INVERTIBLE MATRIX THEOREM

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The invertible matrix theorem (Lay, Section 2.3, Theorem 8) is a really important theorem, and it's really hard to learn and remember 12 conditions. It may be helpful to organize the conditions in the following way.

Let's fix the following notation: Let A be a $m \times n$ matrix, $\mathbf{v}_1, \dots, \mathbf{v}_n$ the columns of A, and $f : \mathbb{R}^n \to \mathbb{R}^m$ the linear function given by $f(\mathbf{x}) = A\mathbf{x}$.

1. Lemmas about injectivity and surjectivity

Injectivity of f means "f doesn't lose any information about the domain $\mathbb{R}^n$".

Lemma 1 (Injectivity)**.** *The following are equivalent conditions:*

(4-a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
(4-b) $\mathbf{v}_1, \dots, \mathbf{v}_n$ *are linearly independent.*
(4-c) f *is one-to-one (injective).*
(4-d) *There is a pivot in every column.*
(4-e) *There is an* $n \times m$ *matrix* C *such that* $CA = I_n$*.*

If the above conditions hold, then $m \geq n$*.*

Surjectivity of f means "f doesn't lose any information about the range $\mathbb{R}^m$".

Lemma 2 (Surjectivity)**.** *The following are equivalent conditions:*

(5-a) $A\mathbf{x} = \mathbf{b}$ *has a solution for all* $\mathbf{b} \in \mathbb{R}^m$*.*
(5-b) $\mathbb{R}^m = \mathit{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$*.*
(5-c) f *is onto (surjective).*
(5-d) *There is a pivot in every row.*
(5-e) *There is an* $n \times m$ *matrix* D *such that* $AD = I_m$*.*

If the above conditions hold, then $m \leq n$*.*

Comments 3. The ith condition of Lemma 1 is analogous to the ith condition of Lemma 2. Conditions (4-a) and (5-a) are phrased in the language of matrix equations. Conditions (4-b) and (5-b) are in the language of vector equations. Conditions (4-c) and (5-c) are in the language of functions. Conditions (4-d) and (5-d) are about the number of pivots; both say that there are as many pivots that the matrix can possibly allow (since pivot positions are in different rows and columns, a $m \times n$ matrix can contain at most $\min\{m, n\}$ pivot positions). Condition (4-e) and (5-e) relate to the invertibility of matrix A.

Warning about (4-e) and (5-e): if $m \neq n$, then (4-e) does not necessarily imply (5-e) and vice versa. Consider the case $m = 2$ and $n = 1$ and $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then (4-e) holds; we can let $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (or $C = \begin{bmatrix} 1 & a \end{bmatrix}$ for any number a). But (5-e) does not hold, since if $D = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ 0 & 0 \end{bmatrix}$, which can never equal I_2.

2. Statement of the Theorem and Applications

In what follows, we assume that $m = n$ (the dimension of the domain and range of f are the same; the number of rows of A and the number of columns of A are the same).

Theorem 4 (Invertible Matrix Theorem)**.** *Suppose $m = n$. The following are equivalent conditions:*

1. *A is invertible.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *{All the conditions of Lemma 1}*
5. *{All the conditions of Lemma 2}*
6. *A^T is invertible.*

Proposition 5. *A linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is injective if and only if it is surjective.*