Problem 1

for (A):

$fl(\bar{x}) = \frac{8090}{4} = 2.02250 \times 10^3$ no rounding here

$fl(\Sigma(x_i - \bar{x})^2) = 5.00000 \times 10^0$ no rounding

$fl(\sigma^2) = \frac{1}{3} \times 5.0 = 1.66667 \times 10^0$

for (B)

$$fl(\sum_{i=1}^{n} x_i)^2 = 4.08444 \times 10^6 + 4.08848 \times 10^6 + 4.09253 \times 10^6 + 4.09658 \times 10^6$$

$$= (8.17292 + 4.09253 + 4.09658) \times 10^6$$

$$= 1.63620 \times 10^7$$

$$fl(n\bar{x}^2) = fl(n) \odot fl(\bar{x}^2)$$

$$= 4 \odot 4.09051 \times 10^6$$

$$= 1.63620 \times 10^7$$

$$fl(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2) = 0$$

$$fl(\sigma^2) = 0$$

Problem 2.

From the output of Python Script it can be shown. explicit representation. shows obvious noise at around $x=3$, this is because expanded polynomial contains too many add/subtract operations

and exponentiation operation

Problem 3

Observation: The exact solution, given by $X_k = \frac{1}{3} \times 4^{1-k}$, shows a consistent decaying trend. But the recursion method, though math very closely for the early values of k, begins to increase for larger values of k (around 20), due to the accumulated rounding errors.

(c) when $k = 1$ and 2,

$$X_1 = \frac{1}{7}(4b-a) + \frac{4}{7}(2a-b) = a.$$

$$X_2 = b$$

$$\therefore X_k = \frac{2^{k-1}}{7}(4b-a) + \frac{4^{2-k}}{7}(2a-b) \text{ for } k=1,2.$$

Denote $X_k = \frac{2^{k-1}}{7}(4b-a) + \frac{4^{2-k}}{7}(2a-b)$ holds true for $k = 1, 2 \cdots n$.

For $k = n+1$, $\frac{2^n}{7}(4b-a) + \frac{4^{1-n}}{7}(2a-b) =$

$$\frac{9}{4} \cdot \frac{2^{n-1}}{7}(4b-a) + \frac{9}{4}\frac{4^{2-n}}{7}(2a-b) - \frac{1}{2}\frac{2^{n-2}}{7}(4b-a) - \frac{1}{2}\frac{4^{3-n}}{7}(2a-b)$$

$\therefore$ For $\forall k$, $X_k = \frac{2^{k-1}}{7}(4b-a) + \frac{4^{2-k}}{7}(2a-b)$

(d) Recursion solution deviates from the exact solution due to accumulated rounding errors

To be specific, when continuously recursing, there are $X_k < UFL$, and exist as UFL in f-p system, then X_{k+1} thus get a bigger value then X_k, and pass on

(e) $X_{k+1} = \frac{9}{4} X_k - \frac{1}{2} X_{k-1}$

I hope $(X_{k+1} - aX_k) = b(X_k - aX_{k-1})$

$$X_{k+1} = (a+b)X_k - abX_{k-1}$$

$$\begin{cases} a+b = \frac{9}{4} \\ ab = \frac{1}{2} \end{cases} \Rightarrow \begin{matrix} a=2 \\ b=\frac{1}{4} \end{matrix} \text{ or } \begin{matrix} a=\frac{1}{4} \\ b=2 \end{matrix}$$

① $X_{k+1} - 2X_k = \frac{1}{4}(X_k - 2X_{k-1})$

$$= \left(\frac{1}{4}\right)^{k-1}(X_2 - 2X_1)$$

$$= \left(\frac{1}{4}\right)^{k-1}(b-2a)$$

② $X_{k+1} - \frac{1}{4}X_k = 2(X_k - \frac{1}{4}X_{k-1})$

$$= 2^{k-1}(X_2 - \frac{1}{4}X_1)$$

$$= 2^{k-1}(b - \frac{1}{4}a)$$

② − ①: $\frac{7}{4}X_k = 2^{k-1}(b - \frac{1}{4}a) - \left(\frac{1}{4}\right)^{k-1}(b-2a)$

$X_k = \frac{2^{k+1}}{7}(b - \frac{1}{4}a) - \frac{4^{2-k}}{7}(b-2a)$, Done.

Problem 4

(a) $\hat{y} = fl((1+\varepsilon_1)a^2 - (1+\varepsilon_2)b^2)$

$$= (1+\varepsilon_3)((1+\varepsilon_1)a^2 - (1+\varepsilon_2)b^2)$$

$$= (1+\varepsilon_3\varepsilon_1+\varepsilon_1+\varepsilon_3)a^2 - (1+\varepsilon_2\varepsilon_3+\varepsilon_2+\varepsilon_3)b^2)$$

$$|\hat{y}-y| = |(\varepsilon_3\varepsilon_1+\varepsilon_1+\varepsilon_3)a^2 - (\varepsilon_2\varepsilon_3+\varepsilon_2+\varepsilon_3)b^2|$$

$$\le |\varepsilon_3+\varepsilon_1+\varepsilon_3\varepsilon_1|a^2 + |\varepsilon_2\varepsilon_3+\varepsilon_2+\varepsilon_3|b^2$$

$$\le |\varepsilon_3+\varepsilon_1+\varepsilon_3\varepsilon_1| \max(a^2,b^2) + | \qquad | \max(a^2,b^2)$$

$\because |\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \le \varepsilon_{mach} \longrightarrow 0$

$\therefore$ $\cancel{\le 2\max(a^2,b^2)}$

$$|\varepsilon_3+\varepsilon_1+\varepsilon_3\varepsilon_1| \le |\varepsilon_3|+|\varepsilon_1|+|\varepsilon_1\varepsilon_3|$$

$$\le 2\varepsilon_{mach} + (\varepsilon_{mach})^2$$

$$= O(\varepsilon_{mach})$$

Similarly, $|\varepsilon_2\varepsilon_3+\varepsilon_2+\varepsilon_3| \le O(\varepsilon_{mach})$

$$\therefore |\hat{y}-y| \le 2O(\varepsilon_{mach})\max\{a^2,b^2\} = O(\varepsilon_{mach})\max\{a^2,b^2\}$$

(b)

we want $|y|$ to be small, so

a and b are close

and we want $|\hat{y}-y|$ to be big,

so $a = 0.6666 = 6.666\times10^{-1}$

$b = 0.6665 = 6.665\times10^{-1}$

$\hat{y} = (a\odot a)\ominus(b\odot b) = 4.444\times10^{-1}\ominus4.442\times10^{-1}$

$= 0.002\times10^{-1}$

$= 0.0002$

$y = [(6.666)^2-(6.665)^2]\times10^{-2}$

$= [44.435556-44.422225]\times10^{-2}$

$= 0.013331\times10^{-2}$

$= 0.00013331$

$|\hat{y}-y| = 0.00006669 = 6.669\times10^{-5}$

$|y| = 1.3331\times10^{-4}$

$$\frac{|\hat{y}-y|}{|y|} = \frac{6.669}{1.3331\times10} = \frac{6.669}{13.331} = 0.5002\cdots > \frac{1}{2}$$

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(c) Let's try $\tilde{y} = \tilde{f}(a, b) = a^2 \ominus b^2$,

then $(1+\varepsilon)(a^2 - b^2) = \tilde{y}$, ε slighly smaller than ε

$$\frac{|\tilde{y} - y|}{|y|} = \frac{|\varepsilon (a^2 - b^2)|}{|a^2 - b^2|} = \varepsilon \le \varepsilon_{mach}$$

$$\therefore \text{ so } \frac{|\tilde{y} - y|}{|y|} \approx O(\varepsilon_{mach})$$

using the $a = 0.6666 = 6.666 \times 10^{-1}$

$b = 0.6665 = 6.6665 \times 10^{-1}$

$$\tilde{y} = 1.333 \times 10^{-4}$$

$$y = 1.3331 \times 10^{-4}$$

$$\frac{|\tilde{y} - y|}{|y|} = \frac{0.0001 \times 10^{-4}}{1.3331 \times 10^{-4}} = \varepsilon \approx O(\varepsilon_{mach})$$