CS281: Advanced ML **September 20, 2017**

Lecture 6: Exponential Families

Lecturer: Sasha Rush *Scribes: Meena Jagadeesan, Yufeng Ling, Tomoka Kan, Wenting Cai*

6.1 Introduction

(Wainwright and Jordan (textbook) presents a more detailed coverage of the material in this lecture.)

This lecture, we will unify all of the fundamentals presented so far:

$p(\theta)$	$p(x)$	$p(y \mid x) / p(x, y)$
Beta, Dir	Discrete	Classification
MVN, IW	MVN	Linear Regression
	Exponential Families	**Generalized Linear Models**
	Undirected Graphic Models	Conditional UGM
	Variational Inference	

We will focus on coming up with a general form for Discrete and MVN through exponential families. We will also come up with a general form for classification and linear regression through generalized linear models.

6.2 Definition of Exponential Family

The definition is

$$\begin{aligned} p(x \mid \theta(\mu)) &= \frac{1}{Z(\theta)} h(x) \exp\{\theta^T \phi(x)\} \\ &= h(x) \exp \theta^T \phi(x) - A(\theta) \end{aligned}$$

where

μ	mean parameters
$\theta(\mu)$	natural / canonical / exponential parameters
$Z(\theta) A(\theta)$	also written as $Z(\theta(\mu))$ or $Z(\mu)$, the partition function and log partition
$\phi(x)$	sufficient statistics of x, potential functions, "features"
$h(x)$	scaling term, in most cases, we have $h(x) = 1$

Note that there is "minimal form" and "overcomplete form".

6.3 Examples of Exponential Families

6.3.1 Bernoulli/Categorical

First, we consider the Bernoulli as an exponential family. Like last lecture, we rewrite the distribution as an exp of log.

$$\begin{aligned} \text{Ber}(x|\mu) &= \mu^x (1-\mu)^{(1-x)} \\ &= \exp x \log \mu + (1-x) \log(1-\mu) \\ &= \underbrace{}_{h(x)} \exp \underbrace{\log\left(\frac{\mu}{1-\mu}\right)}_{\theta} \underbrace{x}_{\phi(x)} + \underbrace{\log(1-\mu)}_{-A(\mu)} \end{aligned}$$

For the **minimal form**, we have

$$
\begin{aligned}
h(x) &= 1 \\
\phi_1(x) &= x \\
\theta_1(\mu) &= \log \frac{\mu}{1-\mu} (\text{“log odds”}) \\
\mu &= \sigma(\theta) \\
A(\mu) &= -\log(1-\mu) \\
A(\theta) &= -\log(1-\sigma(\theta)) = \theta + \log(1+e^{-\theta})
\end{aligned}
$$

For the **overcomplete form**, we have

$$
\begin{aligned}
\phi(x) &= \begin{bmatrix} x \\ 1-x \end{bmatrix} \\
\theta &= \begin{bmatrix} \log \mu \\ \log(1-\mu) \end{bmatrix}
\end{aligned}
$$

For the Categorical/Multinouilli distribution, we have

$$
\theta = \begin{bmatrix} \log \mu_1 \\ \vdots \\ \log \mu_n \end{bmatrix}
$$

where $\sum_c \mu_c = 1$.
Side note: Writing out in overcomplete form usually comes with some restraints.

6.3.2 Univariate Gaussians

$$
\begin{aligned}
\mathcal{N}(x \mid \mu, \sigma^2) &= (2\pi\sigma^2)^{1/2} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\} \\
&= \underbrace{(2\pi\sigma^2)^{-\frac{1}{2}}}_{A(\mu,\sigma^2)} \exp\{\underbrace{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x}_{\theta^T\phi(x)} - \underbrace{\frac{1}{2\sigma^2}\mu^2}_{A(\mu,\theta^2)}\}
\end{aligned}
$$

$$
\begin{aligned}
\phi(x) &= \begin{bmatrix} x \\ x^2 \end{bmatrix} \\
\theta &= \begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix} \\
A(\mu, \sigma^2) &= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\mu^2 \\
\mu &= -\frac{\theta_1}{2\theta_2} \\
\sigma^2 &= -\frac{1}{2\theta_2} \\
A(\theta) &= -\frac{1}{2}\log(-2\theta_2) - \frac{\theta_1^2}{4\theta_2}
\end{aligned}
$$

6.3.3 Bad distributions

Two simple distributions that do not fit this form are the uniform distribution Uniform$(0,1)$ (check this as an exercise), and the Student-T distribution.

6.4 Properties of Exponential Families

Most inference problems involve a mapping between natural parameters and mean parameters, so this is a natural framework.
Here are three properties of exponential families:

Property 1 Derivatives of $A(\theta)$ provide us the cumulants of the distribution $\mathbb{E}(\phi(x))$, $\text{var}(\phi(x))$:

Proof. For univariate, first order:

$$\begin{aligned}
\frac{dA}{d\theta} &= \frac{d}{d\theta}(\log Z(\theta)) \\
&= \frac{d}{d\theta}\log \underbrace{\left(\int \exp\{\theta\phi\}h(x)dx\right)}_{\text{needed to integrate to 1}} \\
&= \frac{\int \phi \exp\{\theta\phi\}h(x)dx}{\int \exp(\theta\phi)h(z)dx} \\
&= \frac{\int \phi \exp\{\theta\phi\}h(x)dx}{\exp(A(\theta))} \\
&= \int \phi(x)\underbrace{\exp(\theta\phi(x) - A(\theta))h(x)}_{p(x)}dx \\
&= \int \phi(x)p(x)dx \\
&= \mathbb{E}(\phi(x))
\end{aligned}$$

The same property holds for multivariates (refer to textbook for proof). □

Bernoulli:

$$A(\theta) = \theta + \log(1 + e^{-\theta})$$
$$\frac{dA}{d\theta} = 1 - \frac{e^{-\theta}}{1+e^{-\theta}} = \frac{1}{\underbrace{1+e^{-\theta}}_{\text{sigmoid}}} = \sigma(\theta) = \mu$$

Univariate Normal Left as exercise.

Property 2 MLE has a nice form (through "moment matching")

Proof.

$$\begin{aligned}
\underset{\theta}{\text{argmax}} \log p(\text{data} \mid \theta) &= \underset{\theta}{\text{argmax}} \left(\sum_d \theta^T\phi(x_d)\right) - NA(\theta) \\
&= \underset{\theta}{\text{argmax}} \theta^T \underbrace{\left(\sum_d \phi(x_d)\right)}_{\text{sum of sufficient statistics}} - \underbrace{NA(\theta)}_{\text{amount of points}}
\end{aligned}$$

We take a derivative to obtain:

$$
\begin{aligned}
\frac{d(.)}{d\theta} &= \sum_d \phi(x_d) - N\frac{dA(\theta)}{d\theta} \\
&= \sum_d \phi(x_d) - N\mathbb{E}(\phi(x)) \\
&= 0
\end{aligned}
$$

$$
E(\phi(x)) = \underbrace{\frac{\sum \phi(x_d)}{N}}_{\text{set mean parameter to sample means that gives us MLE}}
$$

□

Property 3 Exponential families have conjugate priors.

Proof. We first introduce some notations.

$$
\begin{aligned}
&\eta \text{ - parameters} \\
&\bar{s} = \sum_d \phi(x_d)/N \\
p(\text{data} \mid \eta) &\propto \exp[(N\bar{s})\eta - NA(\eta)] \\
p(\eta \mid N_0, s_0) &\propto \exp[(N_0, \bar{s}_0)\eta - N_0 \underbrace{A(\eta)}_{\text{not log partition, which has to be a function strictly of parameters}}] \\
p(\eta|\text{data}) &\propto \exp((N\bar{s} + N_0\bar{s}_0)^T\eta - (N_0 + N)A(\eta))
\end{aligned}
$$

The above two distributions have the same sufficient statistics – so we have a conjugate prior. It also tells us that it is not a coincidence that we kept obtaining pseudo counts. (More references will be put up to describe this).

□

6.5 Definition of Generalized Linear Models

While exponential families generalize $p(x)$, GLMs generalize $p(y|x)$.

$$
p(y|x, w) = h(y)\exp\{\theta(\underbrace{\mu(x)}_{\text{predict mean}})^T\phi(y) - A(\theta)\}
$$

where $\mu(x) = \underbrace{g^{-1}}_{\text{squashing const}}(w^Tx + b)$ where g is an appropriate linear transformation.

This can be summarized through the following sequence of transformations:

$$
x \xrightarrow{g^{-1}(w^Tx+b)} \mu \to \theta \to p(y \mid x).
$$

6.6 Examples of Generalized Linear Models

We present three examples:

Example 1 Exponential family - Normal distribution with $\sigma^2 = 1$ and g^{-1} is the identity function. This gives us the linear regression

$$
\mu = w^Tx + b \qquad \mathbb{R} \to \mathbb{R}.
$$

Example 2 Exponential family - Bernoulli distribution and g^{-1} is the sigmoid function $\sigma : \mathbb{R} \to (0,1)$. Now, $\mu = \sigma(w^T x + b)$ and $\theta = \log\left(\frac{\mu}{1-\mu}\right)$. This is how we define logistic regression. This gives us

$$p(y \mid x) = \sigma(w^T x + b)^y (1 - \sigma(w^T x + b))^{1-y}$$

Example 3 Exponential family - Categorical distribution with g^{-1} as the softmax function.
$\mu_c = \text{softmax}(w_c^T x + b_c)_c$
$\theta_c = \log \mu_c$

Example 4 Exponential family - Gaussian and Multivariate Gaussian distribution
For PDF of univariate Gaussian:

$$\begin{aligned}
p(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\log\sigma - \frac{x^2}{2\sigma^2} + \frac{\mu x}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(\theta^T \phi(x) - \log\sigma - \frac{\mu^2}{2\sigma^2}\right)
\end{aligned}$$

where:

$$\begin{aligned}
\phi(x) &= \begin{pmatrix} x \\ x^2 \end{pmatrix} \\
\theta &= \begin{pmatrix} \mu/\sigma^2 \\ -1/\sigma^2 \end{pmatrix} \\
A(\theta) &= \log\sigma + \frac{\mu^2}{2\sigma^2} \\
&= -\frac{\theta_1^2}{4\theta_2} - \frac{1}{2}\log(-2\theta_2) \\
h(x) &= \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

For PDF of multivariate Gaussian:

$$p(x) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}$$

where:

$$\begin{aligned}
\phi(\mathbf{x}) &= \begin{pmatrix} \mathbf{x} \\ \mathbf{x}\mathbf{x}^T \end{pmatrix} \\
\theta &= \begin{pmatrix} \Sigma^{-1}\mu \\ -\frac{1}{2}\Sigma^{-1} \end{pmatrix} \\
A(\theta) &= \frac{1}{2}\log(|\Sigma|) + \frac{1}{2}\mu^T\Sigma^{-1}\mu \\
&= \frac{1}{2}\log(|-2\theta_2|) + \frac{1}{4}\theta_1^T\theta_2^{-1}\theta_1 \\
h(x) &= (2\pi)^{-D/2}
\end{aligned}$$