# Discrete Fourier Transform of Images

## 图像的离散傅立叶变换

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# Discrete Fourier transform (DFT)

1-D:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

$$u = 0, 1, 2, \ldots, M-1$$

2-D:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

$$u = 0, 1, 2, \ldots, M-1 \text{ and } v = 0, 1, 2, \ldots, N-1$$

# How to program DFT on images?

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$u = 0, 1, 2, \dots, M-1$ and $v = 0, 1, 2, \dots, N-1$

Using Euler's formula, $e^{j\theta} = \cos\theta + j\sin\theta$

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)[\cos[-2\pi(ux/M + vy/N)] + j\sin[-2\pi(ux/M + vy/N)]]$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} [\underbrace{f(x,y)\cos[-2\pi(ux/M + vy/N)]}_{\text{Real part } R(u,v)} + j\,\underbrace{f(x,y)\sin[-2\pi(ux/M + vy/N)]}_{\text{Imaginary part } I(u,v)}]$$

$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Amplitude spectrum： $|F(u,v)| = [R(u,v)^2 + I(u,v)^2]^{1/2}$

Phase spectrum： $\phi(u,v) = \tan^{-1}[\frac{I(u,v)}{R(u,v)}]$

# Before DFT : shift centre

2-D

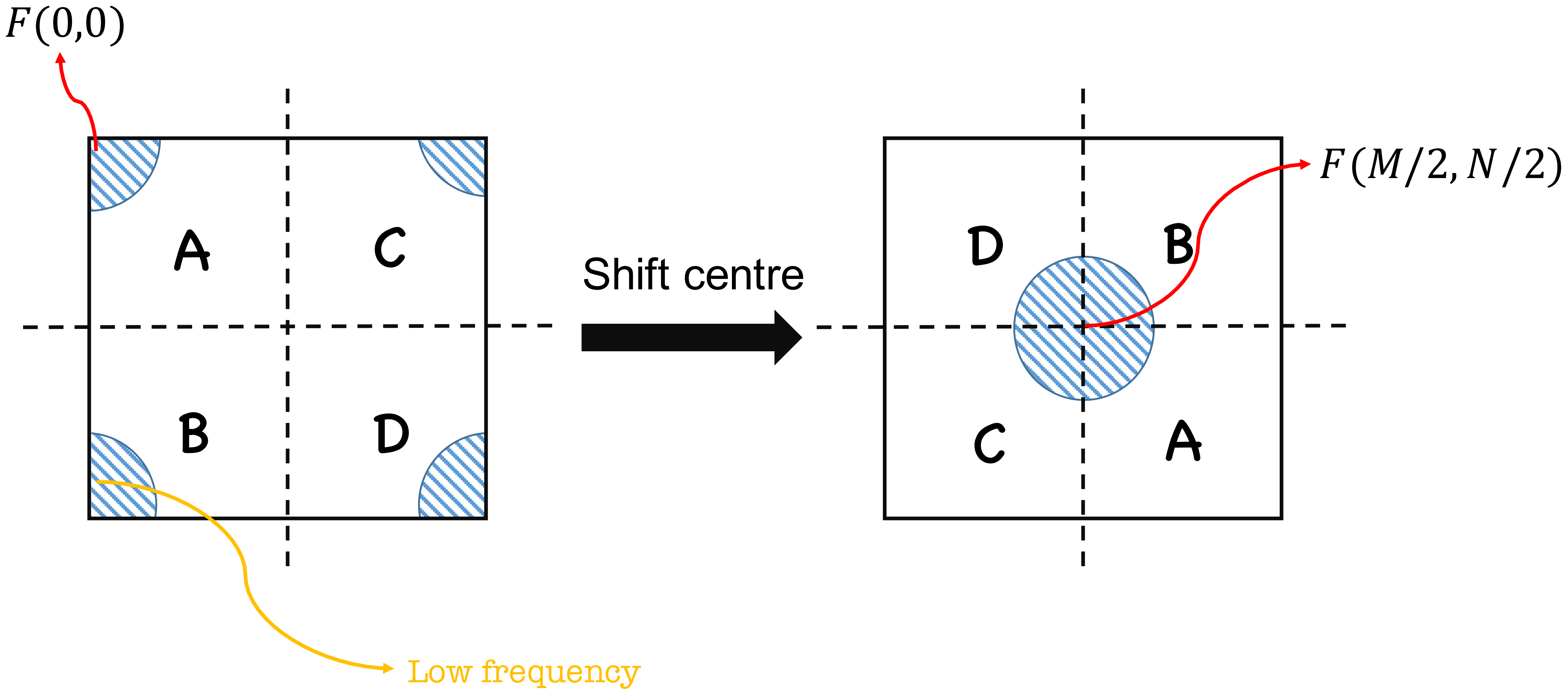
$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-\frac{M}{2}, v-\frac{N}{2})$$

$$f(x,y)e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u-u_0)$$

Let $u_0 = M/2$

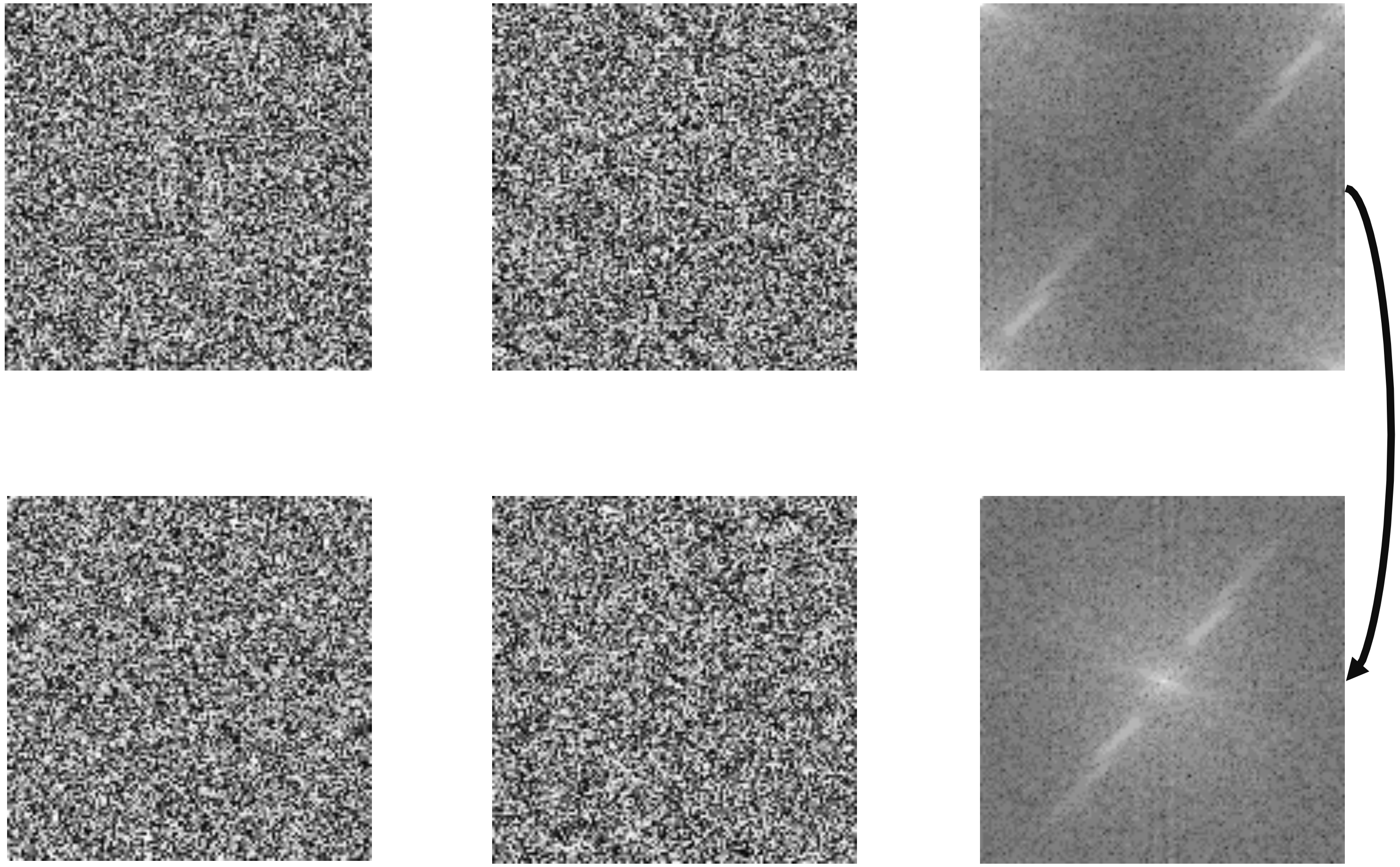
$$f(x,y)(-1)^{x} \Leftrightarrow F(u-M/2)$$

1-D


$F(0,0)$
A
C
B
D
Shift centre
$F(M/2, N/2)$
D
B
C
A
Low frequency

# let's look at an example !

Original image



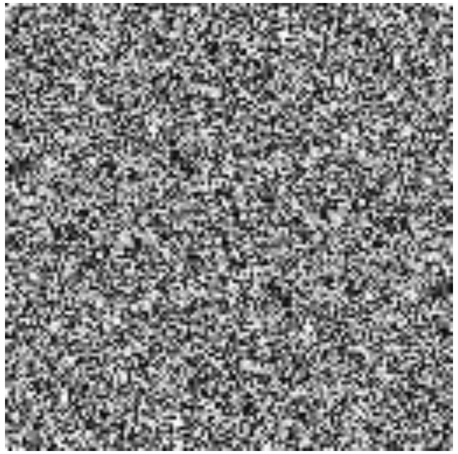
Shift centre

Real part

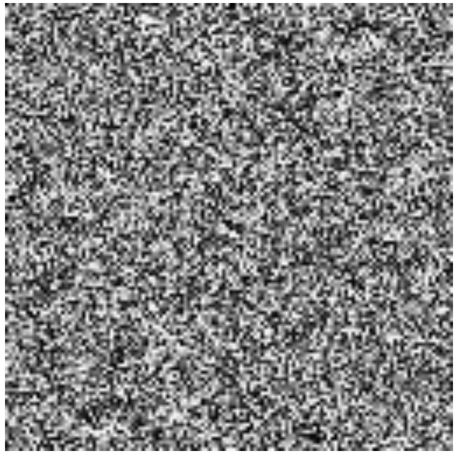
Imaginary part

Final result

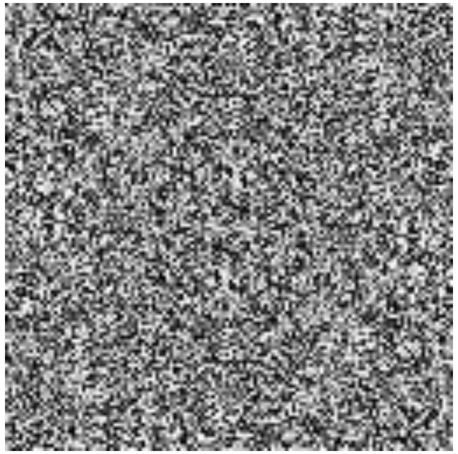
# After DFT : normalization



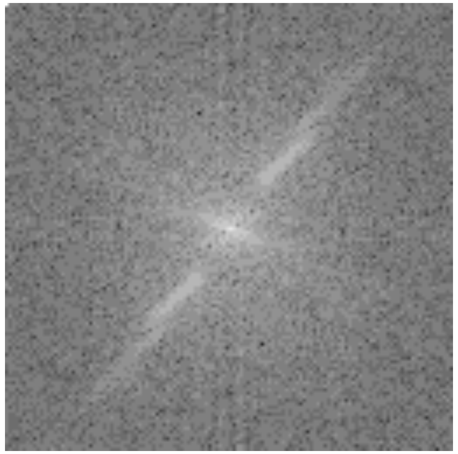
Real part



Imaginary part



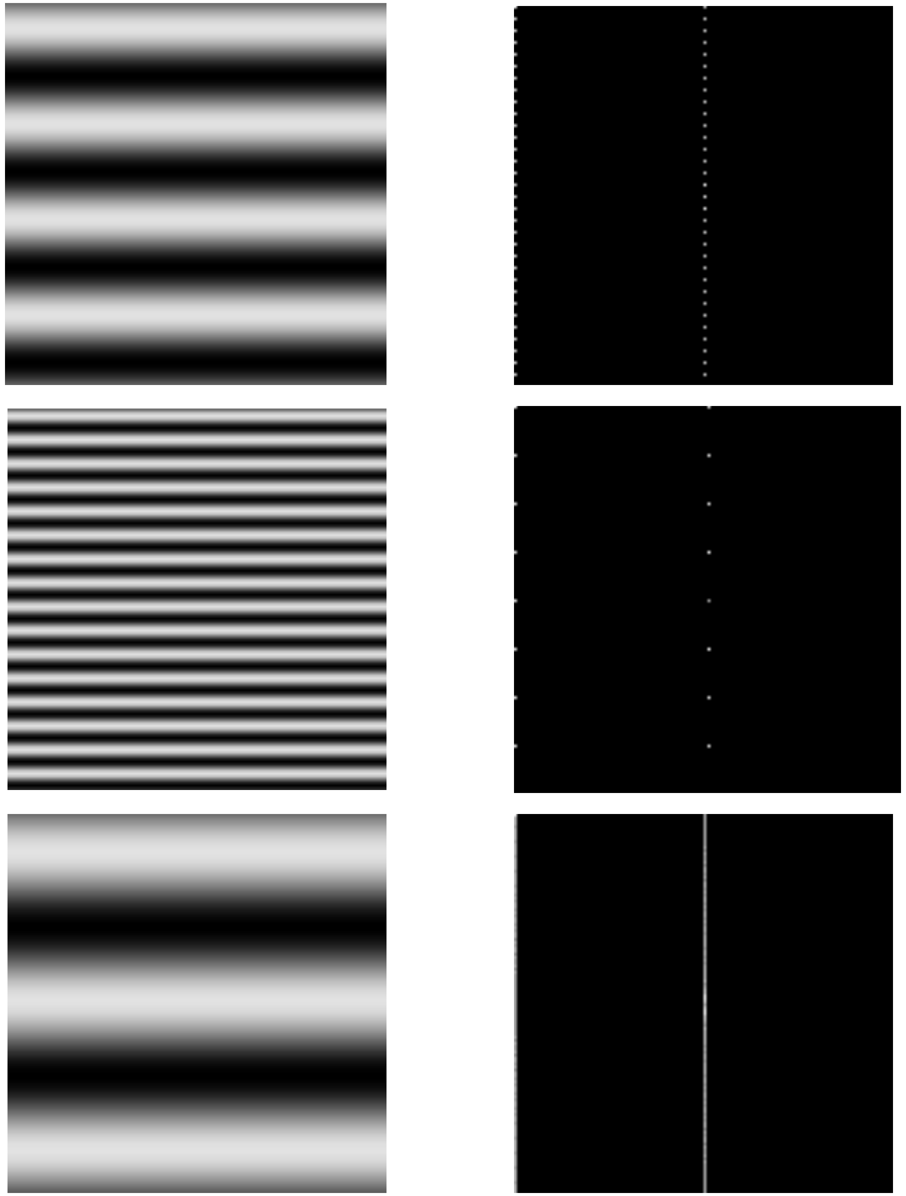
After DFT

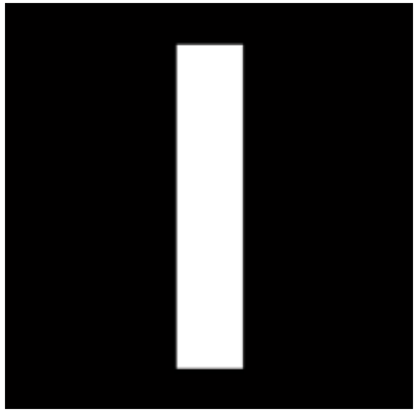


Final result

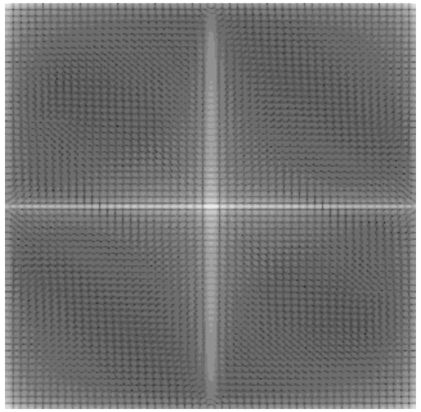
# More examples :

sin image Final result

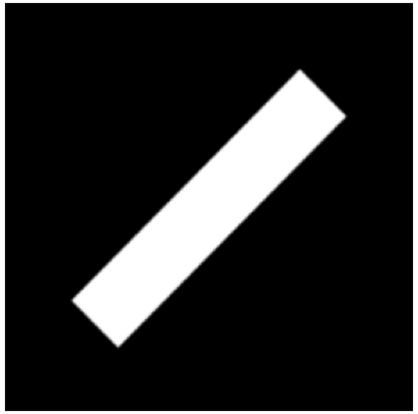




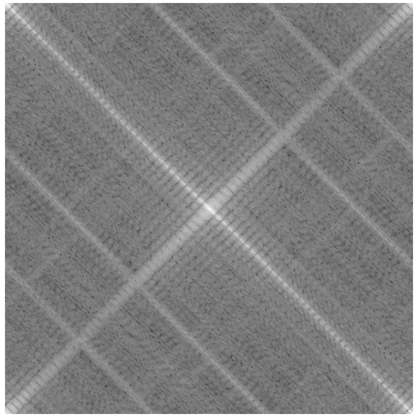
Original image



Final result

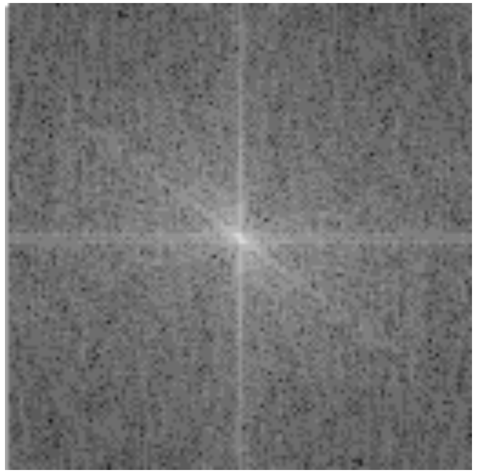


Original image
after rotation



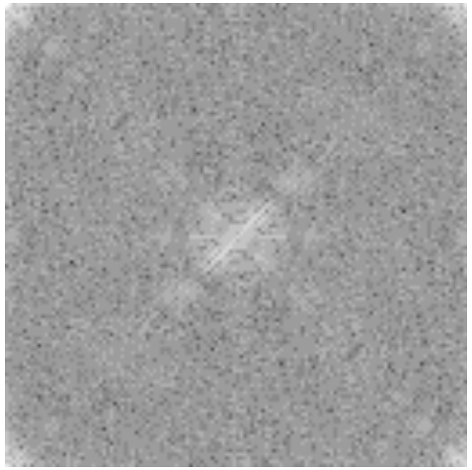
Final result

Original image



Final result

Original image



Final result

# Inverse Discrete Fourier transform (IDFT)

1-D:

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}$$

$$x = 0, 1, 2, \dots, M-1$$

2-D:

$$f(x, y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v)e^{j2\pi(ux/M+vy/N)}$$

$$x = 0, 1, 2, \dots, M-1 \text{ and } y = 0, 1, 2, \dots, N-1$$

# How to program IDFT on images?

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$(\ F(u,v) = R(u,v) + jI(u,v)) = |F(u,v)|e^{j\phi(u,v)})$$

Using Euler's formula, $e^{j\theta} = \cos\theta + j\sin\theta$

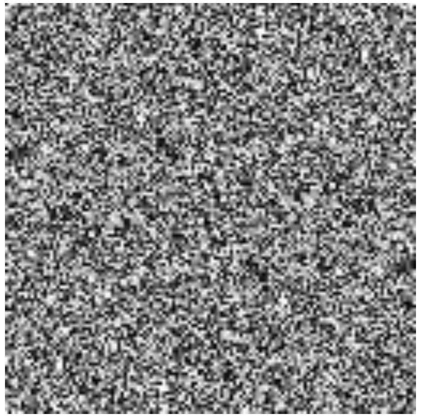
$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}[R(u,v) + jI(u,v)][\cos[2\pi(ux/M + vy/N)] + j\sin[2\pi(ux/M + vy/N)]]$$

$$\Downarrow$$

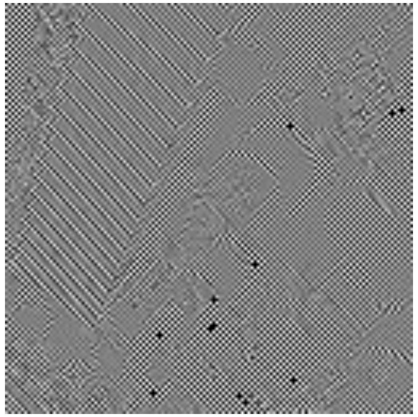
$$f(x,y) = R(x,y) + jI(x,y)$$

$$|f(x,y)| = [R(x,y)^2 + I(x,y)^2]^{1/2}$$

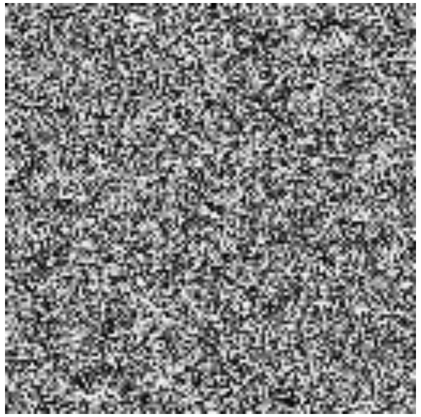
By doing shift to obtain the original image: $f(x,y) = \frac{(-1)^{x+y}|f(x,y)|}{M}$



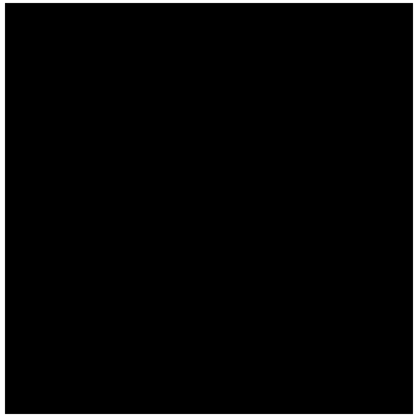
Real part



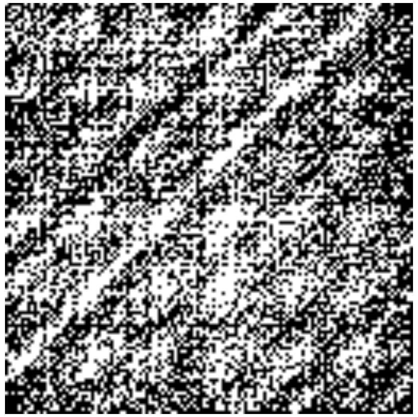
Real part after IDFT



Imaginary part



Imaginary part after IDFT

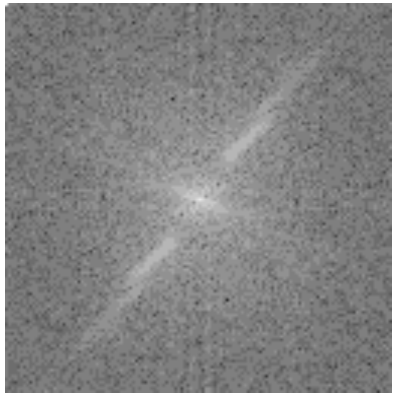


Before shift

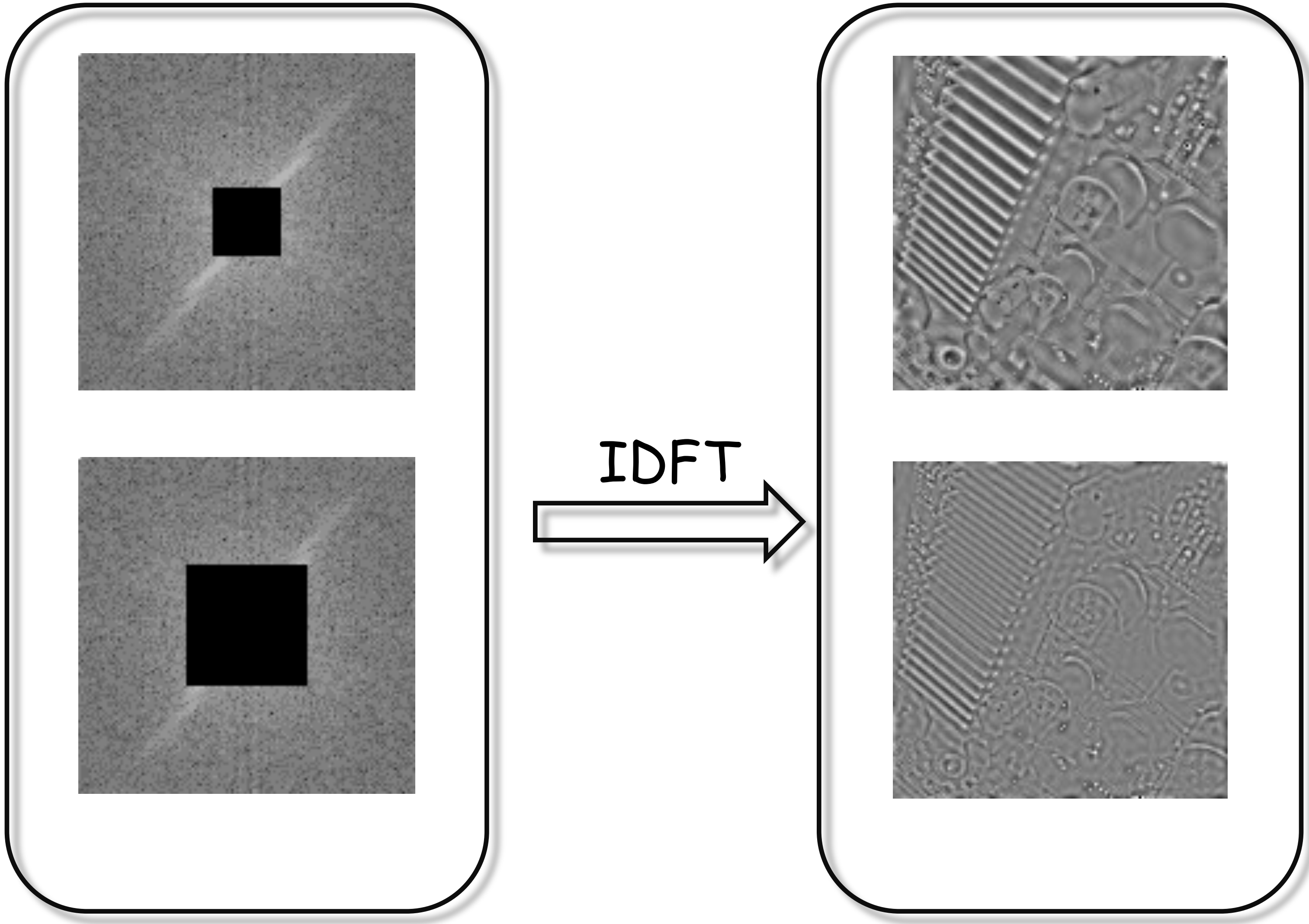
Final result

# The Physical Meaning of FT:

Original image

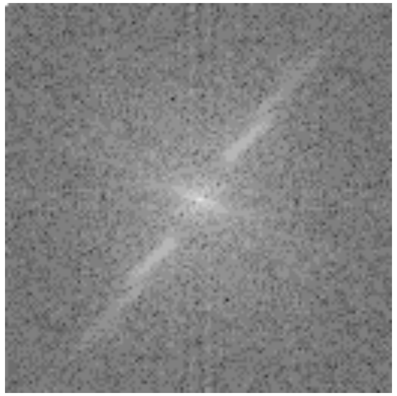


Amplitude map

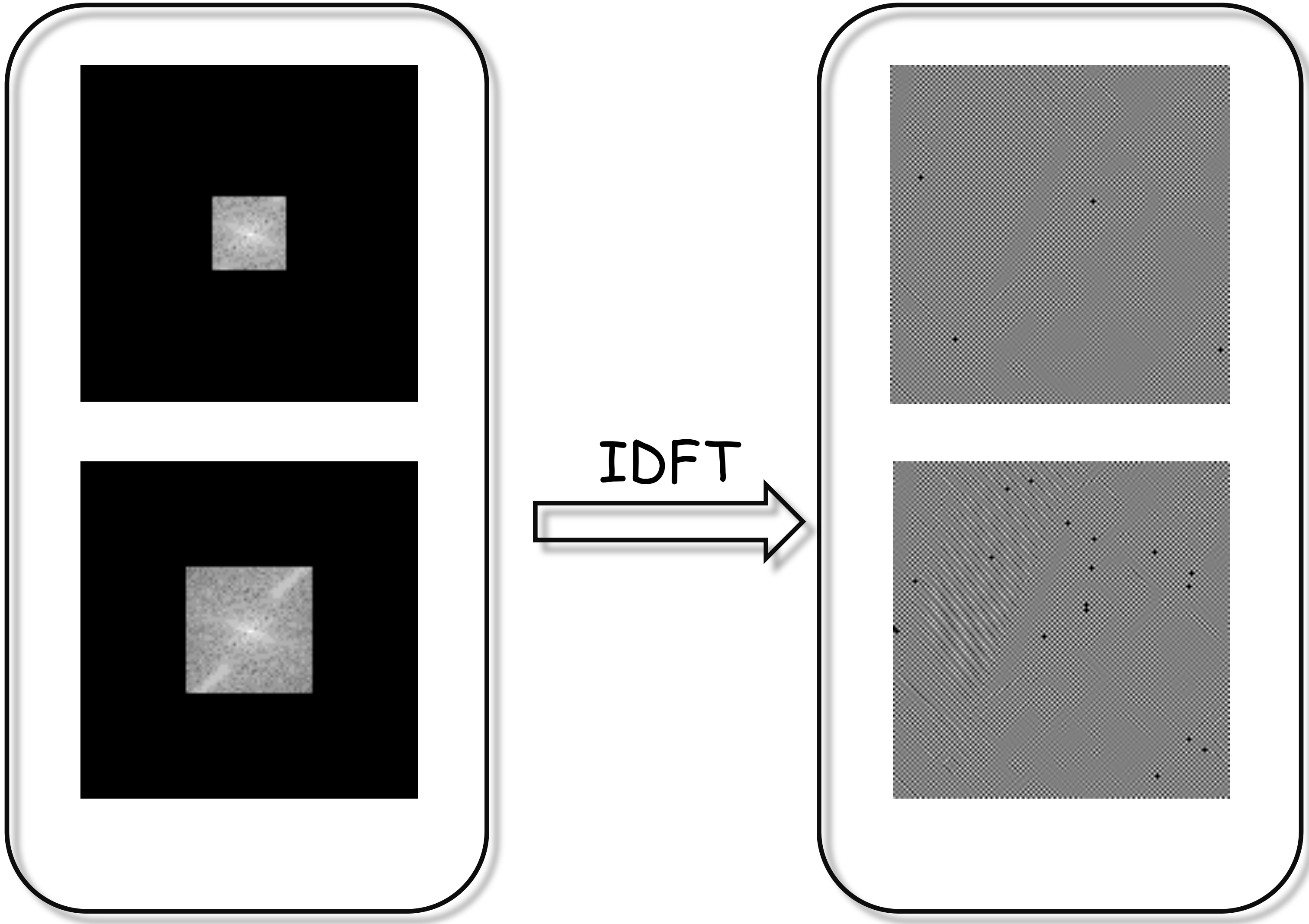


# The Physical Meaning of FT:

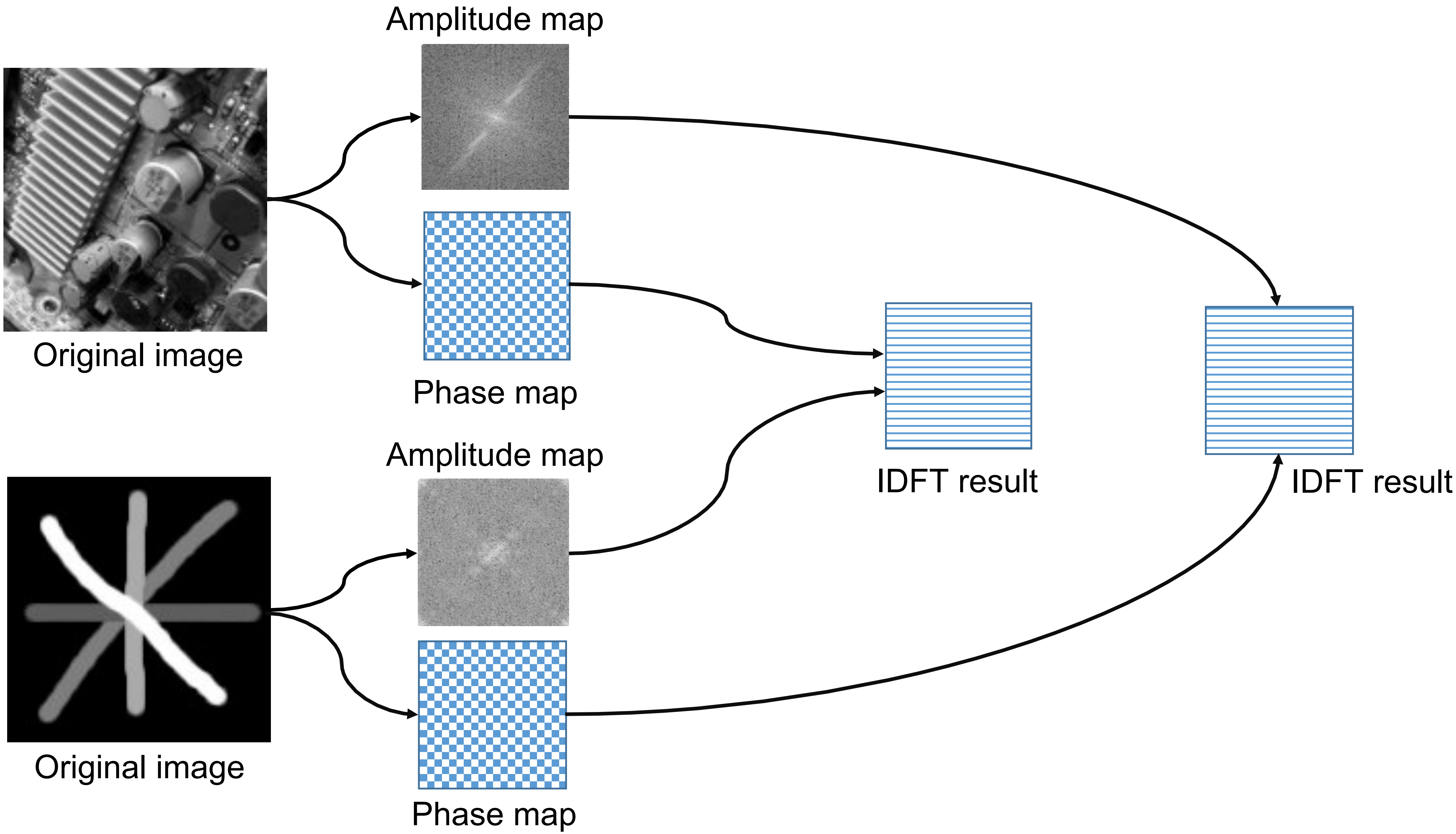
Original image



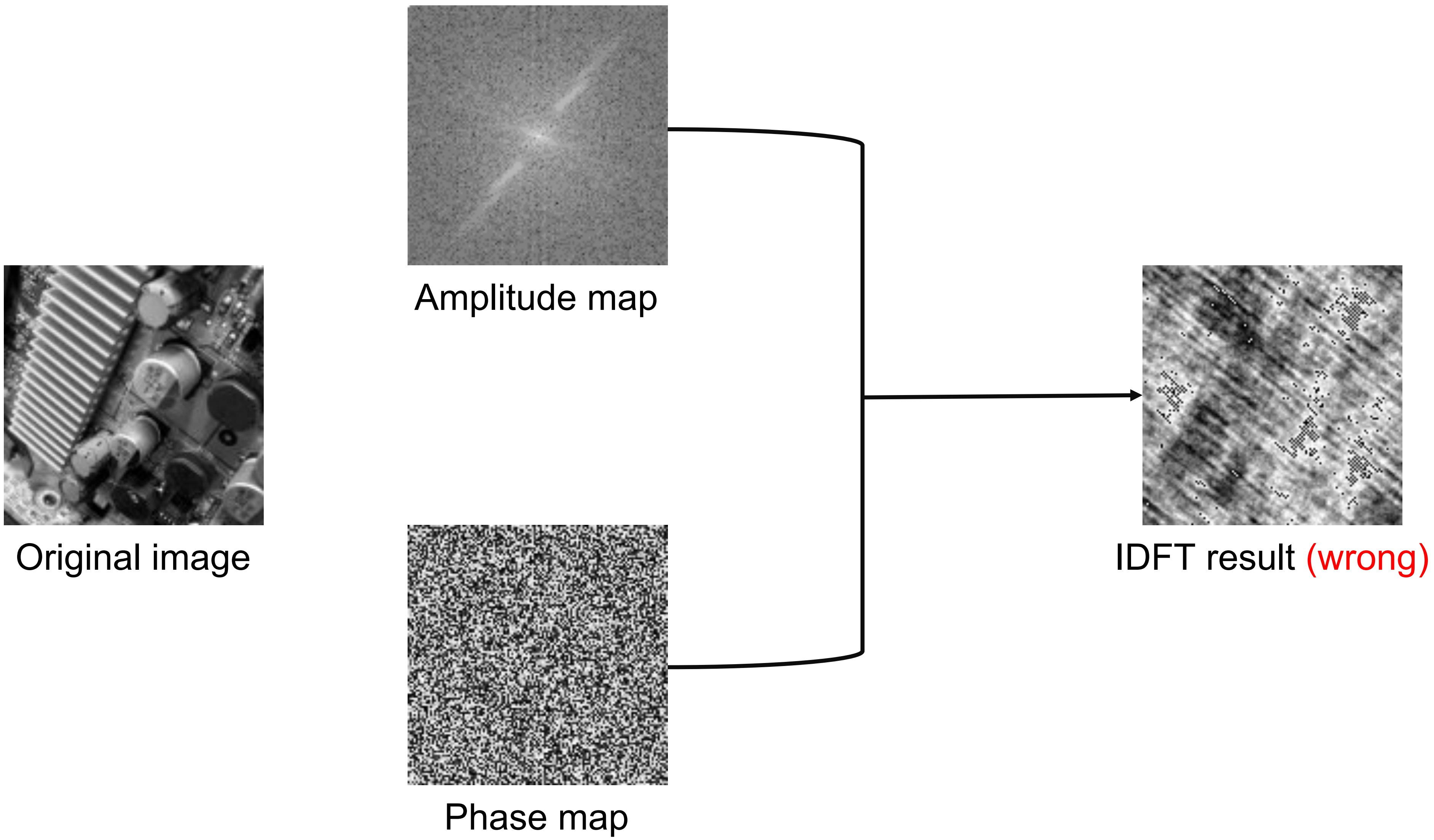
Amplitude map



# The Physical Meaning of FT (to be solved):

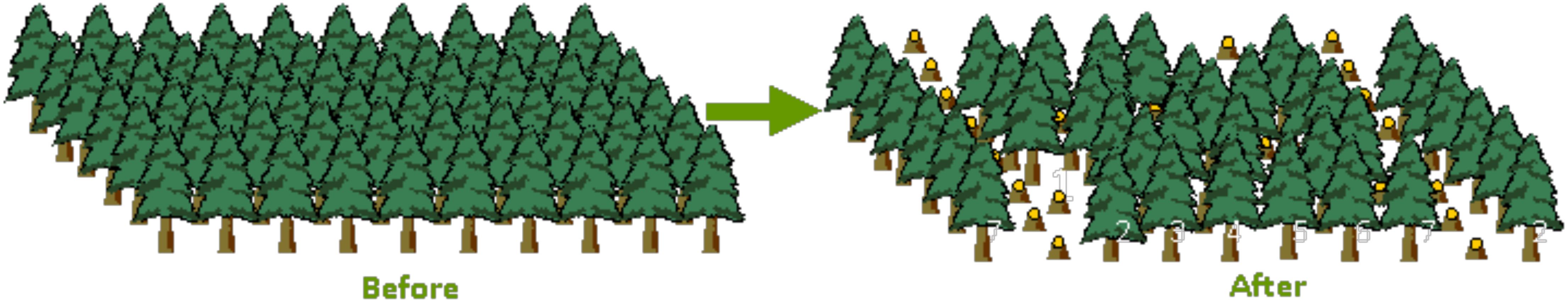


# My Wrong Result:



# Thinning:

Thinning is a morphological operation that is used to remove selected foreground pixels from binary images. It is particularly useful for skeletonization.




Thinning is just like that.

# One iteration:

| | | |
|---|---|---|
| $P_9$ | $P_2$ | $P_3$ |
| $P_8$ | $P_1$ | $P_4$ |
| $P_7$ | $P_6$ | $P_5$ |

In the first subiteration:

(a) $2 \leq B(P_1) \leq 6$

(b) $A(P_1) = 1$

(c) $P_2 * P_4 * P_6 = 0$

(d) $P_4 * P_6 * P_8 = 0$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_9$ |

In the second subiteration:

(a) $2 \leq B(P_1) \leq 6$

(b) $A(P_1) = 1$

(c) $P_2 * P_4 * P_8 = 0$

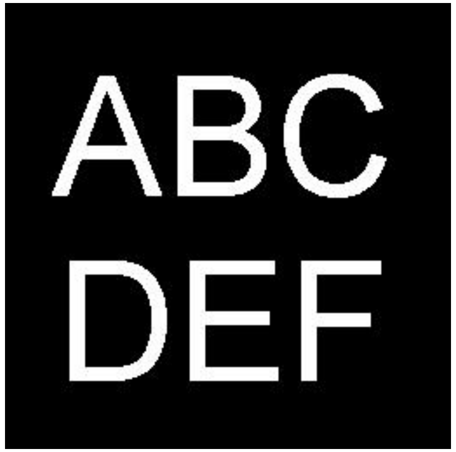
(d) $P_2 * P_6 * P_8 = 0$

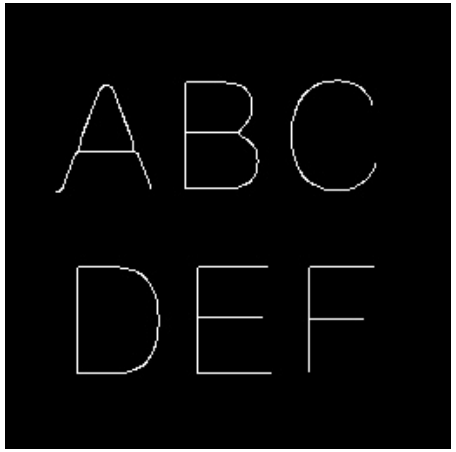
Zhang T Y, Suen C Y. A fast parallel algorithm for thinning digital patterns[J]. Communications of the ACM, 1984, 27(3): 236-239.

# Examples:

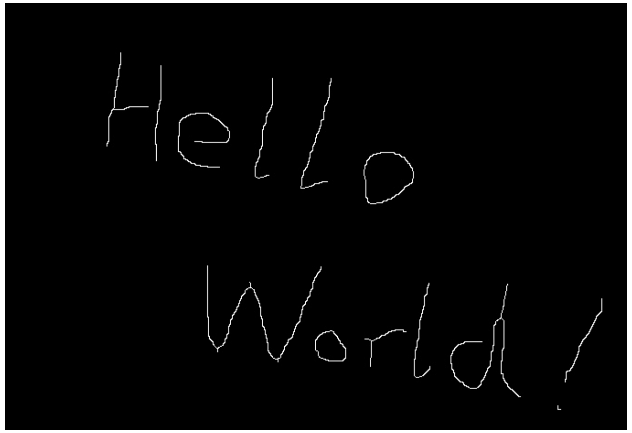
Original image

Final result







Thanks