The topics for this discussion are Exercises 6.6, 6.10, 6.15, 6.28, 6.29, and 6.31 in the textbook.

Exercise 6.6

The joint distribution of amount of pollutant emitted from a smokestack without a cleaning device (Y_1) and a similar smokestack with a cleaning device (Y_2) was given in Exercise 5.10 to be

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 2, \quad 0 \le y_2 \le 1, 2y_2 \le y_1 \\ 0, & \text{elsewhere.} \end{cases}$$

The reduction in amount of pollutant due to the cleaning device is given by $U = Y_1 - Y_2$.

a Find the probability density function for U.

b Use the answer in part (a) to find $E(U)$.

Exercise 6.28

Let Y have a uniform $(0, 1)$ distribution. Show that $U = -2\ln(Y)$ has an exponential distribution with mean 2.

Exercise 6.15

Let Y have a distribution function given by

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0 \end{cases}$$

Find a transformation $G(U)$ such that, if U has a uniform distribution on the interval $(0, 1)$, $G(U)$ has the same distribution as Y.

Exercise 6.10

The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2, were given in Exercise 5.15 with joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find

a The probability density function for U.

b $E(U)$ and $V(U)$.

Exercise 6.29

The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f(v) = av^2 e^{-bv^2}, \quad v > 0$$

where $b = m/2kT$ and k, T, and m denote Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively.

- **a** Derive the distribution of $W = mV^2/2$, the kinetic energy of the molecule.
- **b** Find $E(W)$.

Exercise 6.31

The joint distribution for the length of life of two different types of components operating in a system was given in Exercise 5.18 by

$$f\left(y_1, y_2\right) = \begin{cases} (1/8) y_1 e^{-(y_1+y_2)/2}, & y_1 > 0, y_2 > 0 \\ 0, & \text{elsewhere} \end{cases}$$

The relative efficiency of the two types of components is measured by $U = Y_2/Y_1$. Find the probability density function for U.

Solution

Exercise 6.6

Refer to Ex. 5.10 ad 5.78. Define $F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq Y_2 + u)$.

a For $u \leq 0, F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = 0$.

For $0 \leq u < 1, F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = \int_0^u \int_{2y_2}^{y_2+u} 1 dy_1 dy_2 = u^2/2$.

For $1 \leq u \leq 2, F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = 1 - \int_0^{2-u} \int_{y_2+u}^{2} 1 dy_1 dy_2 = 1 - (2-u)^2/2$.

Thus, $f_U(u) = F_U'(u) = \begin{cases} u & 0 \leq u < 1 \\ 2-u & 1 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$.

b $E(U) = 1$.

Exercise 6.10

Refer to Ex. 5.15 and Ex. 5.108.

a $F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$, so that $f_U(u) = F_U'(u) = e^{-u}, u \geq 0$, so that U has an exponential distribution with $\beta = 1$.

b From part a above, $E(U) = 1$.

Exercise 6.15

Let U have a uniform distribution on $(0,1)$. The distribution function for U is $F_U(u) = P(U \leq u) = u, 0 \leq u \leq 1$. For a function G, we require $G(U) = Y$ where Y has distribution function $F_Y(y) = 1 - e^{-y^2}, y \geq 0$. Note that

$$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P\left[U \leq G^{-1}(y)\right] = F_U\left[G^{-1}(y)\right] = G^{-1}(y).$$

So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$ so that $G(u) = [-\ln(1-u)]^{-1/2}$. Therefore, the random variable $Y = [-\ln(U-1)]^{-1/2}$ has distribution function $F_Y(y)$.

Exercise 6.28

If Y is uniform on the interval $(0,1), f_U(u) = 1$. Then,
$Y = e^{-U/2}$ and $\frac{dv}{du} = -\frac{1}{2}e^{-u/2}$. Then, $f_Y(y) = 1\left|-\frac{1}{2}e^{-u/2}\right| = \frac{1}{2}e^{-u/2}, u \geq 0$ which is exponential with mean 2.

Exercise 6.29

a With $W = \frac{mV^2}{2}, V = \sqrt{\frac{2W}{m}}$ and $\left|\frac{dv}{dw}\right| = \frac{1}{\sqrt{2mw}}$. Then,

$$f_W(w) = \frac{a(2w/m)}{\sqrt{2mw}} e^{-2bw/m} = \frac{a\sqrt{2}}{m^{3/2}} w^{1/2} e^{-w/kT}, w \geq 0.$$

The above expression is in the form of a gamma density, so the constant a must be chosen so that the density integrate to 1 , or simply

$$\frac{a\sqrt{2}}{m^{3/2}} = \frac{1}{\Gamma\left(\frac{3}{2}\right)(kT)^{3/2}}$$

So, the density function for W is

$$f_W(w) = \frac{1}{\Gamma\left(\frac{3}{2}\right)(kT)^{3/2}} w^{1/2} e^{-w/kT}$$

b For a gamma random variable, $E(W) = \frac{3}{2}kT$.

Exercise 6.31

Similar to Ex. 6.25. Fix $Y_1 = y_1$. Then, $U = Y_2/y_1, Y_2 = y_1 U$ and $\left|\frac{dy_2}{du}\right| = y_1$. The joint density of Y_1 and U is

$$f(y_1, u) = \frac{1}{8} y_1^2 e^{-y_1(1+u)/2}, y_1 \geq 0, u \geq 0.$$

So, the marginal density for U is

$$f_U(u) = \int_0^\infty \frac{1}{8} y_1^2 e^{-y_1(1+u)/2} dy_1 = \frac{2}{(1+u)^3}, u \geq 0.$$