

For the recursive relation, we can state as, for a new form

$$a_{np} = a_n a_p - p^{2k-1} a_{n/p};\ a_{nq} = a_n a_p.$$

Denote $i_e(f(\tau)) = f(e\tau)$, $S_2(\Gamma_0(M))^+$ as new space, we have

$$S_2(\Gamma_0(N)) = \bigoplus_{M|N} \bigoplus_{eM|N} i_e(S_2(\Gamma_0(M))^+),$$

$$S_2(\Gamma_0(M))^+ = \bigoplus_{f \text{ new form}} \mathbb{C}f.$$

Attention: i_e may not keep the property of being eigenform or new form, but it's true that if g is an eigenform of level N there must be some new form f of level $M|N$ with $a_n(g) = a_n(f)$ for almost all n. ([**?**] Prop 5.8.4)

In the case $k = 2$, the previous discussion can be interpreted as follows. $S_2(\Gamma_0(N)$ could be identified with $\mathcal{U} = H^0(X(\mathbb{C}), \Omega^1)$ (the holomorphic 1-forms on $\mathbb{C}$) by

$$f \mapsto 2\pi i f(z)dz = \omega_f$$

which is a subspace of the first (deRham) cohomology group $H^1(X(\mathbb{C}))$. Generally, for any compact Riemann surface S, denote the space of degree-1 smooth differential forms on S by $\mathscr{E}^1(S)$. $\mathscr{E}^1(S)$ has a decomposition, say any $w \in \mathscr{E}^1(S)$ can be write as $fdz + gd\overline{z}$. Define the $\star$-operator as

$$\star(fdz + gd\overline{z}) = i(\overline{f}d\overline{z} - \overline{g}dz)$$

and a hermitian product

$$\langle w_1, w_2 \rangle := \int_S w_1 \wedge \star w_2.$$

Under this hermitian product, there is an orthogonal decomposition

$$\mathscr{E}^1(S) = d\mathscr{E}^0(S) \oplus H^0(S(\mathbb{C}), \Omega^1) \oplus \overline{H^0(S(\mathbb{C}), \Omega^1)}.$$

Now the Petersson inner product is the restriction of this hermitian product (up to a positive number,).

Each eigenspace V can also considered as a real character (not Dirichlet) of $\mathbb{T}$ by setting $V(T)$ equal to the corresponding eigenvalues. It's the same (and more usually) we define for every normalized eigenform a character

$$\lambda_f : \mathbb{T} \to \mathbb{R}, \; Tf = \lambda_V(T)f,$$

The image is $\mathbb{Z}[a_n(f) : n \in \mathbb{Z}_{>0}\ ]$, and the kernel is $I_f = \ker(\lambda_f) = \{T \in \mathbb{T} : Tf = 0\}$ then, we have an isomorphism

$$\mathbb{T}/I_f \cong \mathbb{Z}[a_n(f)\ ].$$

The algebra $\mathbb{T} \subset \mathrm{End}_{\mathbb{C}}(S_2(\Gamma_0(N)))$ is obviously free, and seems of infinitely rank, but in fact, let

$$g = \mathrm{genus}(X) = \frac{1}{2}\mathrm{rank}_{\mathbb{Z}} H_1(X, \mathbb{Z}) = \dim_{\mathbb{C}} S_2(\Gamma_0(N)) = \dim \mathcal{U},$$

we have

Theorem 7.1. $\mathbb{T}$ *is of a finitely generated free* $\mathbb{Z}$*-module of rank* g*.*

PROOF: [sketch] Instead of $\mathcal{U}$, we consider it's dual $\mathrm{Hom}(\mathcal{U}, \mathbb{C})$, and the pullback of $\mathbb{T}$. Then $H_1(X, \mathbb{Z})$ is a sublattice in $\mathrm{Hom}(V, \mathbb{C})$ given by

$$\sum a_i \gamma_i \mapsto \sum a_i \int_{\gamma_i}$$

The quotient by $H_1(X, \mathbb{Z})$ defines the Jacobian of X, isomorphic to $\mathrm{Pic}^0(X)$ (Abel-Jacobi). Verifying the actions of T_p are the same under the isomorphism implies $H_1(X, \mathbb{Z})$ is stable under $\mathbb{T}$ ([**?**], sec 6.5). But it seems rather circuitous. In fact, we can express a loop by a pair $\{\tau, \gamma\tau\}$ (a integral modular symbol) independent of the choice of $a \in \mathbb{H}^*$, where $\gamma \in \Gamma_0(N)$. Then he action is given by

$$T_p\{\tau, \gamma\tau\} = \sum \{\tau, \gamma_i \tau\}$$

here γ_i is defined by the double coset operatorT_p. Verify directly $H_1(X, \mathbb{Z})$ is stable under $\mathbb{T}$.[**?**]

Since the action of complex conjugation on $X_0(N)(C)$ induces an action on $H_1(X, \mathbb{Z})$ which commutes with $\mathbb{T}$. Thus, $\mathbb{T}$ in fact act on two submodules of rank g, and so is a commutative subring of $M_g(\mathbb{Z})$. Then its rank is at most g.

For the other side, consider the pairing

$$\mathbb{T} \otimes \mathbb{C} \times (S_2(\Gamma_0(N))), \quad \langle T, f \rangle = a_1(Tf) \tag{8}$$

it's non-degenerate on the right. So $\mathrm{rank}_{\mathbb{Z}} \mathbb{T} = \dim_{\mathbb{C}} \mathbb{T} \otimes \mathbb{C} \geq g$. $\square$ Thus we know that the minimal polynomial of T_p is monic integral. So, the eigenvalues are algebraic integer, and so is the coefficients of an eigenform f. Moreover,the number field of f,

$$K_f = \mathbb{Q} \otimes \mathbb{Z}[a_n(f)] = \mathbb{Q}[a_n(f)]$$

is a finite real extension of $\mathbb{Q}$.

In fact, we know more from the proof. Since the pairing in the proof is non-degenerate on both sides,

Theorem 7.2. *The space of modular forms with integer coefficients in $S_2(\Gamma_0(N))$ is a rank-g lattice. Thus, $S_2(\Gamma_0(N))$ has a basis of forms with integer coefficients*

PROOF: It's just the dual lattice of $\mathbb{T}$ in $\mathbb{T} \otimes \mathbb{R} \subset \mathbb{T} \otimes \mathbb{C}$ □

Surely, these integral forms are usually not newforms, even not eigenforms. But they can be derived from newforms of each level $M|N$ ([?] Cor6.5.6).

8 Eichler-Shimura Construction and rational points on Elliptic curves

A modular parametrization of level N of E is a nonconstant map $\Phi : X = X_0(N) \to E$. Then the pull-back of the invariant differential on E is then of the form ω_f, where $f \in S_2(\Gamma_0(N))$. When E is defined over $\mathbb{Q}$, we hope Φ is defined over $\mathbb{Q}$. Suppose so it is, we can use this map to find rational points on E.

Let K be a imaginary quadratic field as before, $\tau \in K$, so E_τ is CM, but we do not require $P = (E_\tau, E_{N\tau})$ to be heegner. let H be the ring class field of the order $\mathcal{O}_\tau \cap \mathcal{O}_{N\tau}$. Usually, this two orders are not contained in each other, so, the intersection is smaller than both, and the ring class field is bigger. By CM theory, we know that $j(\tau), j(N\tau) \in H$. So, we have a CM point P in $X(H)$. And we have $\Phi(\tau) \in E(H)$.

To find rational point over $\mathbb{Q}$, the standard method is to apply trace map

$$\mathrm{Tr}_{H/\mathbb{Q}} P = \sum_{\sigma \in G'} P^\sigma.$$

Here, the sum is being computed with the group law of E_f, and obvious, it's over $\mathbb{Q}$. Since Φ_f is defined over $\mathbb{Q}$, it communicate with $G' = G_{H/\mathbb{Q}}$ (we have defined $G = G_{H/K}$.) Thus, suppose τ is given by $([\mathfrak{a}], [\mathfrak{b}])$, then

$$\mathrm{Tr}_{H/\mathbb{Q}}(P) = \sum_{\sigma \in G'} \Phi_f(([\mathfrak{a}], [\mathfrak{b}]))^\sigma = \sum_{\sigma \in G'} \Phi_f(([\sigma \mathfrak{a}], [\sigma \mathfrak{b}]))$$

The action of σ on $\mathfrak{a}, \mathfrak{b}$ is just restriction, and could be given by Artin map and the main theorem of CM. Notice that since G' is big, so $\mathrm{Tr}_{H/\mathbb{Q}} P$ is very possible to be 0.

Now, we state the Eichler-Shimura construction, which provides such a modular parametrization.

For a new form f, let N denote it's level, as usual, denote $J_0(N)$ the jacobian of $\Gamma_0(N)$, defined as

$$J_0(N) = \mathrm{Hom}(\mathcal{U}, \mathbb{C})/H_1(X, \mathbb{Z})$$

For $\sigma : K_f \to \mathbb{C}$ an embedding, define

$$f^\sigma = \sum \sigma(a_n(f))q^n$$

it can be show that this action keeps the space of new forms $S_2(\Gamma_0(N))^+$. Define the equivalent class of f

$$[f] = \{f^\sigma : \sigma K_f \to \mathbb{C}\},$$

and the space spanned by this class

$$V_f = \mathbb{C}([\omega_f]) \subset \mathcal{U}.$$

Then the action of $H_1(X, \mathbb{Z})$ can be restricted to V_f, denote it as Λ_f, which is a sublattice of $\mathrm{Hom}(V_f, \mathbb{C})$. The restriction map is a natural homomorphism from $J_0(N)$ to $\mathrm{Hom}(V_f, \mathbb{C})/\Lambda_f$. Use the pairing (3) and other arguments, we can proof

$$A_f := J_0(N)/I_f J_0(N) \xrightarrow{\sim} \mathrm{Hom}(V_f, \mathbb{C})/\Lambda_f.$$

And $\mathrm{Hom}(V_f, \mathbb{C})/\Lambda_f$ is a dimension $[K_f : \mathbb{Q}]$ complex torus. Since $T_p f - a_p(f) = 0$, the Hecke operator T_p acts on A_f by multiplying $a_p(f)$ (i.e. addition for $a_p(f)$ times).

Moreover, if f has integral coefficients, i.e. $[K_f : \mathbb{Q}] = 1$, we get an elliptic curve (not denote it A_f, but) $E_f \cong \mathrm{Hom}(V_f, \mathbb{C})/\Lambda_f$. As we mentioned above, that we could choose a generator of Λ_f to be $\{\tau_0, \gamma(\tau_0)\}$. It's image (denote as Λ_f again) in $\mathrm{Hom}(V_f, \mathbb{C}) \cong \mathbb{C}$ is a sublattice

$$\Lambda_f = \left\{ \int_{\tau_0}^{\gamma(\tau_0)} \omega_f : \gamma \in \Gamma_0(N) \right\}.$$

Now, we have a composition

$$\Phi_f :\ X_0(N) \xrightarrow{\Phi_{x_0}} J_0(N) \xrightarrow{\pi_f} E_f \tag{9}$$

where Φ_{x_0} is the canonical inclusion with some base point x_0

$$\Phi_{x_0} : x \mapsto \int_{x_0}^{x}$$

and π_f is the previous quotient map. One important fact is that, Φ_f gives a pullback of the invariant differential ω on E_f, which is a nonzero multiple of $\omega_f = 2\pi i f(z)dz$, i.e.

$$\Phi_f(\omega) = \omega \circ \Phi_f = c2\pi i f(z)dz, c \neq 0. \tag{10}$$

Besides the fact $X_0(N)$ could be defined over $\mathbb{Q}$, it's well known that Weil and Chow proved that $J_0(N)$ is also defined over $\mathbb{Q}$, and we can choose x_0 such that Φ_{x_0} is also defined over $\mathbb{Q}$. Also, as we will expect, π_f and A_f are both defined over $\mathbb{Q}$, c.f. [?]. So, the previous conposition is defined over $\mathbb{Q}$, known as the modular parametrisation. Also, c in (10) is also in $\mathbb{Q}^*$, it's called the Manin constant.

In particular, take $x_0 = i\infty$, then $\Phi_{i\infty}(x) = \int_\infty^x \in J(X)$. Then the composition

$$\Phi_f : \Gamma_0(N) \setminus \mathbb{H}^* \longrightarrow X_0(N) \longrightarrow J_0(N) \longrightarrow E_f$$

could be given explicitly using(9)(10)

Theorem 8.1. *Let $\Psi_f : \mathbb{C}/\Lambda_f \longrightarrow E_f$ be the Weierstrass uniformisation, then*

$$\Phi_f(\tau) = \Psi_f(z_\tau),\ z_\tau = c\int_{i\infty}^{\tau} 2\pi i f(z)dz = c\sum_{n=1}^{\infty} \frac{a_n}{n} q^n. \tag{11)}$$

The proof is just by definition and change of variables.

From the Eichler-Shimura construction, we can get other modular parametrization. The pullback of $f(\tau) \in S_0(\Gamma_0(N))$ under the canonical covering map

$$\pi : X_0(MN) \to X_0(N), \Gamma_0(N)\tau \mapsto \Gamma_0(MN)\tau$$

is $f(M\tau)$. And the canonical covering map is also defined over $\mathbb{Q}$. Thus Composing with π produces another modular parametrization for E. Also, by composing an isogeny over $\mathbb{Q}$ also works. But in general a modular parametrization can be any morphism from modular curves of any level to E, even not factoring through $X_0(N)$. Of course, something from Eichler-Shimura construction construction is easy to handle.

Some theorem about the Eichler-Shimura construction follows.

Theorem 8.2. *$L(E, s)$ coincides with $L(E_f, s)$.*

$L(E, s)$ It can be proved by the Eichler-Shimura relation about the reductions of E.

Theorem 8.3. *The conductor of E_f is N.*

From this, we know only N is a conductor of some elliptic curves, there might be new forms of level N with integer coefficients. Thus, usually, there no new forms with integer coefficients.

Theorem 8.4. *The modular parametrization of E at the least level must comes from the Eichler-Shimura construction (up to an isogeny over $\mathbb{Q}$.)*

The Eichler-Shimura construction is excellent, but it seems what we want is the other direction: to find modular parametrization of a particular Elliptic curve. This embarrassing condition is solved by the modularity theorem of Wiles.

Theorem 8.5. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. Then there exists a newform $f \in S_2(\Gamma_0(N))$ such that $L(E, s) = L(f, s)$.*

Then, by the isogeny theorem of Faltings, and some other facts about Tate-module, we know E is isogeny to E_f over $\mathbb{Q}$.

9 Heegner points and rational points on elliptic curve

Now, we turn to Heegner points. Suppose $\mathcal{O}_\tau = \mathcal{O}_{N\tau} = \mathcal{O}_D$, the ring class field H is smaller than the general cases. Use the symbol previous, denote the Heegner point of τ as $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$. Let $\Phi_f : X_0(N) \to E$ be a modular parametrization given by Eichler-Shimura construction, define

$$P = \Phi(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}]) \in E(H)$$

Then use formula (3)

$$\begin{aligned}\mathrm{Tr}_{H/K}(P) &= \sum_{\sigma \in G} \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])^\sigma = \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])^\sigma(\mathfrak{b}) \\ &= \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}\mathfrak{b}^{-1}]) = \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{b}]) \in E_f(K)\end{aligned}$$

Similarly, use formula (4) we have

$$\overline{\mathrm{Tr}_{H/K}(P)} = \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f(\mathcal{O}, \bar{\mathfrak{n}}, [\bar{\mathfrak{b}}])$$

Recall our discussion on Atkin-Lehner involution, we see that $w_N(f) = \varepsilon f, \varepsilon = \pm 1$. If $\varepsilon = 1$, by the definition of Φ_f, we have $\Phi_f \circ w_N = \Phi_f$. Thus, by (7),

$$\Phi_f(\mathcal{O}, \bar{\mathfrak{n}}, [\bar{\mathfrak{b}})] = \Phi_f \circ w_N(\mathcal{O}, \mathfrak{n}, [\mathfrak{n}\bar{\mathfrak{b}}]) = \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{n}\bar{\mathfrak{b}}])$$

and

$$\overline{\mathrm{Tr}_{H/K}(P)} = \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{n}\bar{\mathfrak{b}}]) = \sum_{\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})} \Phi_f(\mathcal{O}, \mathfrak{n}, [\mathfrak{b}]) = \mathrm{Tr}_{H/K}(P) \tag{12}$$

So $\overline{\mathrm{Tr}_{H/K}(P)} \in E(\mathbb{Q})$. Similarly, if $\varepsilon = -1$,

$$\overline{\mathrm{Tr}_{H/K}(P)} = -\mathrm{Tr}_{H/K}(P) \qquad (12').$$

Besides, the ε appears in the functional equation of the L- series of f (or of E_f over $\mathbb{Q}$), let

$$\Lambda(f, s) := (2)^s \Gamma(s) N^{s/2} L(f, s)$$

then

$$\Lambda(f, s) = -\Lambda(w_N(f), 2 - s) = -\varepsilon\Lambda(f, 2 - s). \tag{13}$$

Note that $L(E, s)$ vanishes to even (resp. odd) order at $s = 1$ when $\varepsilon = -1$ (resp. $\varepsilon = 1$).

It's also natural to consider $E(H)$ as a G-module, and introduce the characters of G. Generally, Let χ be a 1-dimensional (primitive) complex character of G, define

$$R = \mathbb{Z}[\chi(\sigma) : \sigma \in G].$$

For a $\mathbb{Z}[G]$ module M, we have a $(Z[G], R)$-bimodule $M \otimes R$. Check directly $M^\chi = \{m \in M \otimes R : m^\sigma = \chi(\sigma)m, \forall \sigma \in G\}$ is a sub bimodule of $M \otimes R$. For a sub $\mathbb{Z}[G]$-module N, we have an exact sequence of bimodules

$$0 \longrightarrow N^\chi \longrightarrow M^\chi \longrightarrow (M/N)^\chi .$$

For example, when χ is trivial, the case degenerates to be taking $G-$inviant elements.

Now, take $M = E(H)$, $G = G_{H/K}$. Like trace map, define

$$P_\chi = \sum_{\sigma \in G} \overline{\chi}(\sigma) P^\sigma.$$

It's direct to very that

$$P_\chi \in E(H)^\chi \subset V = E(H) \otimes \mathbb{C}$$

it's called a Heegner vector. By (3)(4),

$$P_\chi = \sum_{\mathfrak{b} \in \mathcal{O}} \chi(\mathfrak{b}) \Phi_f((\mathcal{O}, \mathfrak{n}, [\mathfrak{a}\mathfrak{b}])).$$

If $\chi = 1$, it's just the trace map, providing a point over K.

We have just considered Heegner points for a single order, i.e. in one ring class field, and didn't see much magic. But as the previous examples shows, the trace map can relate the rational points in different fields. It's more meaningful when we consider Heegner points $P_1, P_2, P_3, ...$ on E_f defined over a tower of ring class fields $H_1 \subset H_2 \subset H_3 ...$ of K. They are required to satisfy two properties, known as Euler system conditions. One relating the trace maps and the Hecke operators, called norm compatibility. Rough speaking, these trace maps (on Jacobian or divisor groups) should almost coincide with the Hecke operators. Also, by modular parametrization Φ_f, the Hecke operators T_p induce multiplications by $a_p(f)$, they also should coincide with the trace maps on E_f. The other is the congruence relation, comes from class field theory and Eichler-Shimura congruence relation. It's very big and significant a theme, contributing directly to many progresses in number theory since 1980's. It's direct start is the conjecture of Gross [?] 11.2.

Here, we describe the Euler system of Heegner points introduced by Kolyvagin. To make life easy, assuming the Heegner hypothesis $\gcd(d_K, N) = 1$, or equivalently, all $p|N$ are split in K. Thus there exists an integral ideal of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. Now, for an order $\mathcal{O}_c$ with conductor c coprime to N, define $\mathfrak{n}_c = \mathfrak{n} \cap \mathcal{O}$, so that $\mathcal{O}/\mathfrak{n}_c \cong \mathbb{Z}/N\mathbb{Z}$. Also define H_c be its ring class field. Moreover, we are satisfied with the most easy case. For a Heegner point $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$, we always choose $\mathfrak{a} = \mathcal{O}$, and denote it as

$$Q_c = (\mathcal{O}, \mathfrak{n}_c, \mathcal{O}).$$

Also define

$$P_c = \Phi_f Q_c \in E(H_c)$$

We further define $S = \{c : c \text{ square free, coprime to } N, d_K\}$. Thus for prime $p|c$, $d = c/p \in S$. Define a trace map

$$\mathrm{Tr}_e(Q_c) = \sum_{\sigma \in G_{H_c/H_d}} P_c^\sigma$$

transferring Heegner points defined over H_c to a divisor defined over H_d. Easy to see it coincides with the trace map on $E(H_c)$,

$$\Phi_f(\mathrm{Tr}_e(Q_c)) = \mathrm{Tr}_e(P_c)$$

here we take similar symbol for the trace map of E.

Then we obviously have norm compatibility,

Proposition 9.1. *Let T_e as usual be a Hecke operator, then*

$$Tr_e(Q_c) = T_p(Q_d).$$

Consequently, for E,

$$Tr_e(P_c) = a_p(f)P_d$$

It's almost direct by checking definition. Also we have congruence relation

Proposition 9.2. *Each prime factor λ_c of e in H_c divides a unique prime λ_d in H_d, and we have the congruence $P_c = Frob(\lambda_d)P_d(\mod \lambda_c)$. Here Frob is the Frobenius map.*

Use this Euler system , Kolyvagin could construct certain Galois cohomology classes, which was used to give partial proof of Gross's conjecture.

Theorem 9.3. *Let $P_K = Tr_{H_1/K}P_1 \in E(K)$. Suppose P_K is not torsion, then $E(K)$ is of rank 1..*

The attractiveness of this theorem is that, the existence of a point of infinite order (big) surely implies the the rank$E(K) > 1$, but it unexpectedly bounds the rank to be 1 (small). This is the magic of Heegner points.

In this case, by(12)(12'), $P_K \in E(\mathbb{Q})$ if and only if $\varepsilon = 1$, and by (13) and only if $L(E, 1)$ vanish to an odd order. And $P_K \in E(\mathbb{Q})$ implies the rank$E(\mathbb{Q}) = 1$.

The other side of Gross's conjecture was solved by the Gross-Zagier-Zhang formula, which relates the Euler system to the special value of $L(E/K, s)$. Recall our definition of Heegner vectors.

Theorem 9.4. *Let P_K be defined as in theorem 9.3, $P_{n,\chi}$ be the Heegner vector of P_n, we have*

$$1,\ \langle P_K, P_K\rangle = L'(E/K, 1);$$
$$2,\ \langle P_{n,\chi}, P_{n,\overline{\chi}}\rangle = L'(E/K, \chi, 1).$$

Thus, P_K is not torsion if and only if $L'(E/K, s)$ dose not vanish; $P_{n,\chi}$ non-zero if and only if $L'(E/K, \chi, s)$ dose not vanish .

This two theorem can be combined to prove the following theorem concerning the BSD conjecture

Theorem 9.5. *If E is an elliptic curve over $\mathbb{Q}$ and $ord_{s=1}L(E, s) \leq 1$, then*

$$rank(E(\mathbb{Q})) = ord_{s=1}L(E, s).$$

Part of this theorem is easier. When $ord_{s=1}L(E, s) = 1$, $\varepsilon = 1$ and non-torsion Heegner must belongs to $E(\mathbb{Q})$, so, rank$E(\mathbb{Q}) = 1$. Some results about L-series of elliptic curves guarantee that we can choose a K satisfying heegner Hypothesis and $L(E/K, s)$ vanish to order 1. Then Theroem 9.4 implies P_K is not torsion, so rank$E(\mathbb{Q}) = 1$.

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一类特殊的非线性算子的一致有界定理

孙奥*

一致有界定理，有时也称为共鸣定理，是泛函分析中一个最基本的定理。定理叙述如下

定理 1. 假设 E,F 是两个 *Banach* 空间，设 $\{T_i\}_{i\in I}$ 是一族 $E \to F$ 的连续线性算子。

如果有

$$\sup_{i\in I} \|T_i x\| < \infty \quad \forall x \in E$$

那么就有结论

$$\sup_{i\in I} \|T_i\| < \infty$$

从定理的叙述我们可以明白为什么这个定理被称为一致有界定理。

一致有界定理是对线性算子成立的。那么是否能够推广到一般的（非线性）算子呢？从定理的叙述我们知道，我们应该恰当地定义非线性算子的"有界"。同样，由于非线性算子太过丰富，即使是加上连续性的条件，仍然不一定能够有类似于一致有界定理的条件。所以我们还要恰当地给予非线性算子一些限制。本文即试图从这几个方面出发考察非线性算子的情况。

1 主结果与证明

以下，我们总假设 E, F 是 Banach 空间，并且为了便于说明，均设为实线性空间，$\|\dot{\|}$ 是其上的范数。另外以下提到的所有算子除非特殊声明都是 $E \to F$ 的连续映射。我们用 $B(x,r), x \in E, r > 0$ 表示集合 $\{y : \|y - x\| \leqslant r\}$

定义 1. 我们称一个算子 T 是齐性的，如果它满足对任意 $x \in E$，均有

$$T(x) = \|x\| T(\tfrac{x}{\|x\|})$$

*数 11

容易验证，齐性算子（不一定线性）构成了一个线性空间。而且我们知道，所有的线性算子都是齐性算子。

定义 2. 设 T 是一个齐性算子，我们称 $\sup\limits_{|x|=1}\|T(x)\|$ 为其范数。

并且在不混淆的情况下，也记作 $\|T\|$

事实上，可以验证这个范数就是齐性算子空间的范数。但是我们这里用不到这个结论，所以我们以后提 T 的范数仅仅是用于指其某种意义上的“界”。

定义 3. 我们称一个齐性算子 T 是有界的，当它的范数是有限的。并且此时容易验证，有

$$\|T(x)\| \leqslant \|T\|\|x\| \quad \forall x \in E$$

尽管齐性算子与线性算子有很多相似之处，但是我们仍然不能得出类似于线性算子的齐性算子的一致有界定理。下面是一个例子

例 1. 考虑复数域 $\mathbb{C}$ 作为 2 维的实线性空间。利用 *Urysohn* 引理可以在单位圆周 S^1 上构造连续实值函数 $g_n(n \geqslant 2)$，使得其值域为 $[0, n]$，且满足

$$g_n(e^{\frac{2\pi i}{n}}) = n$$

$$g_n(e^{2\pi i\theta}) = 0, \forall\theta \in [0, \tfrac{1}{2n}]\bigcup[\tfrac{3}{2n}, 1]$$

定义一族非线性算子 $\{T_n\}_{n=1}^{\infty}$ 是 $\mathbb{C} \to \mathbb{R}$ 的算子，满足

$$T_n(x) = \|x\|g_n(\tfrac{x}{\|x\|})$$

于是这一族算子是齐性的，并且每一个都是有界的。由于对于任意的 $x \in \mathbb{C}$，仅有有限个 g_n 使得 $g_n(\frac{x}{\|x\|} \neq 0)$，故

$$\sup_{i\in\mathbb{Z}}\|T_i x\| < \infty \quad \forall x \in E$$

但是由于 $\|T_n\| = n$，我们知道 $\lim\limits_{n} \to \infty\|T_n\| = \infty$。从而这一族算子不满足一致有界定理。

下面我们引入一个特殊的条件。

定义 4. 我们称一个线性算子 T 是 k-控制的，如果 $\forall x, y \in E$，有

$$\|T(x+y)\| \leqslant k\max\{\|T(x)\|, \|T(y)\|\}$$

由于 $\|T(2x)\| = 2\|T(x)\|$，我们知道这里的 $k \geqslant 2$。

特别的，对于线性算子 L，由范数不等式 $\|L(x+y)\| \leqslant \|L(x)\| + \|L(y)\|$，我们知道线性算子是 2-控制的。

可以验证，上一个例子中的算子不是 k-控制的（$\forall k > 0$）。

例 2. 由范数不等式，我们知道 E 上的范数就是一个 2-控制的齐性非线性算子。因此我们知道，以上的定义都不是平凡的（即满足这些性质的算子最终只能是线性算子）。

下面是我们的主定理

定理 2. 如果一族齐性 k-控制算子（$k \geqslant 2$）$\{T_i\}_{i\in I}$ 满足

$$\sup_{i\in I}\|T_i x\| < \infty \quad \forall x \in E$$

那么就有结论

$$\sup_{i\in I}\|T_i\| < \infty$$

$Proof$

先证明一个引理：设 T 是一个齐性 k-控制算子，则 $\forall x \in E, r > 0$ 我们有

$$\sup_{x'\in B(x,r)} \|Tx'\| \geqslant \tfrac{r}{k}\|T\|$$

引理的证明：$\forall x \in E, \forall y$ 使得 $\|y\| \leqslant 1$，由 k-控制的条件，我们有

$$\|T(ry)\| \leqslant k\max\{\|T(ry+x)\|, \|T(x)\| \leqslant k \sup_{x'\in B(x,r)} \|Tx'\|\}$$

上式左端对所有的满足要求的 y 取上确界，即有

$$r\|T\| \leqslant k \sup_{x'\in B(x,r)} \|Tx'\|$$

整理即得引理成立。

回到原命题，现设 $\alpha > 0$ 是使得 $\alpha > 2k^2+1$ 的正实数。我们用反证法，假设结论不对，那么 $\sup\limits_{i\in I}\|T_i\| = \infty$

我们可以从中选出一个子列 $\{T_n\}_{n=1}^{\infty}$，使得 $\|T_n\| \geqslant (\alpha+1)^n$

下面我们选取 E 中的一个子列 $\{x_n\}_{n=1}^{\infty}$ 如下：首先取 $x_0 = 0$。对 $n \geqslant 1$，假设我们已经取好了 $x_0, x_1, \cdots, x_{n-1}$，利用引理，我们可以选择 x_n 使得

$$\|x_n - x_{n-1}\| \leqslant \alpha^{-n}，且 \|T_n x_n\| \geqslant \tfrac{1}{2k}\alpha^{-n}\|T_n\|$$

注意到，由于 $k \geqslant 2$ 我们有 $\alpha > 9$，从而 $\{x_n\}_{n=1}^{\infty}$ 是 *Cauchy* 列。假设其收敛到点 $x \in E$。那么我们有

$$\|x - x_n\| = \lim_{m\to\infty}\|x_m - x_n\| \leqslant \lim_{m\to\infty}\alpha^{-n-1} + \alpha^{-n-2} + \cdots + \alpha^{-m} \leqslant \alpha^{-n}\tfrac{1}{\alpha-1}$$

现在考虑 $\|T_n x_n\| \leqslant k\max\{\|T_n x\|, \|T_n(x_n - x)\|\}$，而我们又有

$$k\|T_n(x_n - x)\| \leqslant k\|T_n\|\|x_n - x\| \leqslant k\alpha^{-n}\tfrac{1}{\alpha-1}\|T_n\|$$

又 $\|T_n x_n\| \geqslant \frac{1}{2k}\alpha^{-n}\|T_n\|$，于是由 $\alpha > 2k^2+1$，我们有

$$\|T_n x_n\| > k\|T_n(x_n - x)\|$$

从而只可能 $\|T_n x_n\| \leqslant k\|T_n x\|$，由此我们得到

$$\|T_n x\| \geqslant \tfrac{1}{k}\|T_n x_n\| \geqslant \tfrac{1}{2k^2}\alpha^{-n}\|T_n\| \geqslant \tfrac{1}{2k^2}(\tfrac{\alpha+1}{\alpha})^n$$

则有 $\lim_{n\to\infty}\|T_nx\|=\infty$，与假设矛盾。故反设不成立。从而定理得证。

推论 1. 由于有界线性算子均是齐性 2-控制算子，所以特别地，我们得到线性算子的一致有界定理。

2 注记

2.1

这个证明主要思想来自参考文献 [1]。齐性、k-控制等名词都是我自己命名的，不知道学术界是否已有固定的名称。

2.2

现在常见的一致有界定理的证明有两种。参考文献 [2] 中列出来的是一个用 Baire 定理的证明。而历史上真正最早出现的是 Hahn 的被称为"gliding hump" 的方法，那是与参考文献 [1] 类似的方法。

事实上，在连续算子的情况下，本文的结论利用参考文献 [1] 的证明是很容易得到的。本文真正有趣的结论在于，主定理的证明完全没有利用连续性。于是完全照抄之前的证明，我们可以得到这样的定理

定理 3. 如果一族齐性有界 k-控制算子（$k\geqslant 2$）$\{T_i\}_{i\in I}$ 满足

$$\sup_{i\in I}\|T_ix\|<\infty\quad\forall x\in E$$

那么就有结论

$$\sup_{i\in I}\|T_i\|<\infty$$

这个结论对<u>不连续的齐性 k-控制算子</u> 也对。

注意，不连续的齐性 k-控制算子是存在的。

例 3. 设 $k>2$，T 是 $\mathbb{C}\to\mathbb{R}$ 的算子，满足

$$T(z)=kz\text{，当 }\Re(z)\Im(z)>0$$
$$T(z)=z\text{，当 }\Re(z)\Im(z)\leqslant 0$$

则 T 是一个不连续的齐性 k-控制算子。而且是有界的。

对于不连续的情况，参考文献 [2] 中的证明由于要用到连续算子的闭集原像仍然是闭集这样一个条件，所以失效了。而本文中的主定理的证明完全没有用到连续性，故依旧成立。

当然，由于对于线性算子，连续与有界是等价条件，从而没有这种区别。但是非线性算子，这两者不再等价。这也体现了非线性算子的特殊之处。

2.3

最后稍微说一下怎么发现这个条件的。参考文献 [2] 中的证明可以得到一个廉价的推广：

只要连续算子 T 满足齐性以及次线性：

$$\|T(x+y)\| \leqslant \|Tx\| + \|Ty\|, \forall x, y \in E$$

那么就有一致收敛定理。

我当时觉得次线性这个推广太廉价了，就想找一个更加强的推广。可以证明 k-控制这个条件比次线性要强。次线性可以推出 2-控制，但是反过来不一定。

例 4. 设 $k > 2$，T 是 $\mathbb{C} \to \mathbb{R}$ 的算子，满足

$$T(z) = kz，\text{当 } \Re(z)\Im(z) > 0$$

$$T(z) = z，\text{当 } \Re(z)\Im(z) \leqslant 0$$

这里 $\Re$ 表示一个复数的实部，$\Im$ 表示一个复数的虚部。则 T 是齐性的 k 控制算子。但是由 $|T(1+i)| = k\sqrt{2} > |T(1)| + |T(i)|$，可知 T 不是次线性算子。

在我寻找各种例子的过程中，发现许多例子都是不连续的算子。这使得我在阅读了参考文献 [1] 后，发现推广到不连续算子的这个结果更加强一些。

当然，除开这一点来说，参考文献 [1] 中的证明也是很漂亮的不利用纲性的证明。

3

关于齐性这个条件，事实上很多文献中指的是下面的定义

定义 5. 称 T 是实线性空间 E 中齐性算子，若对任意的 $\alpha \in \mathbb{R}, x \in E$，有

$$T(\alpha x) = \alpha T(x)$$

注意到这个条件可以推出本文开头的条件，但反过来不对（考虑取范数这个齐性算子）。但如果没有 k-控制这样的条件，这一类算子依旧不能推出类似于一致有界定理。只需将例 1 中的例子稍作修改，使得其中的每个算子的取值关于原点中心反对称即可。

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墨尔本交流感受

林艺儿*

2014 年 7 月 5 日，我们数学系六人来到澳大利亚墨尔本参加为期两周的墨尔本夏令营。本次夏令营主要是为了向我们展示数学、物理、化学中的前沿进展以及不同学科之间的交叉联系。经过长达 11 小时的飞行，抵达了这座世界最宜居住的城市。墨尔本位于南半球，刚下飞机，一阵寒意袭来，赶紧换上冬装。我们的住处是墨尔本大学的学生宿舍，单人间虽然不大，但干净整洁各种设施俱全。

本来双休日食堂是不设午餐的，但考虑到我们旅途劳顿，特意为我们加开了午餐，使我们倍感温暖。之后两天参观了校园和实验室，墨大校园虽然不大，但干净而典雅，由于正值寒假，校园里的学生并不多。

利用晚上的休息时间，我们坐电车去了市中心。墨尔本的交通十分便利，利用电车和火车可以到达城市和郊外的每一个角落。墨尔本是一个融合了各国文化的城市。走在市中心街头，可以看到欧洲、亚洲各种不同的面孔，各国风味的餐馆，在这一个城市里就可以品尝到世界各地的美食。汉语是这里的第二语言，城市里华人很多，走在街上也随处可见各种汉语的标语和招牌，听见人们用中文交谈，倍感亲切。

白天的时间我们主要是在实验室跟着老师做实验，或者上课。数学课上两位教授分别介绍 Markov 链在 PageRank 问题上的应用，以及证明了平面对称群共有 17 种，这两种问题都结合了理论和生活，十分有趣。

双休日是自由时间，因此我们利用这两天游玩了城里城外值得去的地方。第一去了战争纪念馆，皇家植物园和水族馆。战争纪念馆位于皇家植物园旁边，用于悼念为国捐躯的澳洲人。植物园很大，里面各种植物枝繁叶茂，感觉仿佛进入了原始森林，美景令人心旷神怡。值得一提的墨尔本的天气，虽然是蓝天白云，但不时下起淅淅沥沥的小雨。相比之下，水族馆就显得较为一般，除了一些在中国看不到的珍奇动物，比不上国内的水族馆。

晚登上墨尔本最高的建筑 skydeck88，在这座 88 层建筑俯瞰被灯光点缀的整座城市，美不胜收。虽然墨尔本的商店晚上 5 点（周四周五会晚一些）关门，但居然都不关灯。原

*数 23 班

来是政府为了防盗，规定“人走灯亮”。

第二天我们从 Belgrave 乘坐去郊外的 puffing billy 火车，虽然墨尔本的火车很多，铁轨四通八达，但像 puffing billy 这辆以蒸汽作为动力的是独一无二的。火车穿过原始森林，到达郊外。与传统火车不同的是，乘坐时可以坐在“窗沿”上将双脚伸出去，火车运行时，一排排树木飞快地向后移去，清风拂面，可以看到火车头的滚滚蒸汽。火车经过之地，路人都会向火车里的游客招手，十分友好。只不过坐的时间长了难免会感觉寒冷（7 月是墨尔本最冷的时候），郊外 Landlake 十分幽静，向森林深处走去，呼吸新鲜的空气，在北京待久了我都快遗忘了清新空气的味道，因此不管吸多少口都感觉不够。待了两个小时，返程的火车来了，这才恋恋不舍地离开。

每天的晚上都是自由时间，还去了很多有趣的地方，例如世界第三大摩天轮 Melbourne Star，Victory Market。此外我们还用了一整天的时间去了动物园，与袋鼠和树懒等澳洲国宝亲密接触。两周的时间虽然不长，但也使我们很好的感受了墨尔本这座城市的科学、文化和风土人情，我对这座城市有很好的印象，如果有机会，我还会再来的。

征稿启事

《荷思》是清华大学数学系学生自主创办的数学学术刊物，面向各个院系中对数学感兴趣的本科生及研究生。

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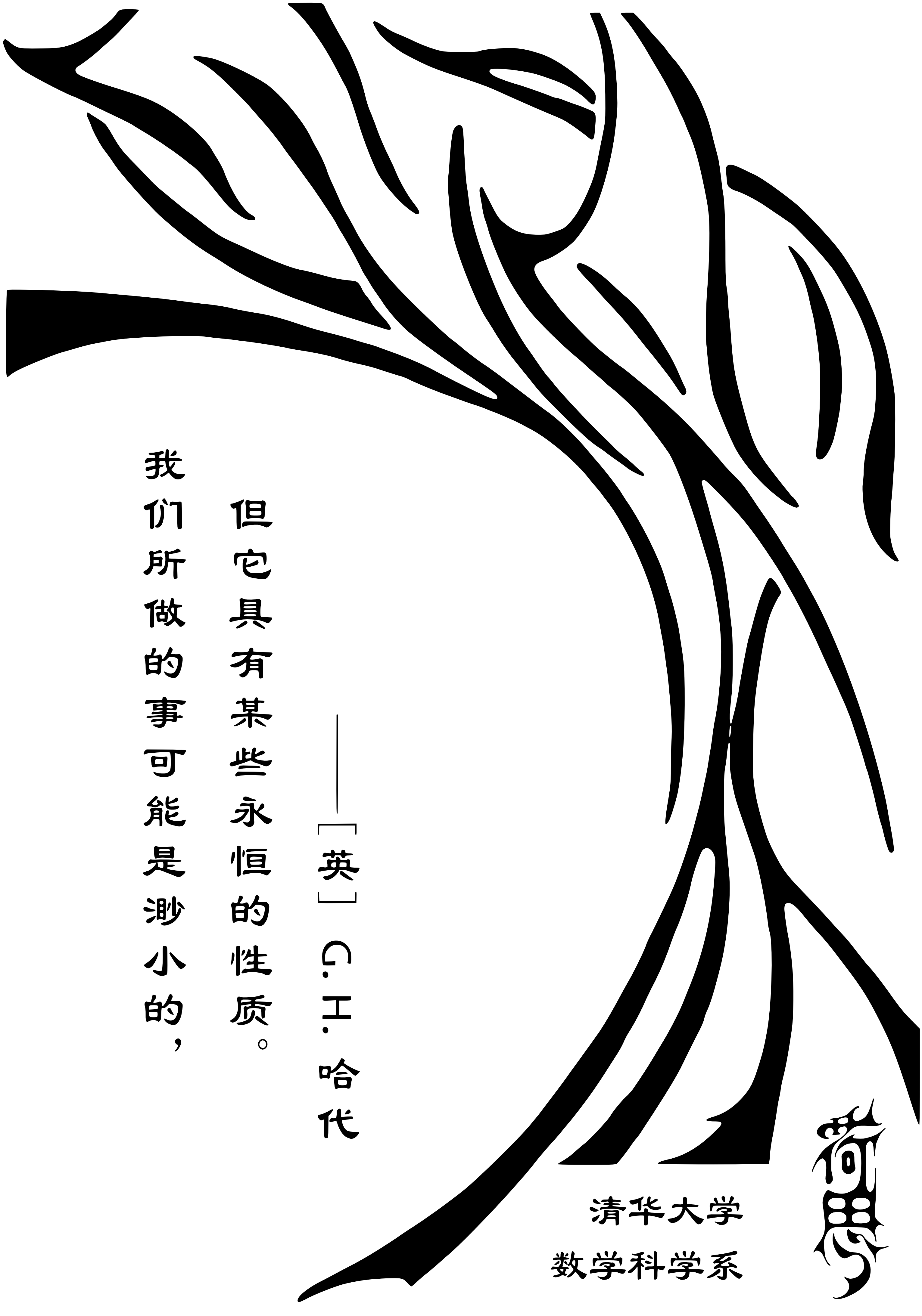

我们所做的事可能是渺小的，
但它具有某些永恒的性质。

——［英］G.H. 哈代

清华大学
数学科学系