Prove Vector Space Properties Using Vector Space Axioms

Problem 711

Using the axiom of a vector space, prove the following properties.
Let V be a vector space over $\mathbb{R}$. Let $u, v, w \in V$.

(a) If $u+v=u+w$, then $v=w$.

(b) If $v+u=w+u$, then $v=w$.

(c) The zero vector $\mathbf{0}$ is unique.

(d) For each $v \in V$, the additive inverse $-v$ is unique.

(e) $0v=\mathbf{0}$ for every $v \in V$, where $0 \in \mathbb{R}$ is the zero scalar.

(f) $a\mathbf{0}=\mathbf{0}$ for every scalar a.

(g) If $av=\mathbf{0}$, then $a=0$ or $v=\mathbf{0}$.

(h) $(-1)v=-v$.

The first two properties are called the **cancellation law**.

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(h) $(-1)v = -v$.

The Axioms of a Vector Space

Recall the axioms of a vector space:

A set V is said to be a **vector space** over $\mathbb{R}$ if

(1) an addition operation "+" is defined between any two elements of V, and

(2) a scalar multiplication operation is defined between any element of K and any element in V.

Moreover, the following properties must hold for all $u, v, w \in V$ and $a, b \in \mathbb{R}$:

Closure Properties

(c1) $u + v \in V$.

(c2) $av \in V$.

Properties of Addition

(a1) $u + v = v + u$.

(a2) $u + (v + w) = (u + v) + w$.

(a3) There is an element $\mathbf{0} \in V$ such that $\mathbf{0} + v = v$ for all $v \in V$.

(a4) Given an element $v \in V$, there is an element $-v \in V$ such that $v + (-v) = \mathbf{0}$.

Properties of Scalar Multiplication

(m1) $a(bv) = (ab)v$.

(m2) $a(u + v) = au + av$.

(m3) $(a + b)v = av + bv$.

(m4) $1v = v$ for all $v \in V$.

The element $\mathbf{0} \in V$ is called the **zero vector**, and for any $v \in V$, the element $-v \in V$ is called the **additive inverse** of v.

Solution.

(a) If $u + v = u + w$, then $v = w$.

We know by $(a4)$ that there is an additive inverse $-u \in V$. Then

$$\begin{aligned}
u + v = u + w \implies & -u + (u + v) = -u + (u + w) \\
\overset{(a2)}{\implies} & (-u + u) + v = (-u + u) + w \\
\overset{(a1)}{\implies} & (u + (-u)) + v = (u + (-u)) + w \\
\overset{(a4)}{\implies} & \mathbf{0} + v = \mathbf{0} + w \\
\overset{(a3)}{\implies} & v = w \,.
\end{aligned}$$

(b) If $v + u = w + u$, then $v = w$.

Now suppose that we have $v+u=w+u$. Then by (a1), we see that $u+v=u+w$. Now, it follows from (a) that $v=w$.
(Alternatively, you may prove this just like part (a).)

(c) The zero vector 0 is unique.

Suppose that $\mathbf{0}'$ is another zero vector satisfying axiom (a3). That is, we have $\mathbf{0}'+v=v$ for any $v \in V$.
Since $\mathbf{0}$ is also satisfy $\mathbf{0}+v=v$, we have

$$\mathbf{0}'+v=v=\mathbf{0}+v,$$

where v is any fixed vector (for example $v=\mathbf{0}$ is enough).
Now by the cancellation law (see (b)), we obtain $\mathbf{0}'=\mathbf{0}$.
Thus, there is only one zero vector $\mathbf{0}$.

(d) For each $v \in V$, the additive inverse $-v$ is unique.

Since $-v$ is the additive inverse of $v \in V$, we have $v+(-v)=\mathbf{0}$. (This is just (a4).)
Now, suppose that we have a vector $w \in V$ satisfying $v+w=\mathbf{0}$. So, w is another element satisfying axiom (a4).
Then we have

$$v+(-v)=0=v+w\,.$$

By the cancellation law (see (a)), we have $-v=w$. Thus, the additive inverse is unique.

(e) $0v=\mathbf{0}$ for every $v \in V$, where $0 \in \mathbb{R}$ is the zero scalar.

Note that 0 is a real number and $\mathbf{0}$ is the zero vector in V. For $v \in V$, we have

$$0v=(0+0)v \overset{(m3)}{=} 0v+0v\,.$$

We also have

$$0v \overset{(a3)}{=} \mathbf{0}+0v\,.$$

Hence, combining these, we see that

$$0v+0v=\mathbf{0}+0v,$$

and by the cancellation law, we obtain $0v=\mathbf{0}$.

(f) $a\mathbf{0}=\mathbf{0}$ for every scalar a.

Note that we have $\mathbf{0}+\mathbf{0}=\mathbf{0}$ by (a3).
Thus, we have

$$a\mathbf{0}=a(\mathbf{0}+\mathbf{0}) \overset{(m2)}{=} a\mathbf{0}+a\mathbf{0}\,.$$

We also have

$$a\mathbf{0} = \mathbf{0} + a\mathbf{0}$$

by (a3). Combining these, we have

$$a\mathbf{0} + a\mathbf{0} = \mathbf{0} + a\mathbf{0},$$

and the cancellation law yields $a\mathbf{0} = \mathbf{0}$.

(g) If $av = \mathbf{0}$, then $a = 0$ or $v = \mathbf{0}$.

For this problem, we use a little bit logic. Our assumption is $av = \mathbf{0}$. From this assumption, we need to deduce that either $a = 0$ or $v = \mathbf{0}$.
Note that if $a = 0$, then we are done as this is one of the consequence we want. So, let us assume that $a \neq 0$. Then we want to prove $v = 0$.
Since a is a nonzero scalar, we have a^{-1}. Then we have

$$a^{-1}(av) = a^{-1}\mathbf{0}\,.$$

The right hand side $a^{-1}\mathbf{0}$ is $\mathbf{0}$ by part (f).
On the other hand, the left hand side can be computed as follows:

$$a^{-1}(av) \overset{(m1)}{=} (a^{-1}a)v = 1v \overset{(m4)}{=} v\,.$$

Therefore, we have $v = 0$.
Thus, we conclude that if $av = \mathbf{0}$, then either $a = 0$ or $v = \mathbf{0}$.

(h) $(-1)v = -v$**.**

Note that $(-1)v$ is the scalar product of -1 and v. On the other hand, $-v$ is the additive inverse of v, which is guaranteed to exist by (a4).
We show that $(-1)v$ is also the additive inverse of v:

$$v + (-1)v \overset{(m4)}{=} 1v + (-1)v \overset{(m3)}{=} (1 + (-1))v = 0v \overset{(e)}{=} \mathbf{0}\,.$$

So $(-1)v$ is the additive inverse of v. Since by part (d), we know that the additive inverse is unique, it follows that $(-1)v = -v$.

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