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Quasi-Polynomial-Based Robust Stability of Time-Delay Systems can be Less Conservative than Lyapunov-Krasovskii Approaches

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***Abstract*—In this investigation, stability of uncertain time-delay unforced systems is analyzed through a generalized characteristic quasi-polynomial, which is exactly rewritten as a polytope whose interpolating functions exhibit mutual dependency. It is shown that an adequate combination of Kharitonov-like results and Pólya-like relaxations leads to progressively better results at the price of more demanding computational burden. Stability methodologies are developed and proved advantageous when compared with traditional Lyapunov-based approaches, both for structured and unstructured uncertainty.**

I. Introduction

This work is inspired in the approach adopted in [1], where the problem of robust stability analysis of unforced linear time-delay systems is considered via the characteristic quasi-polynomial instead of the direct Lyapunov method. Uncertainties are considered to be frozen and to lie within a polytope, after which some frequency-based tests allow guaranteeing the whole family of quasi-polynomials lie in the left half complex plane $\mathbb{C}^-$. Such root location is known to be equivalent to the stability of a quasi-polynomial [2].

To illustrate our goals, consider a particular case of the model of a metal cutting process in [3] $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = \theta y(t) - \theta y(t-\tau) + u(t)$ where $m = k = 1$, $c = -0.1$, and $u(t) = -y(t) + y(t-\tau)$; its state-space representation is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ \theta-2 & 0.1\end{bmatrix}}_{A(\theta)}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ -\theta+1 & 0\end{bmatrix}}_{D(\theta)}\begin{bmatrix}x_1(t-\tau)\\ x_2(t-\tau)\end{bmatrix}, \quad (1)$$

where $x_1 = y$, $x_2 = \dot{y}$. From now on, delayed states $x_i(t-\tau)$ will be denoted as $x_{i\tau}$.

The nominal system of the previous family is usually used as a prior step for analysis, i.e., instead of (1) the model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & 0.1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_{1\tau}\\ x_{2\tau}\end{bmatrix} \quad (2)$$

is employed. Frequency domain analysis allows determining the range of delays $\tau \in [\tau_0, \tau_1]$ guaranteeing stability of the system (2) [4]; tests based on linear matrix inequalities (LMIs) provide sufficient conditions leading to even shorter guaranteed ranges of stability [5]. Naturally, these values may not correspond to those valid for the original uncertain system (1); the analysis thus made carries then some loss of generality and conservativeness.

To consider the whole uncertain model, a common solution is to employ integral inequalities [6], [7] along with Lyapunov methods: the uncertain model is embedded in a polytope of time-delay linear systems (i.e., a linear differential-difference inclusion) which is combined with a Lyapunov-Krasovskii [8] or Lyapunov-Razumikhin [9] functional in order to derive conditions in the form of LMIs [10]. The polytope is obtained via the sector nonlinearity approach which exactly rewrites uncertainties as convex sums of their bounds [11]. For the example under consideration, if $\theta \in [\theta^0, \theta^1]$ then

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \sum_{i=0}^{1} w_i(\theta)\left(\underbrace{\begin{bmatrix}0 & 1\\ \theta^i-2 & 0.1\end{bmatrix}}_{A_i}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ -\theta^i+1 & 0\end{bmatrix}}_{D_i}\begin{bmatrix}x_{1\tau}\\ x_{2\tau}\end{bmatrix}\right), \quad (3)$$

with $w_0(\theta) = \left(\theta^1 - \theta\right) / \left(\theta^1 - \theta^0\right) \geq 0$, $w_1(\theta) = 1 - w_0(\theta) \geq 0$. Clearly, $A(\theta) = \sum_{i=0}^{1} w_i(\theta)A_i$ and $D(\theta) = \sum_{i=0}^{1} w_i(\theta)D_i$, so (3) is algebraically equivalent to (1). A variety of results for linear parameter varying (LPV) as well as quasi-LPV systems can be used to obtain LMI conditions which establish stability of the whole polytope based on the vertex system pairs (A_i, D_i); for instance, [12].

Yet another way to use the original setup is presented in this note: it intends to recover the frequency-based analysis that has been used for time-delay linear systems. Resuming our motivating example, consider its characteristic quasi-polynomial

$$\begin{aligned} q(s,\theta) &= \left|sI - A(\theta) - D(\theta)e^{-\tau s}\right| \\ &= \underbrace{s^2 - 0.1s - \theta + 2}_{d(s,\theta)} + \underbrace{(\theta - 1)}_{n(s,\theta)}e^{-\tau s} \quad (4) \\ &= \sum_{k=0}^{2}\sum_{j=0}^{1} a_{kj}(\theta)s^k e^{\tau_j s}, \quad (5) \end{aligned}$$

where $a_{00}(\theta) = -\theta + 2$, $a_{10}(\theta) = -0.1$, $a_{20}(\theta) = 1$, $a_{01}(\theta) = \theta - 1$, and $a_{11}(\theta) = a_{21}(\theta) = 0$, $\tau_0 = 0$, $\tau_1 = -\tau$. The notation in (4) can be found in [13] whereas that in (5) is the one adopted in [1]: they are used for certain systems, i.e., all a_{kj} are constants and $d(\cdot)$, $n(\cdot)$ are polynomials in s, no θ. Importantly, the cluster treatment of characteristic roots (CTCR) constitutes a non-polytopic alternative for stability analysis via the characteristic quasi-polynomial; the interested reader is referred to [14], [15], [16] for details.

The quasi-polynomial above can be written in a convex form by means of the same techniques employed for the

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matrix case (3), i.e., defining w_0 and w_1 as above, (4) or (5) can be rewritten as

$$q(s,\theta) = \sum_{i=0}^{1} w_i(\theta) q_i(s), \tag{6}$$

where $q_0(s) = s^2 - 0.1s + 2 - \theta^0 + (\theta^0 - 1)e^{-\tau s}$ and $q_1(s) = s^2 - 0.1s + 2 - \theta^1 + (\theta^1 - 1)e^{-\tau s}$ are the vertex quasi-polynomials of a polytope (in this case, as there are only two vertices, the polytope reduces to a segment in the coefficient space).

Some questions arise: (a) can the steps above be translated into a consistent modelling methodology leading to a polytope of quasi-polynomials such as (6)?, (b) can stability of an uncertain time-delay system such as (1) be established through the vertex quasi-polynomials in a representation like (6)?, (c) can results via the direct Lyapunov method be overcome by the suggested quasi-polynomial-based analysis? The first question is addressed in section II, both for structured and unstructured uncertainties; section III aims to answer the second question using results from the field of robust analysis of linear time-delay systems as well as Pólya-like relaxations for which the proposed structures are suited; the third question is addressed through illustrative examples in section IV; some conclusions are gathered in section V.

II. Polytopic modelling of uncertain quasi-polynomials

Consider an uncertain time-delay unforced system:

$$\dot{x}(t) = A(\theta)x + D(\theta)x(t-\tau), \tag{7}$$

with $x \in X \subset \mathbb{R}^n$ being the state vector, $\theta \in \Theta \subset \mathbb{R}^o$ a vector grouping the different bounded uncertainties in the system, $A(\theta)$ and $D(\theta)$ matrices whose entries are bounded functions in a Θ; $\tau > 0$ being a constant time delay.

The characteristic quasi-polynomial of (7) is given by $\det(sI - A(\theta) - e^{-s\tau}D(\theta))$, whose stability is the same of

$$\begin{aligned} q(s,\theta) &= e^{\tau_m s} \det\left(sI - A(\theta) - e^{-s\tau}D(\theta)\right) \\ &= \sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}(\theta) s^k e^{\tau_l s}, \end{aligned} \tag{8}$$

where $0 < \tau_0 < \tau_1 < \cdots < \tau_m$, $a_{kl}(\theta)$ are the coefficients of the polynomial which depend on the uncertainties θ; clearly, as they come from bounded expressions in $A(\theta)$ and $D(\theta)$, these coefficients have a finite set of bounded uncertainties.

An exact polytopic rewriting of (8) can be systematically obtained in two steps:

Step 1: Denote each of the r different uncertain terms in coefficients $a_{kl}(\theta)$ of (8) as $z_i(\theta) \in [z_i^0, z_i^1]$, $i \in \{1, 2, \ldots, r\}$ and write each of them as a convex sum of its bounds $z_i(\theta) = w_0^i(\theta) z_i^0 + w_1^i(\theta) z_i^1$, where:

$$w_0^i(\theta) = \frac{z_i^1 - z_i(\theta)}{z_i^1 - z_i^0} \geq 0, \quad w_1^i(\theta) = 1 - w_0^i(\theta) \geq 0. \tag{9}$$

Step 2: Convexity of functions $w_j^i(\theta)$, $i \in \{1, 2, \ldots, r\}$, $j \in \{0, 1\}$, allows writing (8) as the following convex sum of quasi-polynomials:

$$q(s,\theta) = \sum_{\mathbf{i}\in\mathbb{B}^r} \mathbf{w}_{\mathbf{i}}(\theta) \underbrace{\sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}^{\mathbf{i}} s^k e^{\tau_l s}}_{q_{\mathbf{i}}(s)}, \tag{10}$$

with $\mathbf{i} = (i_1, i_2, \ldots, i_r) \in \mathbb{B}^r$, $\mathbb{B} = \{0, 1\}$, $\mathbf{w}_{\mathbf{i}}(\theta) = w_{i_1}^1(\theta) w_{i_2}^2(\theta) \cdots w_{i_r}^r(\theta)$, and $a_{kl}^{\mathbf{i}}$ being the constant coefficients of the vertex quasi-polynomial $q_{\mathbf{i}}(s)$. Clearly, (10) can be viewed as a polytope of quasi-polynomials[1].

Very often, structured uncertainty may consist of different powers of the same term $z_i(\theta)$. In that case, powers of convex sums may appear which may indeed increase the number of vertices while enriching the relationships between the weighting functions. For instance, instead of modelling z_i and $(z_i)^2$ as two unrelated terms, the latter can be written in terms of the former as[2]

$$(z_i)^2 = \left(\sum_{i_1=0}^{1} w_{i_1}^i z_i^{i_1}\right)\left(\sum_{i_2=0}^{1} w_{i_2}^i z_i^{i_2}\right) = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^i w_{i_2}^i z_i^{i_1} z_i^{i_2}.$$

In general, if a convex representation of z_i is raised to power p, its convex representation will consist of p nested convex sums whose terms have products of the form $w_{i_1}^i w_{i_2}^i \cdots w_{i_p}^i$ with $(i_1, i_2, \ldots, i_p) \in \mathbb{B}^p$, all of them associated with the same weighting function. Thus, if p_i denotes the maximum power of z_i in (8), the resulting convex sum of quasi-polynomials is

$$q(s,\theta) = \prod_{j=1}^{r} \sum_{\mathbf{i}_j\in\mathbb{B}^{p_j}} \mathbf{w}_{\mathbf{i}_j}^j(\theta) \underbrace{\sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}^{\mathbf{i}} s^k e^{\tau_l s}}_{q_{\mathbf{i}}(s)}, \tag{11}$$

where $\mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_r)$, $\mathbf{i}_j = (i_1^j, i_2^j, \cdots, i_{p_j}^j)$, $\mathbf{w}_{\mathbf{i}_j}^j(\theta) = w_{i_1}^j(\theta) w_{i_2}^j(\theta) \cdots w_{i_{p_j}}^j(\theta)$. This representation can associate similar terms according to the weighting functions, thus producing new vertices in the quasi-polynomial coefficient space. Clearly, due to the associations, this new polytope fits more closely the actual coefficient space [17].

Another common representation of uncertain time-delay systems is that based on interval matrices, i.e., instead of beginning with (7), we have:

$$\dot{x}(t) = (A + \Delta A)x + (D + \Delta D)x(t-\tau), \tag{12}$$

where A and D represent the nominal system while interval matrices ΔA and ΔD (with entries $\Delta a_{ij} \in [\Delta a_{ij}^0, \Delta a_{ij}^1]$, $\Delta d_{ij} \in [\Delta d_{ij}^0, \Delta d_{ij}^1]$, $i, j \in \{1, 2, \ldots, n\}$) represent unstructured uncertainty. Defining $z_{ij} = a_{ij} + \Delta a_{ij} \in [a_{ij} + \Delta a_{ij}^0, a_{ij} + \Delta a_{ij}^1]$ and $\zeta_{ij} = d_{ij} + \Delta d_{ij} \in [d_{ij} + \Delta d_{ij}^0, d_{ij} + \Delta d_{ij}^1]$, the characteristic polynomial of (12) is $\det(sI - (A + \Delta A) - e^{-s\tau}(D + \Delta D))$, whose stability is equivalent to that of the following :

$$\begin{aligned} q(s,\Delta) &= e^{n\tau s} \det\left(sI - (A + \Delta A) - e^{-s\tau}(D + \Delta D)\right) \\ &= \det \begin{bmatrix} e^{\tau s}(s - z_{11}) - \zeta_{11} & \cdots & -e^{\tau s} z_{1n} - \zeta_{1n} \\ \vdots & \ddots & \vdots \\ -e^{\tau s} z_{n1} - \zeta_{n1} & \cdots & e^{\tau s}(s - z_{nn}) - \zeta_{nn} \end{bmatrix} \\ &= \sum_{\mathbf{j}} (-1)^{f(\mathbf{j})} \prod_{i=1}^{n} \left(e^{\tau s}(g(i, j_i)s - z_{ij_i}) - \zeta_{ij_i}\right), \end{aligned} \tag{13}$$

[1] Indeed, if multi-index notation such as $\mathbf{i}$ is to be avoided, the polytope in (10) can be rewritten as in [1], i.e., as a single sum $\sum_l^N \mu_l q_l(s)$, provided μ_i is adequately defined as a product of weights $w_{\mathbf{i}}(\theta)$ such that i is the decimal equivalent of the binary number $\mathbf{i}$; clearly, due to convexity of the weighting functions $0 \leq \mu_i \leq 1$, $\sum_{i=1}^N \mu_i = 1$, $N = 2^r$.

[2] Following notation in the quasi-LPV literature, terms z_i^0 and z_i^1 denoting minimum and maximum of z_i, respectively, are not to be mistaken by powers of z_i which are denoted as $(z_i)^p$.

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where $f(\mathbf{j})$ gives the number of inversions in the permutation $\mathbf{j} = (j_1, j_2, \ldots, j_n)$, $j_i \in \{1, 2, \ldots, n\}$, $j_i \neq j_l$ for $i \neq l$, and $g(i, j_i) = 1$ when $j_i = i$, otherwise $g(i, j_i) = 0$. There are at most $2n^2$ different bounded uncertainties coming from ΔA and ΔD in (13): z_{ij}, ζ_{ij}, $i, j \in \{1, 2, \ldots, n\}$; they can be exactly modelled using only a fixed set of weights[3]:

$$w_0^{ij}(z) = \frac{z_{ij}^1 - z_{ij}}{z_{ij}^1 - z_{ij}^0},\ w_1^{ij}(z) = 1 - w_0^{ij}(z), \tag{14}$$

$$\omega_0^{ij}(\zeta) = \frac{\zeta_{ij}^1 - \zeta_{ij}}{\zeta_{ij}^1 - \zeta_{ij}^0},\ \omega_1^{ij}(\zeta) = 1 - \omega_0^{ij}(\zeta), \tag{15}$$

with $z_{ij}^k = a_{ij} + \Delta a_{ij}^k$, $\zeta_{ij}^k = d_{ij} + \Delta d_{ij}^k$, $z_{ij} = w_0^{ij} z_{ij}^0 + w_1^{ij} z_{ij}^1$, and $\zeta_{ij} = \omega_0^{ij} \zeta_{ij}^0 + \omega_1^{ij} \zeta_{ij}^1$, $i, j \in \{1, 2, \ldots, n\}$, $k = \{0, 1\}$. Since each term in the main sum of (13) cannot repeat factors[4], by convexity of the weights in (14)-(15), (13) can be rewritten as:

$$q(s, \Delta) = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{B}^n \times \mathbb{B}^n} \mathbf{w_k}(z) \boldsymbol{\omega}_\mathbf{l}(\zeta) q_{\mathbf{kl}}(s), \tag{16}$$

with $q_{\mathbf{kl}}(s) = \sum_{\mathbf{j}} (-1)^{f(\mathbf{j})} \prod_{i=1}^n \left(e^{\tau s} (g(i, j_i) s - z_{i j_i}^{k_{i j_i}}) - \zeta_{i j_i}^{l_{i j_i}} \right)$, $\mathbf{w_k}(z) = w_{k_{11}}^{11}(z_{11}) w_{k_{12}}^{12}(z_{12}) \cdots w_{k_{nn}}^{nn}(z_{nn})$, $\boldsymbol{\omega}_\mathbf{l}(\zeta) = \omega_{l_{11}}^{11}(\zeta_{11}) \omega_{l_{12}}^{12}(\zeta_{12}) \cdots \omega_{l_{nn}}^{nn}(\zeta_{nn})$. In contrast with model (11) (structured uncertainty), (16) (unstructured uncertainty) cannot benefit from powers of uncertainties as they only appear once; in other words, this model is already a box-like one.

Stability of polytopes such as (10), (11) (structured uncertainty) and (16) (unstructured uncertainty) can be established via frequency-based tests from the linear robust control field [20] which are the subject of the next subsection.

III. Stability of Polytopes of Quasi-Polynomials

Stability of a polytope of quasi-polynomials can be established via the Edge Theorem: it is based on guaranteeing the stability of the polytope edges in the coefficient space [21]. An edge is a convex combination of two vertex polynomials; if these vertices are stable, a stability test can be found in [1]. Finally, stability of a single quasi-polynomial can be established via the interlacing condition in [20]. Except for the first one, these results are all frequency-based and are now reviewed in the next lemmas.

Lemma 1. [21] The polytope of quasi-polynomials $f(s) = \sum_{l=1}^N \mu_l f_l(s)$, $\mu_l \geq 0$, $l \in \{1, 2, \ldots, N\}$, $\sum_{l=1}^N \mu_l = 1$, where $f_l(s) = \sum_{k=0}^n \sum_{j=0}^m a_{kj}^l s^k e^{\tau_j s}$, $a_{nm}^l > 0$, is stable if and only if for every pair of quasi-polynomials $f_i(s)$, $f_j(s)$, $i \neq j$, the corresponding one-parameter edge $f_j(s) + \mu g(s)$, $g(s) = f_i(s) - f_j(s)$, $\mu \in [0, 1]$, is stable.

Lemma 2. [1] Consider a pair of quasi-polynomials $f_i(s)$ and $f_j(s)$. The one-parameter edge $f_j(s) + \mu g(s)$, $g(s) = f_i(s) - f_j(s)$, $\mu \in [0, 1]$, is stable if $f_j(s)$ and $f_j(s) + g(s)$ are stable and the following inequality is satisfied $\forall \omega > 0$ wherever the expressions involved are well defined:

$$\frac{\partial \arg(g(j\omega))}{\partial \omega} \leq \frac{\tau_0 + \tau_m}{2} + \left| \frac{\sin(2 \arg(g(j\omega)) - (\tau_0 + \tau_m)\omega)}{2\omega} \right|. \tag{17}$$

[3]This is a consequence of using the tensor-product approach for exact modelling. The interested reader is referred to [18] for details.

[4]See definition of determinant in expression (8), pg. 26 of [19].

Lemma 3. [20] A quasi-polynomial in the form $f(s) = \sum_{k=0}^n \sum_{j=0}^m a_{kj} s^k e^{\tau_j s}$, $a_{nm} \neq 0$, is stable if and only if $\tau_0 + \tau_m \geq 0$, $h(\omega)$ and $g(\omega)$ in $f(j\omega) = h(\omega) + j g(\omega)$ have real, simple, interlacing roots, and $g'(\omega) h(\omega) - g(\omega) h'(\omega) > 0$ holds $\forall \omega \in \mathbb{R}$.

We are now ready to state our main results:

Theorem 1. Assume the characteristic polynomial (8) of the uncertain time-delay system (7) has an exact convex representation (11) for $\theta \in \Theta$ (structured uncertainty). Such system is robustly asymptotically stable if

(i) the quasi-polynomials $\sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k})} q_\mathbf{i}(s)$, $\mathbf{k} \in \mathcal{Q}$, where $\mathcal{Q} = \{(k_1, k_2, \ldots, k_r) : 0 \leq k_j \leq p_j, 1 \leq j \leq r\}$ and $\mathcal{P}(\mathbf{k}) = \{(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_r) : i_1^j + i_2^j + \cdots + i_{p_j}^j = k_j\}$ hold $\tau_0 + \tau_m \geq 0$, with $h(\omega)$ and $g(\omega)$ in $\sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k})} q_\mathbf{i}(j\omega) = h(\omega) + j g(\omega)$ having real, simple, interlacing roots, and $g'(\omega) h(\omega) - g(\omega) h'(\omega) > 0$ $\forall \omega \in \mathbb{R}$,

(ii) each convex combination of quasi-polynomials $\sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k}_1)} q_\mathbf{i}(s)$ and $\sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k}_2)} q_\mathbf{i}(s)$, with $\mathbf{k}_1$, $\mathbf{k}_2 \in \mathcal{Q}$, $\mathbf{k}_1 \neq \mathbf{k}_2$, satisfies $\partial \arg(g(j\omega))/\partial \omega \leq (\tau_0 + \tau_m)/2 + \left| (\sin(2 \arg(g(j\omega)) - (\tau_0 + \tau_m)\omega))/(2\omega) \right|$, $\forall \omega > 0$ with $g(s) = \sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k}_1)} q_\mathbf{i}(s) - \sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k}_2)} q_\mathbf{i}(s)$ wherever the expressions involved are well defined.

Moreover, should each variable $z_i(\cdot)$ be independent of the others, there exists a finite number $\kappa \geq \max_{1 \leq j \leq r} p_j$ such that conditions (i) and (ii) become sufficient and necessary if $\mathcal{P}(\mathbf{k}, \rho)$, standing for the set of all the last $\rho = \sum_{j=1}^r p_j$ bits of $\mathcal{P}_\kappa(\mathbf{k}) = \{(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_r) : i_1^j + i_2^j + \cdots + i_\kappa^j = k_j\}$, $\mathbf{k} \in \mathcal{Q}_\kappa$, $\mathcal{Q}_\kappa = \{(k_1, k_2, \ldots, k_r) : 0 \leq k_j \leq \kappa, 1 \leq j \leq r\}$, replaces any occurrence of $\mathcal{P}(\mathbf{k})$, $\mathcal{P}(\mathbf{k}_1)$, and $\mathcal{P}(\mathbf{k}_2)$, therein.

Proof: The exact convex representation (11), $\theta \in \Theta$, can be transformed as follows by associating similar terms sharing the same $\mathbf{w_i}$-decomposition:

$$\begin{aligned} q(s, \theta) &= \prod_{j=1}^r \sum_{\mathbf{i}_j \in \mathbb{B}^{p_j}} \mathbf{w}_{\mathbf{i}_j}^j(\theta) \sum_{k=0}^n \sum_{l=0}^m a_{kl}^\mathbf{i} s^k e^{\tau_l s} \\ &= \sum_{\mathbf{k} \in \mathcal{Q}} \underbrace{\prod_{j=1}^r (w_0^j)^{p_j - k_j} (w_1^j)^{k_j} \sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k})}}_{\mathbf{w_k}(\theta)} \sum_{k=0}^n \sum_{l=0}^m a_{kl}^\mathbf{i} s^k e^{\tau_l s}, \end{aligned} \tag{18}$$

where $\mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_r)$, $\mathbf{i}_j = (i_1^j, i_2^j, \cdots, i_{p_j}^j) \in \mathbb{B}^{p_j}$, $\mathcal{Q} = \{(k_1, k_2, \ldots, k_r) : 0 \leq k_j \leq p_j, 1 \leq j \leq r\}$ and $\mathcal{P}(\mathbf{k}) = \{(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_r) : i_1^j + i_2^j + \cdots + i_{p_j}^j = k_j\}$, as those indexes $\mathbf{i}$ belonging to $\mathcal{P}(\mathbf{k})$ are similar in $\mathbf{w_k}$ and can thus be associated. From the convex sum properties, it is clear that (18) is a polytope of quasi-polynomials whose vertices are given by

$$\sum_{\mathbf{i} \in \mathcal{P}(\mathbf{k})} \sum_{k=0}^n \sum_{l=0}^m a_{kl}^\mathbf{i} s^k e^{\tau_l s}.$$

By hypothesis (i) along with lemma 3, each of these vertex quasi-polynomials is stable. Moreover, by hypothesis (ii) along with lemma 2 each edge of the polytope is also stable. Thus, conditions in lemma 1 are fulfilled and the polytope of quasi-polynomials is stable, which in turn implies the system is robustly asymptotically stable.

Necessity of conditions (i) and (ii) can be ensured via Pólya-like argumentations [18], since independency of the set of z_j guarantees the coefficient space of the quasi-polynomials will be reached as the number of nested convex sums is increased. Indeed, notice that (18) can be multiplied by any number of nested convex sums (i.e., by 1), let say by κ_j nested convex sums of the convex sum $w_0^j + w_1^j$, $j \in \{1, 2, \ldots, r\}$, where $\kappa_j = \kappa - p_j$; thus:

$$\begin{aligned} q(s,\theta) &= \sum_{\mathbf{k}\in\mathcal{Q}} \mathbf{w_k}(\theta) \sum_{\mathbf{i}\in\mathcal{P}(\mathbf{k})} \sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}^{\mathbf{i}} s^k e^{\tau_l s} \\ &= \prod_{j=1}^{r}\left(\sum_{\ell=0}^{1} w_\ell^j\right)^{\kappa_j} \sum_{\mathbf{k}\in\mathcal{Q}} \mathbf{w_k}(\theta) \sum_{\mathbf{i}\in\mathcal{P}(\mathbf{k})} \sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}^{\mathbf{i}} s^k e^{\tau_l s} \\ &= \sum_{\mathbf{k}\in\mathcal{Q}_\kappa} \mathbf{w_k}(\theta) \sum_{\mathbf{i}\in\mathcal{P}(\mathbf{k},\rho)} \sum_{k=0}^{n}\sum_{l=0}^{m} a_{kl}^{\mathbf{i}} s^k e^{\tau_l s} \qquad (19) \end{aligned}$$

which produces further associations in the polytope quasi-polynomial vertices which get progressively closer to the actual coefficient space, thus concluding the proof. ■

Remark 1. The number of vertices and borders in the resulting polytope is at most 2^κ (without associations). Thus, the number of conditions to be verified is bounded by $2^{\kappa+1}$.

The characteristic quasi-polynomial (13) for systems with unstructured uncertainty (12) already fulfills the requirement of having independent z_{ij}, ζ_{ij}, $i, j \in \{1, 2, \ldots, n\}$; thus, conditions directly deduced from its convex form (16) are sufficient and necessary to represent the coefficient space.

Theorem 2. Assume the characteristic polynomial (13) of the uncertain time-delay system (12) has an exact convex representation (16), for interval matrices ΔA, ΔD (unstructured uncertainty). Such system is robustly asymptotically stable if and only if

(i) the quasi-polynomials $q_{\mathbf{kl}}(s)$, $\mathbf{k}, \mathbf{l} \in \mathbb{B}^n \times \mathbb{B}^n$, hold $h(\omega)$ and $g(\omega)$ in $q_{\mathbf{kl}}(j\omega) = h(\omega) + jg(\omega)$ having real, simple, interlacing roots, and $g'(\omega)h(\omega) - g(\omega)h'(\omega) > 0\ \forall\omega \in \mathbb{R}$,

(ii) each convex combination of quasi-polynomials $q_{\mathbf{k}_1\mathbf{l}_1}(s)$ and $q_{\mathbf{k}_2\mathbf{l}_2}(s)$, with $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{l}_1$, $\mathbf{l}_2 \in \mathbb{B}^n \times \mathbb{B}^n$, $\mathbf{k}_1\mathbf{l}_1 \neq \mathbf{k}_1\mathbf{k}_2$, satisfies $\partial\arg(g(j\omega))/\partial\omega \leq (n\tau)/2 + \left|(\sin(2\arg(g(j\omega)) - (n\tau)\omega))/(2\omega)\right|$, $\forall\omega > 0$ with $g(s) = q_{\mathbf{k}_1\mathbf{l}_1}(s) - q_{\mathbf{k}_2\mathbf{l}_2}(s)$ wherever the expressions involved are well defined.

Proof: As with theorem 1, hypotheses (i) and (ii) along with lemmas 3 and 2 guarantee that each vertex quasi-polynomial $q_{\mathbf{kl}}(s)$ and one-parameter edge in (16) is stable. Thus, conditions in lemma 1 are satisfied and the polytope of quasi-polynomials is stable, which in turn implies the system is robustly asymptotically stable. Necessity comes from the fact that the set of z_{ij}, ζ_{ij}, $i, j \in \{1, 2, \ldots, n\}$ is independent of each other as the model is already a box-like one with no further associations to be made (in other words, no Pólya-like argumentations are needed). ■

Remark 2. Theorems 1 and 2 rely on lemmas 1, 2, and 3, which require normalized quasi-polynomials; they can be obtained dividing each vertex expression by its coefficient of highest power in s, as the weighting functions are unaffected.

IV. Numerical Examples

The examples below illustrate the advantages of the proposed approach against a variety of well-known Lyapunov-based techniques: in example 1 for the motivational example, in example 2 for structured uncertainties, and finally in example 3 for asymptotical exactness.

Example 1. Resuming the motivating example given in (3), it has been seen that two exact convex representations can be found: on the one hand, a matrix form given by (3) with pairs (A_i, D_i) as defined therein; on the other hand, a polytope of quasi-polynomials (5) with a_{ij} as defined therein. In order to apply the results of the previous section, polynomial $q(s,\theta)$ should be rewritten as to avoid having $a_{21} = 0$ since otherwise theorems 1 and 2 cannot be applied, i.e.:

$$\begin{aligned} q^*(s,\theta) &= (s^2 - 0.1s - \theta + 2)e^{\tau s} + (\theta - 1) \\ &= \sum_{k=0}^{2}\sum_{l=0}^{1} a_{kl}^*(\theta) s^k e^{\tau_l s}, \qquad (20) \end{aligned}$$

with $a_{00}^* = \theta - 1$, $a_{01}^* = -\theta + 2$, $a_{10}^* = 0$, $a_{11}^* = -0.1$, $a_{20}^* = 0$, and $a_{21}^* = 1$ will be considered instead. Again, the single uncertainty $z_1(\theta) = \theta \in [\theta^0, \theta^1]$ is to be rewritten as in (3) and (5), i.e., $\theta = w_0(\theta)\theta^0 + w_1(\theta)\theta^1$. This allows obtaining vertex quasi-polynomials $q_0^*(s) = (s^2 - 0.1s + 2 - \theta^0)e^{\tau s} + (\theta^0 - 1)$ and $q_1^*(s) = (s^2 - 0.1s + 2 - \theta^1)e^{\tau s} + (\theta^1 - 1)$ which in turn provide an exact convex form of $q^*(s,\theta)$:

$$q^*(s,\theta) = \sum_{i=0}^{1} w_i(\theta) q_i^*(s). \qquad (21)$$

Let us take advantage of the fact that τ has not been specified in order to compare different methodologies against the one proposed in the previous section: which of them reaches the largest first delay stable pocket (interval) as defined in [22]? Considering $\theta \in [0, 0.5]$, the vertex polynomials are $q_0^*(s) = (s^2 - 0.1s + 2)e^{\tau s} - 1$ and $q_1^*(s) = (s^2 - 0.1s + 1.5)e^{\tau s} - 0.5$, which have the stable delay intervals $[0.1002, 1.7178]$ and $[0.2014, 2.0300]$, respectively.

Table I compares the maximum stable delay interval obtained via theorem 1 (or 2, which can be used instead in this case due to the appearance of only one uncertainty) against others obtained from Lyapunov-Krasovskii functional approaches. Clearly, sufficiency and necessity is met as the family of quasi-polynomials $q(s,\theta)$ is guaranteed stable for $\tau \in [0.2014, 1.7178]$, which is the largest common interval as deduced from the original ones above.

For the sake of comparison, now consider fixed $\theta^0 = 0$ and let vary $\theta^1 \in [0, 1]$. Feasibility of different approaches is examined in figure 1: that based on the Lyapunov-Krasovskii functional is more conservative than the proposed quasi-polynomial approach. Similarly, figure 2 shows that feasibility of our proposal in the coefficient space θ^0, θ^1 for $\tau = 1$ surpasses that of the Lyapunov-Krasovskii approach while the latter is unable to render feasible solutions for $\tau = 1.7$ and $\tau = 2$. Fig. 1 can be obtained via exhaustive single-parameter search [16]; yet, this is not a finite set of tests as proposed.

TABLE I: Range of delay obtained with LMIs.

Approach	τ_{max}	
Based on Jensen's inequality [12]	–	
Based on [23]	[0.2049	1.2548]
Based on [24]	[0.2020	1.4502]
Theorems 1 or 2	[0.2014	1.7178]

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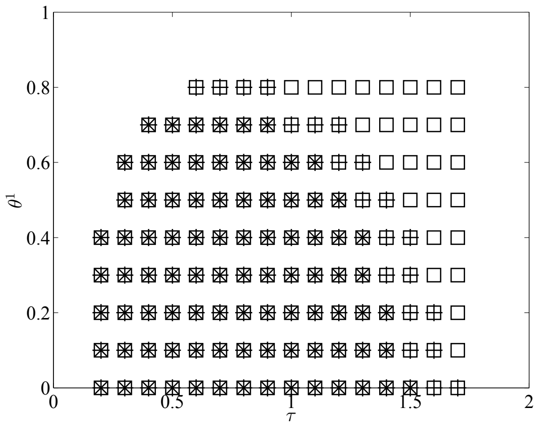

Figure 1: (×) based on [23], (+) based on [24], (□) quasi-polynomial.

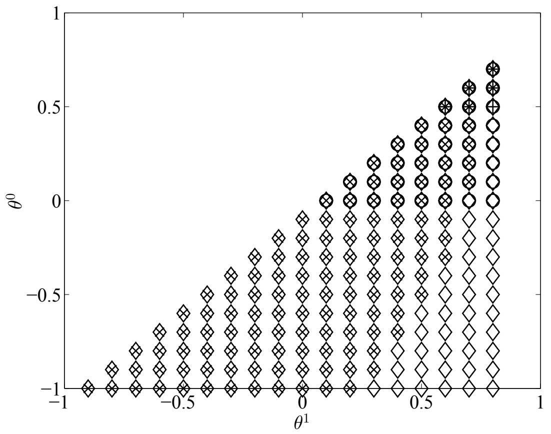

Figure 2: Feasibility in the coefficient space: (×) for $\tau = 1$ based on [23]; (⋄) for $\tau = 1$, (○) for $\tau = 1.7$ and (+) for $\tau = 2$ based on our proposal.

Example 2. Consider the following system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1-\theta & -2 \end{bmatrix}}_{A(\theta)} x + \underbrace{\begin{bmatrix} 0 & 0 \\ \theta & -\theta^2 \end{bmatrix}}_{D(\theta)} x_\tau, \tag{22}$$

where $\theta_1 = \theta \in [\theta_1^0, \theta_1^1]$ and $\theta_2 = \theta^2 \in [\theta_2^0, \theta_2^1]$; its characteristic quasi-polynomial is given by $q(s,\theta) = s^2 + 2s + 1 + \theta + (\theta^2 s - \theta)e^{-\tau s}$.

A convex form of the characteristic quasi-polynomial is $q(s,\theta) = s^2 + 2s + 1 + b_0 + (b_1 s - b_0)e^{-\tau s}$, where b_0 is a coefficient dependent on θ and b_1 is a coefficient dependent on θ^2:

$$\begin{aligned} b_0 &= w_0^1 w_0^2 \theta_1^0 + w_0^1 w_1^2 \theta_1^0 + w_1^1 w_0^2 \theta_1^1 + w_1^1 w_1^2 \theta_1^1, \\ b_1 &= w_0^1 w_0^2 \theta_2^0 + w_0^1 w_1^2 \theta_2^0 + w_1^1 w_0^2 \theta_2^1 + w_1^1 w_1^2 \theta_2^1. \end{aligned}$$

The corresponding quasi-polynomial vertices are:

$$\begin{aligned} q_{00} &= s^2 + 2s + b_0^0 + 1 + (b_1^{00} s - b_0^0)e^{-\tau s}, \\ q_{01} &= s^2 + 2s + b_0^0 + 1 + (b_1^{01} s - b_0^0)e^{-\tau s}, \\ q_{10} &= s^2 + 2s + b_0^1 + 1 + (b_1^{10} s - b_0^1)e^{-\tau s}, \\ q_{11} &= s^2 + 2s + b_0^1 + 1 + (b_1^{11} s - b_0^1)e^{-\tau s}, \end{aligned}$$

where $b_0^0 = \theta_1^0$, $b_1^{00} = \theta_2^0$, $b_1^{10} = \theta_2^0$, $b_0^1 = \theta_1^1$, $b_1^{01} = \theta_2^1$, $b_1^{11} = \theta_2^1$.

Considering $\theta_1 \in [-1, 0]$ and $\theta_2 \in [0, 1]$, and without considering any relationship between the terms, vertex quasi-polynomials are $q_{00}(s) = s^2 + 2s + e^{-s\tau}$, $q_{01}(s) = s^2 + 2s + (s+1)e^{-s\tau}$, $q_{10}(s) = s^2 + 2s + 1$, $q_{11}(s) = s^2 + 2s + 1 + se^{-s\tau}$. Analyzing the stability with theorem 1 without any Pólya relaxation (thus, theorem 2 would be equally valid) the maximum delay yielding feasible solutions is $\tau = 2.7424$.

Now, if the relationship between the parameters is taken into account ($\kappa = 2$), we can make some associations as in (18) as to obtain another convex sum of quasi-polynomials, i.e., $q(s,\theta) = s^2 + 2s + c_0 + 1 + (c_1 s - c_0)e^{-\tau s}$ with

$$\begin{aligned} c_0 &= w_0^1 w_0^2 \theta_1^0 + w_0^1 w_1^2 \theta_1^0 + w_1^1 w_0^2 \theta_1^1 + w_1^1 w_1^2 \theta_1^1 \\ &= (w_0)^2 \theta_1^0 + (w_0^1 w_1^2 + w_1^1 w_0^2)\left(\frac{\theta_1^0 + \theta_1^1}{2}\right) + (w_1)^2 \theta_1^1, \\ c_1 &= w_0^1 w_0^2 \theta_1^0 \theta_1^0 + w_0^1 w_1^2 \theta_1^0 \theta_1^1 + w_1^1 w_0^2 \theta_1^1 \theta_1^0 + w_1^1 w_1^2 \theta_1^1 \theta_1^1, \\ &= (w_0)^2 (\theta_1^0)^2 + (w_0^1 w_1^2 + w_1^1 w_0^2)\left(\frac{\theta_1^0 \theta_1^1 + \theta_1^1 \theta_1^0}{2}\right) + (w_1)^2 (\theta_1^1)^2. \end{aligned}$$

Thus, new vertex quasi-polynomials emerge $q_{00} = s^2 + 2s + c_0^0 + 1 + (c_1^{00} s - c_0^0)e^{-\tau s} = s^2 + 2s + (s+1)e^{-\tau s}$, $0.5(q_{01} + q_{10}) = s^2 + 2s + 0.5 + 0.5e^{-\tau s}$, and $q_{11} = s^2 + 2s + c_0^1 + 1 + (c_1^{11} s - c_0^1)e^{-\tau s} = s^2 + 2s + 1$, where $c_0^0 = \theta_1^0$, $c_0^1 = \theta_1^1$, $c_1^{00} = \theta_1^0\theta_1^0$, $c_1^{01} = \theta_1^0\theta_1^1$, $c_1^{10} = \theta_1^1\theta_1^0$, $c_1^{11} = \theta_1^1\theta_1^1$, $q_{01} = s^2 + 2s + c_0^0 + 1 + (c_1^{01} s - c_0^0)e^{-\tau s} = s^2 + 2s + e^{-\tau s}$ and $q_{10} = s^2 + 2s + c_0^1 + 1 + (c_1^{10} s - c_0^1)e^{-\tau s} = s^2 + 2s + 1$.

Conditions in theorem 1 (theorem 2 cannot be longer applied) are feasible for a maximum delay of $\tau = 3.2809$, which improves over the results without associations. In [24], the maximum time delay using the Lyapunov-Krasovskii approach is $\tau = 1.1186$, which is significantly lower than the quasi-polynomial approach. The results are gathered in table II. The convex hull for each vertex quasi-polynomials is shown in figure 3: clearly, new associations result in closer

TABLE II: Maximum delay $\tau_{\max}$ allowed

Approach	τ_{max}
Without any association $\kappa = 2$ (th.2)	2.7424
With association $\kappa = 2$ (th.1)	3.2809
With association $\kappa = 3$ (th.1)	3.2809
With association $\kappa = 4$ (th.1)	3.2809
Based on [23]	1.1186

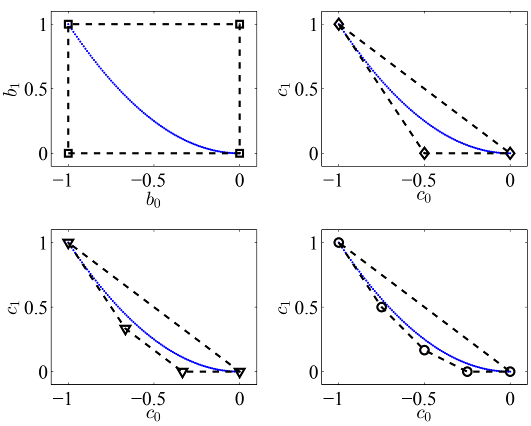

Figure 3: Coefficient space of (22). Non-associated vertices: (□); Associated vertices: (⋄) $\kappa = 2$, (▽) $\kappa = 3$, (○) $\kappa = 4$.

polytopes to the actual coefficient space, yet, once some fixed number of sums is reached, improvements are no longer possible, which is consistent with Pólya-like asymptotic sufficient and necessary conditions [18].

Example 3. Consider the following system:

$$\dot{x}=\underbrace{\begin{bmatrix}0 & 1\\ \theta_1-13 & -2\end{bmatrix}}_{A(\theta)}x+\underbrace{\begin{bmatrix}0 & 0\\ \theta_1{}^2+4\theta_1-36-\theta_2{}^2+2\theta_2-1 & 0\end{bmatrix}}_{D(\theta)}x_\tau, \tag{23}$$

where $\theta_1 \in [-6,3]$, $\theta_2 \in [-1,1]$. The generalized quasi-polynomial is $q(s,\theta) = s^2 + 2s + b_0 + b_1 e^{-\tau s}$, where $b_0 = -\theta_1 + 13$ and $b_1 = -\theta_1{}^2 - 4\theta_1 + 36 + \theta_2{}^2 - 2\theta_2 + 1$.

Considering no associated terms, 16 vertex quasi-polynomials are obtained but two of them are unstable: $s^2 + 2s + 10 - 12e^{-\tau s}$ and $s^2 + 2s + 10 - 13e^{-\tau s}$; therefore, conditions in theorems 1 or 2 are infeasible. However, associating similar terms ($\kappa = 4$) the polytope reduces to six vertex quasi-polynomials (resulting from association):

$$s^2+2s+19+24e^{-\tau s},\ s^2+2s+19+28e^{-\tau s},$$
$$s^2+2s+14.5+60e^{-\tau s},\ s^2+2s+14.5+64e^{-\tau s},$$
$$s^2+2s+10+19e^{-\tau s},\ s^2+2s+10+15e^{-\tau s},$$

for which theorem 1 is feasible up to $\tau = 0.0316$.

Furthermore, introducing Pólya relaxations via nested convex sums with $\kappa = 6$ and $\kappa = 8$ in theorem 1 (leading to 12 and 20 different vertex quasi-polynomials, respectively), the maximum delay with feasible solutions can be increased as shown in table III. The corresponding polytopes in the coefficient space for each of the considered cases is shown in figure 4 along with marks representing the vertex quasi-polynomials. Importantly, the Lyapunov-Krasovskii approach is always infeasible.

TABLE III: Maximum delay $\tau_{\max}$ allowed

Approach	τ_{max}
Without any association $\kappa = 4$ (th.2)	Unstable
With association $\kappa = 4$ (th.1)	0.0316
With association $\kappa = 6$ (th.1)	0.0391
With association $\kappa = 8$ (th.1)	0.0402
Based on [23]	Not feasible

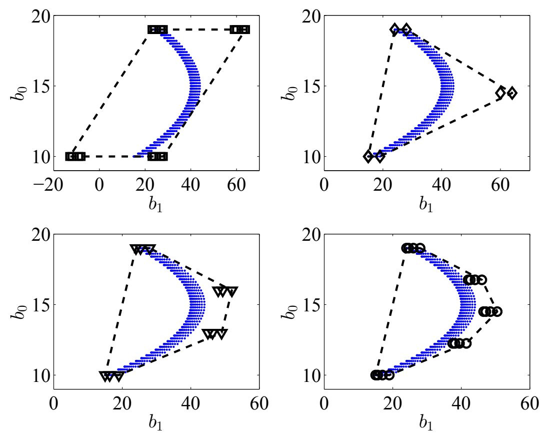

Figure 4: Coefficient space of (23). Non-associated vertices: ($\square$) $\kappa = 4$; associated vertices: ($\diamond$) $\kappa=4$, ($\triangledown$) $\kappa=6$, ($\circ$) $\kappa=8$.

V. Conclusions

A novel approach for stability analysis of uncertain time-delay systems has been presented which is based on an exact convex rewriting of the generalized characteristic quasi-polynomial. It has been shown that the modelling methodology allows exploiting dependency between coefficients of the system quasi-polynomial via Pólya-like relaxations. Examples have been provided where the proposed techniques outperform Lyapunov-Krasovskii approaches.

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