LEFT/RIGHT INVERTIBLE MATRICES

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Definition 1. Let A be an $m \times n$ matrix. We say that A is *left invertible* if there exists an $n \times m$ matrix C such that $CA = I_n$. (We call C a *left inverse* of A.[1]) We say that A is *right invertible* if there exists an $n \times m$ matrix D such that $AD = I_m$. (We call D a *right inverse* of A.[2]) We say that A is *invertible* if A is both left invertible and right invertible.

Notice that $CA = I_n$ doesn't imply that $AC = I_m$, since matrix multiplication is not commutative.

Suppose that A is invertible. By definition, it is both left invertible and right invertible. Thus there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. It turns out that in this case $C = D$:

$$C = CI_m = C(AD) = (CA)D = I_nD = D.$$

Example 2. The matrix $\begin{bmatrix} 1 & 0 \end{bmatrix}$ is right invertible but not left invertible. Indeed, $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$ but $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 \\ a_{2,1} & 0 \end{bmatrix}$, which can never be I_2.

Example 3. The matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is left invertible but not right invertible. Indeed, $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ 0 & 0 \end{bmatrix}$, which can never be I_2.

These examples illustrate a general phenomenon regarding nonsquare matrices.

Proposition 4. *Let m, n be two positive integers.*

(1) *Suppose $m < n$ (more columns than rows).*
  (i) *There are no left invertible $m \times n$ matrices.*
  (ii) *An $m \times n$ matrix is right invertible if and only if its REF has pivots in every row.*[3]
(2) *Suppose $m > n$ (more rows than columns).*
  (i) *There are no right invertible $m \times n$ matrices.*
  (ii) *An $m \times n$ matrix is left invertible if and only if its REF has pivots in every column.*[4]

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[1]Some matrices A have more than one left inverse, so I write "a" rather than "the". (Such matrices with more than one left inverse are necessarily nonsquare (why?).)

[2]Some matrices A have more than one right inverse, so I write "a" rather than "the". (Such matrices with more than one right inverse are necessarily nonsquare (why?).)

[3]Notice that this is the maximum number of pivots.

[4]Notice that this is the maximum number of pivots.

Outline of proof. (1,i) and (2,i): Assume that $m > n$ and there exists an $m \times n$ matrix A and an $n \times m$ matrix B such that $AB = I_m$. Show that the linear transformation $\mathbb{R}^m \to \mathbb{R}^n$ defined by B is injective. But derive a contradiction from the fact that $D\mathbf{x} = 0$ has a nontrivial solution (since the REF of D is bound to have a nonpivot column, corresponding to a free variable).

(1,ii) and (2,ii): Assume that $m < n$ and there exists an $m \times n$ matrix A and an $n \times m$ matrix B such that $AB = I_m$. Show that the columns of A span $\mathbb{R}^m$ and the columns of B are linearly independent. This implies that the REF of A has pivots in every row and the REF of B has pivots in every column.

Conversely, if the REF of A has pivots in every row, then the columns of A span $\mathbb{R}^m$, so the equation $A\mathbf{x} = \mathbf{e}_i$ has a solution for all $i = 1, \ldots, m$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be the solutions, i.e. $A\mathbf{b}_i = \mathbf{e}_i$. Let $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_m]$. Then $AB = I_m$.

If the REF of B has pivots in every column, then the columns of B are linearly independent, so the rows of B^T are linearly independent, so the REF or B^T has pivots in every row, so by the above there exists some $n \times m$ matrix C such that $B^T C = I_m$. Then $C^T B = (B^T C)^T = I_m$. □

For square matrices, we have the following proposition, which is immediate from the IMT:

Proposition 5. *Suppose $m = n$ and let A be an $n \times n$ matrix. Then the following conditions are equivalent:*

(i) *A is left invertible.*
(ii) *A is right invertible.*