MATHEMATICS

# Rotational invariance of cross product

Asked 11 years, 1 month ago Modified 1 year, 3 months ago Viewed 15k times

21

I'm looking for a proof that

$$(Ra \times Rb) = R(a \times b)$$

where $\times$ is the three-dimensional cross product, and $R$ is a rotational matrix (such that $\det R = 1$ and $R^T R = I$).

I've already found a proof of it on planetmath, but I'm very new to using Levi-Civita etc. For example, I don't get why

$$\epsilon^{imk} R_{ij} R_{mn} u^j v^n = \epsilon^{iml} \delta_{kl} R_{ij} R_{mn} u^j v^n$$

I'm either looking for a list of identities for the Levi-Civita Symbol which would help me understand this proof or someone who could explain the single steps to me because I'm really quite lost.

rotations cross-product

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edited Jan 13, 2022 at 6:06
Calvin Khor
34.8k 6 43 93

asked Jan 15, 2013 at 10:02
Howdy Ho
645 2 6 11

The planetmath link is dead; I link here to a related question that has the Levi-Civita computations (and more): [math.stackexchange.com/questions/859836/...](math.stackexchange.com/questions/859836/...) – Calvin Khor Jan 9, 2022 at 13:48

Well, the equation

$$(Ra \times Rb) = R(a \times b)$$

shows precisely that the cross product is not invariant under rotations. But it allows us to prove the rotational invariance of some equations that involve the cross product. – Filippo Aug 18, 2022 at 10:05

## 4 Answers

Sorted by: Highest score (default)

21

This is most easily proved without coordinates. The cross product is the unique vector that is orthogonal to both factors, has length given by the area of the parallelogram they form and forms a right-handed triple with them. These properties are all invariant under rotations, and thus so is the cross product.

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answered Jan 15, 2013 at 10:43

joriki

236k 14 297 516

17

You can use $\boldsymbol{a}^T(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ to prove it. Let $j$ be any component of the vector $R(\boldsymbol{a}\times\boldsymbol{b})$, then

$$\begin{aligned}
(R(\boldsymbol{a}\times\boldsymbol{b}))_j &= \boldsymbol{e}_j\cdot R(\boldsymbol{a}\times\boldsymbol{b}) \\
&= \boldsymbol{e}_j^T R\,(\boldsymbol{a}\times\boldsymbol{b}) \\
&= (R^T\boldsymbol{e}_j)^T(\boldsymbol{a}\times\boldsymbol{b}) \\
&= \det(R^T\boldsymbol{e}_j,\ \boldsymbol{a},\ \boldsymbol{b}) = \det R\ \det(R^T\boldsymbol{e}_j,\ \boldsymbol{a},\ \boldsymbol{b}) \\
&= \det(RR^T\boldsymbol{e}_j,\ R\boldsymbol{a},\ R\boldsymbol{b}) = \det(\boldsymbol{e}_j,\ R\boldsymbol{a},\ R\boldsymbol{b}) \\
&= \boldsymbol{e}_j\cdot(R\boldsymbol{a}\times R\boldsymbol{b}) \\
&= (R\boldsymbol{a}\times R\boldsymbol{b})_j
\end{aligned}$$

During the calculation we used $\det R = 1$ and $RR^T = I$, each once.

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edited Aug 3, 2016 at 12:19

answered Aug 3, 2016 at 11:22

Léreau

3,014 1 13 34

1 This is Christian's answer written down the other way round – Bananach Aug 3, 2016 at 11:35

1 @Bananach That's tight. The approach is slightly different though, I do not discuss a volume form etc. so it is less detailed in that regard. But my calculation is a little bit more detailed and I point out, where the properties of $R$ are used. But if it is really considered to close to his answer, I might delete mine. – Léreau Aug 3, 2016 at 12:19

1 Only wanted to point that out. Indeed, I can imagine your answer is easier to digest to some – Bananach Aug 3, 2016 at 12:57

I like that you highlighted the steps where you used the defining properties of rotations (+1). – Filippo Aug 18, 2022 at 10:13

13

Here is an explanation which is nearer to linear algebra:

In $\mathbb{R}^3$ we have a *volume form*

$$\epsilon: \quad (\mathbb{R}^3)^3 \to \mathbb{R}, \qquad (a,b,c) \mapsto \epsilon(a,b,c)\ ,$$

which produces for any three given vectors $a$, $b$, $c$ the signed volume of the parallelotope spanned by them. It is linear in all three entries, and when $a$, $b$, $c$ are given with respect to orthonormal coordinates then $\epsilon(a,b,c)$ is the determinant of the matrix $[a\ b\ c]$ with $a$, $b$, $c$ in its columns.

Given any two vectors $a$, $b$ the function

$$\phi: \quad \mathbb{R}^3 \to \mathbb{R}, \qquad x \mapsto \epsilon(a,b,x)$$

is linear in $x$, so there is a unique vector $q \in \mathbb{R}^3$ such that

$$\epsilon(a,b,x) = \phi(x) = q \cdot x \qquad \forall x \in \mathbb{R}^3\ .$$

This vector $q$ depends in a skew bilinear way on the given vectors $a$ and $b$, and is called the *cross product* of $a$ and $b$. It is allowed to denote this $q$ by $a \times b$, so that we have

$$\epsilon(a,b,x) = (a \times b) \cdot x \qquad \forall x \in \mathbb{R}^3\ .$$

Assume now that a rotation $R$ is given. Then

$$\begin{aligned}(Ra \times Rb) \cdot x &= \epsilon(Ra, Rb, x) = \det[Ra\ Rb\ x] = \det\bigl(R\,[a\ b\ R'x]\bigr) \\ &= \det R\ \det[a\ b\ R'x] = \epsilon(a,b,R'x) = (a \times b) \cdot R'x \\ &= R(a \times b) \cdot x\ .\end{aligned}$$

Since this is true for all $x \in \mathbb{R}^3$ the stated identity follows.

Share Cite Follow edited Jan 15, 2013 at 13:20 answered Jan 15, 2013 at 11:36

Christian Blatter
226k 13 187 464

If we prove $c^T a = c^T b \ \forall \ c \neq 0$ then $a = b$ is not obvious to me. Consider my attempt at a proof. If $a \neq b$ then since $|a|\hat{a} \neq |b|\hat{b}$, choose $c = |b|(\hat{b} - \hat{a})$ and note that $\cos\theta < 1$. Since $a \cdot c = |a||b|\cos\theta - |a||b| < 0$ and $b \cdot c = |b|^2 - |b|^2 \cos\theta > 0$, we can conclude that $a \cdot c \neq b \cdot c \ \forall \ c \neq 0$. We have proved the contrapositive by contradiction. Is this ok? – Aditya P Jan 16, 2020 at 8:08

2 @AdityaP: If $c^T a = c^T b$ for all $c$ put $c := a - b$, and obtain $|a - b|^2 = (a - b)^T (a - b) = 0$. It follows that $a = b$. – Christian Blatter Jan 16, 2020 at 9:17

That is what I was looking for, so simple and now it's clear. Thanks a lot! – Aditya P Jan 16, 2020 at 9:24

5

Let's represent the rotation matrix $R$ in terms of its row vectors:

$$R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

From this, we get $Ra$ and $Rb$ as the following:

$$Ra = \begin{bmatrix} R_1 \cdot a \\ R_2 \cdot a \\ R_3 \cdot a \end{bmatrix}$$

$$Rb = \begin{bmatrix} R_1 \cdot b \\ R_2 \cdot b \\ R_3 \cdot b \end{bmatrix}$$

By the analytical definition of the cross product, we have

$$Ra \times Rb = \begin{bmatrix} (R_2 \cdot a)(R_3 \cdot b) - (R_3 \cdot a)(R_2 \cdot b) \\ (R_3 \cdot a)(R_1 \cdot b) - (R_1 \cdot a)(R_3 \cdot b) \\ (R_1 \cdot a)(R_2 \cdot b) - (R_2 \cdot a)(R_1 \cdot b) \end{bmatrix}$$

It can be then shown that the following identity is true for vectors $A$, $B$, $C$, and $D$:

$$(A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) = (A \times B) \cdot (C \times D)$$

allowing for

$$\begin{bmatrix} (R_2 \cdot a)(R_3 \cdot b) - (R_3 \cdot a)(R_2 \cdot b) \\ (R_3 \cdot a)(R_1 \cdot b) - (R_1 \cdot a)(R_3 \cdot b) \\ (R_1 \cdot a)(R_2 \cdot b) - (R_2 \cdot a)(R_1 \cdot b) \end{bmatrix} = \begin{bmatrix} (R_2 \times R_3) \cdot (a \times b) \\ (R_3 \times R_1) \cdot (a \times b) \\ (R_1 \times R_2) \cdot (a \times b) \end{bmatrix}$$

Now, for a 3x3 matrix $M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$, we know

$$det(M) = (M_1 \times M_2) \cdot M_3$$

Since performing one row swapping on $M$ negates the determinant, taking two row swaps will double-negate the determinant. From this, we can show that

$$det(M) = (M_1 \times M_2) \cdot M_3 = (M_2 \times M_3) \cdot M_1 = (M_3 \times M_1) \cdot M_2$$

Additionally, since $R$ is a rotation matrix, we know that the length of each row of $R$ is 1. This means that $R_i \cdot R_i = 1$ for each ith row of $R$. Taking this into consideration,

$$(R_1 \times R_2) \cdot R_3 = R_3 \cdot R_3$$

and thus

$$(R_1 \times R_2) = R_3$$

$$(R_1 \times R_2) = R_3$$

Baring this in mind without loss of generality, this allows

$$\begin{bmatrix} (R_2 \times R_3) \cdot (a \times b) \\ (R_3 \times R_1) \cdot (a \times b) \\ (R_1 \times R_2) \cdot (a \times b) \end{bmatrix} = \begin{bmatrix} (R_1) \cdot (a \times b) \\ (R_2) \cdot (a \times b) \\ (R_3) \cdot (a \times b) \end{bmatrix} = R(a \times b)$$

.

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edited Feb 10, 2013 at 21:32

answered Feb 10, 2013 at 19:15

DoggoDougal

**151** 2