The questions for this discussion are Exercises 6.17, 6.52, and 6.88 from textbook.

## Exercise 6.17

A member of the power family of distributions has a distribution function given by

$$F\left(y\right)=\begin{cases}0, & y<0,\\ \left(\frac{y}{\theta}\right)^{\alpha}, & 0\leq y\leq\theta,\\ 1, & y>\theta\end{cases}$$

where $\alpha,\theta>0$.

**a** Find the probability density function.

**b** For fixed values of $\theta$ and $\alpha$, find a transformation $G(U)$ so that $G(U)$ has a distribution function of $F$ when $U$ has a uniform distribution on the interval $(0,1)$.

**c** Given that a random sample of size 5 from a uniform distribution on the interval $(0,1)$ yielded the values .2700, .6901, .1413, .1523, and .3609, use the transformation derived in part **b** to give values associated with a random variable with a power family distribution with $\alpha=2$, $\theta=4$.

**b** and **c** can be used to sample from a known strictly increasing distribution function (Inverse transform sampling): We can first uniformly sample $u$ from the interval $(0,1)$, then apply the transformation $G(\cdot)$ and $G(u)$ is equivalent to sampling directly from the distribution $F$.

# Exercise 6.52 (correlated to exercise 6.54)

Let $Y_1$ and $Y_2$ be independent Poisson random variables with means $\lambda_1$ and $\lambda_2$, respectively. Find the

**a** probability function of $Y_1 + Y_2$.

**b** conditional probability function of $Y_1$, given that $Y_1 + Y_2 = m$.

# Exercise 6.88

Suppose that the length of time $Y$ it takes a worker to complete a certain task has the probability density function given by

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta, \\ 0, & \textit{elsewhere}, \end{cases}$$

where $\theta$ is a positive constant that represents the minimum time until task completion. Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of completion times from this distribution. Find

**a** the density function for $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$.

**b** $E\left(Y_{(1)}\right)$.

# Supplement content:

1. Let $Y_1, \ldots, Y_n$ be a random i.i.d sample with mean $\mu$ and variance $\sigma^2$. Find $E[(Y_i - \bar{Y})^2]$.
2. From 1., find $E(S^2)$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$.
   i. The sum of the squares of $Y_1, \ldots, Y_n$ is: $\sum_{i=1}^{n} Y_i^2 = (n-1)S^2 + n\bar{Y}^2$
   ii. From 2.i., we have $\sum_{i=1}^{n}(Y_i - \mu)^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2 + n(\bar{Y} - \mu)^2$.
3. Further consider that all samples $Y_1, \ldots, Y_n$ come from a normal distribution:
   i. How to prove that $\bar{Y}$ and $Y_i - \bar{Y}$ are independent for all $i = 1, \ldots, n$?
   ii. From 3.i., how to prove $\bar{Y}$ and $S^2$ are independent?
   iii. From 2.ii. and 3.ii., how to prove $\dfrac{(n-1)S^2}{\sigma^2}$ has a $\chi^2$ distribution with $(n-1)$ degrees of freedom? (Hint: moment generating function)
4. – Suppose $U \sim U(0,1)$, $\beta > 0$. Then $-\beta \ln(U) \sim \mathit{Exp}(\beta) = \mathit{Gamma}(1, \beta)$.
   – Suppose all $Z_i$ are independent standard normal random variables, then $\sum_{i=1}^{k} Z_i^2 \sim \chi^2(k) = \mathit{Gamma}(\frac{k}{2}, 2)$.
5. About Gamma distribution:

Summation: If $X_i$ has a $\mathit{Gamma}(\alpha_i, \beta)$ distribution for $i = 1, \ldots, m$ and all $X_i$ are independent, then $\sum_{i=1}^{m} X_i \sim \mathit{Gamma}(\sum_{i=1}^{m} \alpha_i, \beta)$

Scaling: If $X \sim \mathit{Gamma}(\alpha, \beta)$, then for any $c > 0$: $cX \sim \mathit{Gamma}(\alpha, c\beta)$.

# Solution

## Exercise 6.17

**a** Taking the derivative of $F(y), f(y)=\frac{\alpha y^{\alpha-1}}{\theta^{\alpha}}, 0 \leq y \leq \theta$.

**b** Following Ex.6.15 and 6.16 , let $u=\left(\frac{y}{\theta}\right)^{\alpha}$ so that $y=\theta u^{1/\alpha}$. Thus, the random variable $Y=\theta U^{1/a}$ has distribution function $F_Y(y)$.

**c** From part (b), the transformation is $y=4\sqrt{u}$. The values are $2.0785, 3.229, 1.5036,\ 1.5610, 2.403$.

## Exercise 6.52

The mgfs for $Y_1$ and $Y_2$ are, respectively, $m_{Y_1}(t)=e^{\lambda_1\left(e^t-1\right)}, m_{Y_2}(t)=e^{\lambda_2\left(e^t-1\right)}$.

**a** Since $Y_1$ and $Y_2$ are independent, the mgf for $Y_1+Y_2$ is $m_{Y_1}(t)\times m_{Y_2}(t)=e^{\left(\lambda_1+\lambda_2\right)\left(e^t-1\right)}$. This is the mgf of a Poisson with mean $\lambda_1+\lambda_2$.

**b** From Ex. 5.39 , the distribution is binomial with $m$ trials and $p=\frac{\lambda_1}{\lambda_1+\lambda_2}$.

## Exercise 6.88

This is somewhat of a generalization of Ex. 6.87. The distribution function of $Y$ is

$$F(y)=P(Y \leq y)=\int_{\theta}^{y} e^{-(t-\theta)} dy=1-e^{-(y-\theta)}, y>\theta$$

**a** $g_{(1)}(y)=n\left[e^{-(y-\theta)}\right]^{n-1} e^{-(y-\theta)}=n e^{-n(y-\theta)}, y>\theta$.

**b** $E\left(Y_{(1)}\right)=\frac{1}{n}+\theta$.