A SPECIAL VERSION OF EXCESS INTERSECTION FORMULA

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In this note, we discuss a special version of the excess intersection formula [2, Theorem 6.3] which is used in the computation of rational Chow rings of Hurwitz spaces with marked ramification [?], and sketch a proof of [1, Theorem 13.9].

1. Excess intersection formula

We assume all varieties are smooth to avoid viewing the Chern classes as operations via cap products. Consider a fiber diagram

$$\begin{tikzcd}
X'' \arrow[r, "i''"] \arrow[d, "q"] & Y'' \arrow[d, "p"] \\
X' \arrow[r, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}$$

where both i and i' are regular embeddings with codimension d and d' respectively. The excess intersection formula reveals how two refined Gysin maps $i^!$ and $i'^!$ are related in terms of the excess normal bundle of the bottom square, which is $g^*N_{X/Y}/N_{X'/Y'}$. To be explicit, for any class $\alpha \in A_b(Y'')$, the excess intersection formula reads

$$i^!\alpha = q^*c_{d-d'}(g^*N_{X/Y}/N_{X'/Y'})(i'^!\alpha) \in A_{b-d}(X'').$$

In particular, if the top square is the identity square, *i.e.,* both q and p are identity maps and thus $i'' = i'$, then for any class $\beta \in A_b(Y')$,

$$i^!\beta = c_{d-d'}(g^*N_{X/Y}/N_{X'/Y'})(i'^*\beta).$$

Thus if X' has expected codimension in Y', meaning that $d' = d$, or equivalently, the excess normal bundle $g^*N_{X/Y}/N_{X'/Y'}$ is the trivial bundle on X', then the refined Gysin pullback $i^!$ agrees with the standard pullback i^*, *i.e.,* $i^!\beta = i'^*\beta$, which is an important case of the compatibility of refined Gysin maps. In general, X' may have a connected component C which does not have expected codimension in Y' (but regularly embedded in Y'), which is the case when the corresponding excess normal bundle has positive rank. In this case, such component C is called an "excess component" and the class $c(g^*N_{X/Y}/N_{C/Y'})$ provides the corrected term in the refined Gysin pullback $i^!\beta$. If C is not regularly embedded in Y' then we replace the normal bundle $N_{C/Y'}$ by the normal cone $C_{C/Y}$ and consider its Segre class.

Moreover, if we assume $Y'' = Y'$ has pure dimension k, then

$$i^![Y'] = c_{d-d'}(g^*N_{X/Y}/N_{X'/Y'}) \in A^{d-d'}(X') \cong A_{k-d}(X'),$$

which is the precisely how the refined Gysin pullback $i^![Y']$ is computed.

The refined Gysin map can be further carried over to local complete intersection morphisms. If i and i' are local complete intersection morphisms, the excess intersection formula still holds, but

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with different definition for excess normal bundle. Consider a fiber diagram

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

where we only assume that f is a local complete intersection morphism of constant codimension d. Then f admits a factorization $f = p \circ \iota : Y' \xrightarrow{\iota} W \xrightarrow{p} Y$ where $\iota : Y' \to W$ is a regular embedding and $p : W \to Y$ is a smooth morphism. Thus we have the following fiber diagram where ι and τ are both regular embeddings and p and q are smooth morphisms:

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "\tau"] \arrow[dd, bend right=60, "g"'] & Y' \arrow[d, "\iota"'] \arrow[dd, bend left=60, "f"] \\ V \arrow[r, "j"] \arrow[d, "q"] & W \arrow[d, "p"'] \\ X \arrow[r, "i"] & Y. \end{tikzcd}$$

The refined Gysin map for f is defined to be $f^! = \iota^! \circ q^*$. In order to make the same formula for excess intersection holds, the excess normal bundle of the outer square with respect to f is now defined to be the excess normal of the top square with respect to ι:

$$i'^* N_{Y'/W} / N_{X'/V}.$$

Therefore for any class $\beta \in A_b(X)$, the excess intersection formula for local complete intersection morphisms reads

$$f^! \beta = \left\{ c(i'^* N_{Y'/W} / N_{X'/V})(g^* \beta) \right\}_{b-d} \in A_{b-d}(X').$$

2. Proof of [1, Theorem 13.9]

In [1], Theorem 13.9 give a (weaker) version of excess intersection formula, which mixes refined Gysin maps for both regular embeddings and local complete intersection morphisms. The key point for the theorem to be true is that once the refined Gysin maps are defined, the excess normal bundle is independent of the orientation of its corresponding fiber square.

Theorem. [1, Theorem 13.9] *Suppose that $Z \subseteq X$ is a smooth subvariety of a smooth variety X, and that $\pi : X' \to X$ is a morphism from another smooth variety. Let $Z' = \pi^{-1}(Z)$ and assume that Z' is smooth, with connected components C_α (of dimension c_α). Write $i : Z \to X$ and $i'_\alpha : C_\alpha \to X'$ for the inclusion maps, and likewise π_α for the restriction of π to C_α. For any class $\beta \in A_b(Z)$,*

$$\pi^*(i_* \beta) = \sum_\alpha (i'_\alpha)_* \{ \pi_\alpha^* (\beta c(N_{Z/X})) s(C_\alpha, X') \}_{b + \dim X' - \dim X}.$$

Proof. Note that $\pi : X' \to X$ is a morphism between two smooth varieties, it is a local complete intersection morphism, thus it admits a factorization $\pi = \rho \circ \delta : X' \to Y \to X$ where $\delta : X' \to Y$ is a regular embedding and $\rho : Y \to X$ is a smooth morphism. Consider the following cartesian diagram:

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "\delta'"] \arrow[dd, bend right=60, "\pi'"'] & X' \arrow[d, "\delta"'] \arrow[dd, bend left=60, "\pi"] \\ Y' \arrow[r] \arrow[d, "\rho'"] & Y \arrow[d, "\rho"'] \\ Z \arrow[r, "i", "\text{regular}"'] & X. \end{tikzcd}$$

For any class $\beta \in A_b(Z)$, by the compatibility of refined Gysin maps with proper pushforward and flat pullback, we have

$$\pi^*(i_*\beta) = i'_*\pi^!\beta = i'_*(\delta^!\rho'^*\beta) \tag{1}$$

where $\pi^!$ is given by the refined Gysin map extended to local intersection morphisms, and the excess normal bundle with respect to π of the outer square is $i'^*N_{X'/Y}/N_{Z'/Y'}$. In particular if i' is also a regular embedding, for example, when Z' is irreducible, then the excess normal bundle of π in fact equals to the excess normal bundle with respect to the regular embedding i, which is given by $\pi'^*N_{Z/X}/N_{Z'/X'}$. These two quotient bundles have the same rank because smooth morphism ρ and its base change ρ' share the same (constant) relative dimension. In other words, the excess normal bundle doesn't depend on which map we take its Gysin pullback. In this case by the excess intersection formula, the expression (1) can be further written as

$$\begin{aligned}
i'_*(\delta^!\rho'^*\beta) &= i'_*\left\{c\left(i'^*N_{X'/Y}/N_{Z'/Y'}\right)\left(\delta'^*(\rho'^*\beta)\right)\right\}_{b-(\dim X-\dim X')} \\
&= i'_*\left\{c\left(i'^*N_{X'/Y}/N_{Z'/Y'}\right)\pi'^*(\beta)\right\}_{b-(\dim X-\dim X')} \\
&\overset{(\star)}{=} i'_*\left\{c\left(\pi'^*N_{Z/X}/N_{Z'/X'}\right)\pi'^*(\beta)\right\}_{b-(\dim X-\dim X')} \\
&= i'_*\left\{\pi'^*\left(\beta c(N_{Z/X})\right)c(N_{Z'/X'})^{-1}\right\}_{b-(\dim X-\dim X')}.
\end{aligned}$$

The key equality $(\star)$ is precisely due to the fact that the excess normal bundle is independent of the orientation of the (outer) fiber square provided that the refined Gysin maps for the bottom and rightmost morphisms are defined. Therefore in general, when $Z' = \sqcup_\alpha C_\alpha$ is decomposed as a disjoint union of connected components C_α (where these C_α's are not necessarily regularly embedded in X'), applying the formula to each component we obtain

$$\pi^*(i_*\beta) = \sum_\alpha (i'_\alpha)_*\left\{\pi'^*_\alpha\left(\beta c(N_{Z/X})\right)s(C_\alpha, X')\right\}_{b-(\dim X-\dim X')}. \qquad \square$$

3. A special version of excess intersection

In this section we give a special version of excess intersection formula which is used in the computation of rational Chow ring of the Hurwitz space $\mathcal{H}_{3,g}((3))$ with marked triple ramification [?].

The fiber diagram considered is the following

$$\begin{tikzcd}
T \arrow[r, "\tau"] \arrow[d, "q"] \arrow[dd, bend right, "j|_T"'] & S \arrow[d, "p"'] \arrow[dd, bend left, "f_2|_S"] \\
W \arrow[r, "i"] \arrow[d, "j"] & X_2 \arrow[d, "f_2"'] \\
X_1 \arrow[r, "f_1"] & Y,
\end{tikzcd}$$

where both f_1, f_2 are regular embeddings with codimension d_1 and d_2, and X_1, X_2 have pure dimension k_1 and k_2 respectively. Thus $k_1 + d_1 = k_2 + d_2 = \dim Y$. In addition we assume that p, q are both open immersions. We also assume for now that i and j are regular embeddings which will help state the results.

As in [1, Theorem 13.9], we aim to find the formula of $(f_2|_S)^*(f_1)_*[X_1]$ using refined Gysin maps. First note that [1, Theorem 13.9] can be directly applied here by viewing $f_2|_S$ as a morphism between smooth varieties S and Y. On the other hand, we can also write $f_2|_S = f_2 \circ p$ and apply excess intersection to the bottom square. To be explicit,

$$(f_2|_S)^*(f_1)_*[X_1] = \tau_*(f_2|_S)^![X_1] \tag{2}$$

$$= p^* f_2^*(f_1)_*[X_1] = p^* i_*(f_2^![X_1]). \tag{3}$$

Applying [1, Theorem 13.9] to the expression (2), we obtain

$$\begin{aligned}(f_2|_S)^*(f_1)_*[X_1] = \tau_*(f_2|_S)^![X_1] &= \tau_* \left\{c(\tau^* N_{S/X_2}/N_{T/W})\right\}_{k_1-d_2} \\ &= \tau_* \left\{c(j|_T^* N_{X_1/Y}) s(C_{T/S})\right\}_{k_1-d_2}.\end{aligned}$$

But if we consider the expression (3), then there is no need to use the refined Gysin maps for local complete intersection morphisms. Here is the reason. Since both f_1 and f_2 are regular embeddings, the excess normal bundle is independent of the orientation of the bottom square, *i.e.,*

$$j^* N_{X_1/Y}/N_{W/X_2} = i^* N_{X_2/Y}/N_{W/X_1}.$$

Then

$$f_2^![X_1] = \left\{c(i^* N_{X_2/Y}/N_{W/X_1})\right\}_{k_1-d_2} = \left\{c(j^* N_{X_1/Y}/N_{W/X_2})\right\}_{k_2-d_1} = f_1^![X_2],$$

which means that we have the symmetry between two refined intersections $X_1 \cdot_Y X_2 = X_2 \cdot_Y X_1$ on $A_{k_1-d_2}(W)$. Together with the compatibility of refined Gysin maps with flat pullback and proper pushforward, we can further write expression (3) as

$$\begin{aligned}(f_2|_S)^*(f_1)_*[X_1] = p^* i_*(f_2^![X_1]) = p^* i_*(f_1^![X_2]) = \tau_*(q^* f_1^![X_2]) = \tau_*(f_1^! p^*[X_2]) = \tau_*(f_1^![S]) \\ = \tau_* \left\{c(j|_T^* N_{X_1/Y}) s(C_{T/S})\right\}_{k_2-d_1}.\end{aligned}$$

Therefore, if $T = \sqcup_\lambda C_\lambda$ is decomposed as disjoint union of connected components C_λ with codimension l_λ in S, then applying the above formula of $(f_2|_S)^*(f_1)_*[X_1]$ to each connected component gives

$$(f_2|_S)^*(f_1)_*[X_1] = \sum_\lambda (\tau_{C_\lambda})_* \alpha_{C_\lambda} = \sum_\lambda (\tau_{C_\lambda})_* \left\{c(j_{C_\lambda}^* N_{X_1/Y}) s(C_{C_\lambda/S})\right\}^{d_1-l_\lambda} \in A^{d_1}(S),$$

where τ_{C_λ} is the restriction of τ to the component C_λ, and j_{C_λ} is the restriction of $j \circ q$ to C_λ. Note that $d_1 - l_\lambda$ is the discrepancy of the codimension of C_λ in S from being "expected" and the corrected term comes from the codimension $(d_1 - l_\lambda)$ part of the Chern class of "excess normal bundle".

References

[1] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR3617981

[2] William Fulton, *Intersection theory*, Second, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323