Exercise 9.22

Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a Poisson-distributed population with mean λ. Again, assume that $n = 2k$ for some integer k. Consider

$$\hat{\lambda} = \frac{1}{2k} \sum_{i=1}^{k} (Y_{2i} - Y_{2i-1})^2$$

a Show that $\hat{\lambda}$ is an unbiased estimator for λ.

b Show that $\hat{\lambda}$ is a consistent estimator for λ.

Exercise 9.27

Use the method described in Exercise 9.26 to show that, if $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$ when $Y_1, Y_2, \ldots, Y_n$ are independent uniform random variables on the interval $(0, \theta)$, then $Y_{(1)}$ is not a consistent estimator for θ. [Hint: Based on the methods of Section 6.7, $Y_{(1)}$ has the distribution function

$$F_{(1)}(y) = \begin{cases} 0, & y < 0, \\ 1 - (1 - y/\theta)^n, & 0 \le y \le \theta, \\ 1, & y > \theta.] \end{cases}$$

Exercise 9.35

Let $Y_1, Y_2, \ldots$ be a sequence of random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma_i^2$. Notice that the σ_i^2,s are not all equal.

- **a** What is $E\left(\bar{Y}_n\right)$?
- **b** What is $V\left(\bar{Y}_n\right)$?
- **c** Under what condition (on the σ_i^2 's) can Theorem 9.1 be applied to show that $\bar{Y}_n$ is a consistent estimator for μ?

Exercise 9.40

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Rayleigh distribution with parameter θ. Show that $\sum_{i=1}^{n} Y_i^2$ is sufficient for θ.

Consistency

For a statistical estimation problem:

- $X_1, \ldots, X_n$ i.i.d. P_θ,
- $q(\theta)$ is the target of estimation,
- $\hat{q}_n = \hat{q}(X_1, \ldots, X_n)$ is an estimator of $q(\theta)$.

We have the following definition of consistent estimator:

Definition 1 (weakly consistent). *$\hat{q}_n$ is a consistent estimator of $q(\theta)$, i.e., $\hat{q}_n(X_1, \ldots, X_n) \xrightarrow{p} q(\theta)$ if for every $\epsilon > 0$,*

$$\lim_{n\to\infty} P\left(\left|\hat{q}_n(X_1, \ldots, X_n) - q(\theta)\right| > \epsilon\right) = 0.$$

Corollary 1. *Consider P_θ such that $E[X_1] = q(\theta) = \theta$, $\hat{q}_n = \bar{X}$. Then*

- *If $\mathrm{Var}(X_1) = \sigma^2 < \infty$, apply Chebychev's Inequality. For any $\epsilon > 0$:*

$$0 \leq P_\theta\left(\left|\hat{q}_n - \theta\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}(\hat{q}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{n\to\infty} 0.$$

- *If $\mathrm{Var}(X_1) = \infty$, $\hat{q}_n$ is (weakly) consistent if $E[|X_1|] < \infty$. Proof: by Levy Continuity Theorem, or read chapter 10 section 2 of the first volume of Feller's book (an introduction to probability theory and its applications). The proof is not required for this class.*
  Example: X_i has Pareto distribution with pdf $f(x) = \frac{2a^2}{x^3} I\{x \geq a\}$, where $a > 0$ is a fixed constant. Then $E[|X_1|] = E[X_1] = 2a$, $\mathrm{Var}(X_1) = \infty$. $\bar{X}$ is a weakly consistent estimator of $2a$.

Definition 2 (uniformly consistent). *$\hat{q}_n$ is a uniformly consistent estimator of $q(\theta)$, if for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \left(\sup_{\theta\in\Theta} \left[P\left(\left|\hat{q}_n - q(\theta)\right| > \epsilon\right) \right] \right) = 0.$$

Corollary 2. *Consider P_θ such that $E[X_1] = q(\theta) = \theta$, $\hat{q}_n = \bar{X}$. Then the proof of weak consistency of $\hat{q}_n$ extends to uniform consistency if*

$$\sup_{\theta\in\Theta} \mathrm{Var}(\hat{q}_n) \leq M < \infty. \tag{1}$$

Here M is a fixed constant. Examples satisfying (1):

- *X_i i.i.d. Bernoulli(θ).*
- *X_i i.i.d. Noremal(μ, σ^2), where $\theta = (\mu, \sigma^2)$ and $\sigma^2 \leq M < \infty$.*

Definition 3 (strongly consistent). *$\hat{q}_n$ is a strongly consistent estimator of $q(\theta)$, if*

$$P\left(\lim_{n\to\infty} \left[\left|\hat{q}_n - q(\theta)\right| \leq \epsilon\right]\right) = 1, \text{ for every } \epsilon > 0.$$

$$\hat{q}_n \xrightarrow{a.s.} q(\theta). \quad (a.s. \equiv \text{"almost surely"})$$

Corollary 3 (relationships between consistency).

strongly consistent $\Longrightarrow$ uniformly consistent $\Longrightarrow$ weakly consistent

Exercise 9.22

We have that the estimator $\hat{\lambda}$ can be written as

$$\hat{\lambda} = \frac{1}{k}\left[\frac{(Y_2 - Y_1)^2}{2} + \frac{(Y_4 - Y_3)^2}{2} + \frac{(Y_6 - Y_5)^2}{2} + \ldots + \frac{(Y_n - Y_{n-1})^2}{2}\right]$$

For Y_i, Y_{i-1}, we have that:

$$\frac{E\left[(Y_i - Y_{i-1})^2\right]}{2} = \frac{E\left(Y_i^2\right) - 2E\left(Y_i\right)E\left(Y_{i-1}\right) + E\left(Y_{i-1}^2\right)}{2} = \frac{(\lambda + \lambda^2) - 2\lambda^2 + (\lambda + \lambda^2)}{2} = \lambda$$
$$\frac{V\left[(Y_i - Y_{i-1})^2\right]}{4} < \frac{V\left(Y_i^2\right) + V\left(Y_{i-1}^2\right)}{4} = \frac{2\lambda + 12\lambda^2 + 8\lambda^3}{4} = \gamma, \text{ since } Y_i \text{ and } Y_{i-1} \text{ are}$$

independent and non-negative (the calculation can be performed using the Poisson mgf).

a From the above, $E(\hat{\lambda}) = (k\lambda)/k = \lambda$. So $\hat{\lambda}$ is an unbiased estimator of λ.

b Similarly, $V(\hat{\lambda}) < k\gamma/k^2$, where $\gamma < \infty$ is defined above. Since $k = n/2, V(\hat{\lambda})$ goes to 0 with n and $\hat{\lambda}$ is a consistent estimator.

Exercise 9.27

$$P\left(\left|Y_{(1)} - \theta\right| \leq \varepsilon\right) = P\left(\theta - \varepsilon \leq Y_{(1)} \leq \theta + \varepsilon\right) = F_{(1)}(\theta + \varepsilon) - F_{(1)}(\theta - \varepsilon) = 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n = \left(\frac{\varepsilon}{\theta}\right)^n.$$

But, $\lim_{n\to\infty}\left(\frac{\varepsilon}{\theta}\right)^n = 0$ for $\varepsilon < \theta$. So, $Y_{(1)}$ is not consistent.

Exercise 9.35

a $E\left(\bar{Y}_n\right) = \frac{1}{n}(\mu + \mu + \cdots + \mu) = \mu$, so $\bar{Y}_n$ is unbiased for μ.

b $V\left(\bar{Y}_n\right) = \frac{1}{n^2}\left(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2\right) = \frac{1}{n^2}\sum_{i=1}^n \sigma_i^2$.

c In order for $\bar{Y}_n$ to be consistent, it is required that $V\left(\bar{Y}_n\right) \to 0$ as $n \to \infty$. Thus, it must be true that all variances must be finite, or simply $\max_i\left\{\sigma_i^2\right\} < \infty$.

Exercise 9.40

The likelihood is $L(\theta) = 2^n\theta^{-n}\prod_{i=1}^n y_i \exp\left(-\sum_{i=1}^n y_i^2/\theta\right)$. By Theorem 9.4, $U = \sum_{i=1}^n Y_i^2$ is sufficient for θ with $g(u, \theta) = \theta^{-n}\exp(-u/\theta)$ and $h(y) = 2^n\prod_{i=1}^n y_i$.