Assignment 3

1. $$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 1 \Leftrightarrow 3$$

$$P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 2 & -1 & -1 & 0 \\ 1 & \frac{5}{3} & 1 & 1 \\ -2 & 1 & -4 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & \frac{1}{3} & -4 & \frac{2}{3} \end{bmatrix}}_{U_1}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 2 \Leftrightarrow 3$$

$$P_2 P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & \frac{5}{3} & 1 & 1 \\ 2 & -1 & -1 & 0 \\ -2 & 1 & -4 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & \frac{1}{3} & -4 & \frac{2}{3} \end{bmatrix}}_{U_1}$$

$\cancel{P_3 = I_4}$

$$\cancel{P_3 P_2 P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & \frac{5}{3} & 1 & 1 \\ 2 & -1 & -1 & 0 \\ -2 & 1 & -4 & 0 \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{6} & 1 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{5}{6} & -\frac{5}{9} \\ 0 & 0 & -\frac{25}{6} & \frac{5}{9} \end{bmatrix}}_{U_2}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad 3 \Leftrightarrow 4$$

$$P_3 P_2 P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & \frac{5}{3} & 1 & 1 \\ -2 & 1 & -4 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 1 & 0 \\ \frac{2}{3} & -\frac{1}{6} & 0 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{25}{6} & +\frac{5}{9} \\ 0 & 0 & -\frac{5}{6} & -\frac{5}{9} \end{bmatrix}}_{U_2}$$

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$$P_3 P_2 P_1 A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 1 & 0 \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{5} & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{25}{6} & \frac{5}{9} \\ 0 & 0 & 0 & -\frac{2}{3} \end{bmatrix}}_{U_3}$$

$$\cancel{P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$PA = L_3 U_3$$

Problem 2

(a) $n=1$, obvious

we assume it's true for $n \times n$

Consider $(n+1) * (n+1)$ matrix A

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & a_{n+1,n+1} \end{pmatrix}$$

A_{11} is a $n \times n$'s upper triangular, A_{12} is a $n*1$ vector, $a_{n+1,n+1}$ is a non-zero value.

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} a_{n+1,n+1}^{-1} \\ 0 & a_{k+1,k+1}^{-1} \end{pmatrix}$$

According to induction hypothesis, A_{11}^{-1} is an upper Δ, then obviously A^{-1} is an upper Δ

(b) A is an lower Δ

then A^T is an upper trangular

$$A=(A^T)^T$$

$$A^{-1}=\left[(A^T)^T\right]^{-1}=\left[(A^T)^{-1}\right]^T$$

Because $(A^T)^{-1}$ is an upper Δ

$\left[(A^T)^{-1}\right]^T$ is a lower Δ

the diagonal element will not change because there is no row exchange or row multiply.

Problem 3.

According to Hint:

$$[A\ b]=[Q_r\ q]\begin{pmatrix} R & p \\ 0 & \rho \end{pmatrix} \Longrightarrow \begin{cases} Q_r R = A \\ Q_r p + \rho q = b \end{cases}$$

$\therefore \min \|Ax-b\|_2 \Longrightarrow$

$$\min \|Q_r R_x - (Q_r p + \rho q)\|_2 = \min \|Q_r(Rx-p) - \rho q\|_2$$

$\because Q_r^T(\rho q) = \rho Q_r^T q = 0$

$\therefore Q_r(Rx-p) \perp \rho q$

$\therefore \|Q_r(R_x-p)-\rho q\|_2 = \sqrt{\|Q_r(Rx-p)\|_2^2+\|\rho q\|_2^2}$

to minimize it, we need $\|Q_r(R_x-p)\|_2^2$ to reach its minimum. Because Q_r is Orthogonal Matrix, $\|Q_r(Rx-p)\|_2^2 = \|Rx-p\|_2^2$

$\|\rho q\|_2^2 = \rho^2 \|q\|_2^2 = \rho^2$

$\therefore \min \|Ax-b\|_2 = \min \|Rx-p\|_2^2 + \rho^2$

when $Rx=p$, we reach the minimum and Now $\|Ax-b\|_2 = |\rho|$

Problem 4

(a) the first colum of L is obviously all -1 because all the element below 1 is -1. and we will find the second column doesn't change after we do $E_1 W = U_1$, so the second column, so the second column in L is $\begin{bmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$. then we happily find column 3 of U_2 is no different than the original W, so the third column in L is $\begin{bmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$ as well, ... till the last column. But when we reach the last column, the i-th element doubles every time I do elimination and it double i times, so the value is 2^{i-1} for i-th element.

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(b) $\|W\|_\infty = n$ clearly

$$\|W^{-1}\|_\infty = \left| \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} \right|$$

$$= \left| \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}} + \frac{1}{2^{n-1}} \right|$$

$$= 1$$

$\therefore$ Condition $(W) = n$, ill-conditioned because n can $\to \infty$

(c) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

So follow this mode

L^{-1} would be $$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 2 & 1 & 1 & \cdots & 0 \\ 4 & 2 & 1 & \cdots & 0 \\ 2^{n-2} & 2^{n-1} & \cdots & \cdots & 1 \end{bmatrix}$$

$\|L\|_\infty = n$ clearly,

$$\|L^{-1}\|_\infty = 2^{n-2} + 2^{n-1} + \cdots + 1 = 2^{n-1}$$