Exercise 9.79

Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed random variables from a Pareto distribution with parameters α and β, where β is known. Then, if $\alpha > 0$,

$$f(y \mid \alpha, \beta) = \begin{cases} \alpha\beta^{\alpha}y^{-(\alpha+1)}, & y \geq \beta \\ 0, & \text{elsewhere} \end{cases}$$

Show that $E(Y_i) = \alpha\beta/(\alpha - 1)$ if $\alpha > 1$ and $E(Y_i)$ is undefined if $0 < \alpha < 1$. Thus, the method-of-moments estimator for α is undefined.

Exercise 9.94

Suppose that $\hat{\theta}$ is the MLE for a parameter θ. Let $t(\theta)$ be a function of θ that possesses a unique inverse [that is, if $\beta = t(\theta)$, then $\theta = t^{-1}(\beta)$]. Show that $t(\hat{\theta})$ is the MLE of $t(\theta)$.

Exercise 9.92

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with density function

$$f(y \mid \theta) = \begin{cases} \frac{3y^2}{\theta^3}, & 0 \leq y \leq \theta \\ 0, & \text{elsewhere} \end{cases}$$

Exercise 9.52 showed that $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ is sufficient for θ.

a Find the MLE for θ.

b Find a function of the MLE in part (a) that is a pivotal quantity.

c Use the pivotal quantity from part (b) to find a $100(1-\alpha)\%$ confidence interval for θ.

Exercise 9.108

Suppose that a random sample of length-of-life measurements, $Y_1, Y_2, \ldots, Y_n$, is to be taken of components whose length of life has an exponential distribution with mean θ.

a Find the MLE of $\bar{F}(t) = 1 - F(t) = e^{-t/\theta}$. Is the MLE unbiased?

b Use the Rao-Blackwell theorem to find the MVUE of $e^{-t/\theta}$.

Exercise 9.109

Suppose that n integers are drawn at random and with replacement from the integers $1, 2, \ldots, N$. That is, each sampled integer has probability $1/N$ of taking on any of the values $1, 2, \ldots, N$, and the sampled values are independent.

a Find the method-of-moments estimator $\hat{N}_1$ of N. Find $E(\hat{N}_1)$ and $V(\hat{N}_1)$.

b Find the MLE $\hat{N}_2$ of N. Show that $E\left(\hat{N}_2\right)$ is approximately $[n/(n+1)]N$. Adjust $\hat{N}_2$ to form an estimator $\hat{N}_3$ that is approximately unbiased for N.

c Find an approximate variance for $\hat{N}_3$ by using the fact that for large N the variance of the largest sampled integer is approximately

$$\frac{nN^2}{(n+1)^2(n+2)}$$

d Show that for large N and $n > 1, V\left(\hat{N}_3\right) < V\left(\hat{N}_1\right)$.

Exercise 9.79

For Y following the given Pareto distribution,

$$E(Y) = \int_{\beta}^{\infty} \alpha\beta^{\alpha} y^{-\alpha} dy = \alpha\beta^{\alpha} \frac{y^{-\alpha+1}}{-\alpha+1}\bigg|_{\beta}^{\infty} = \alpha\beta/(\alpha-1).$$

The mean is not defined if $\alpha < 1$. Thus, a generalized MOM estimator for α cannot be expressed.

Exercise 9.94

Let $\beta = t(\theta)$ so that $\theta = t^{-1}(\beta)$. If the likelihood is maximized at $\hat{\theta}$, then $L(\hat{\theta}) \geq L(\theta)$ for all θ. Define $\hat{\beta} = t(\hat{\theta})$ and denote the likelihood as a function of β as $L_1(\beta) = L\left(t^{-1}(\beta)\right)$. Then, for any β,

$$L_1(\beta) = L\left(t^{-1}(\beta)\right) = L(\theta) \leq L(\hat{\theta}) = L\left(t^{-1}(\hat{\beta})\right) = L_1(\hat{\beta}).$$

So, the MLE of β is $\hat{\beta}$ and so the MLE of $t(\theta)$ is $t(\hat{\theta})$.

Exercise 9.92

a The MLE of θ is $\hat{\theta} = Y_{(n)}$.

b From Ex. 9.63, $f_{(n)}(y) = 3ny^{3n-1}/\theta^{3n}, 0 \leq y \leq \theta$. The distribution of $T = Y_{(n)}/\theta$ is

$$f_T(t) = 3nt^{3n-1}, 0 \leq t \leq 1.$$

Since this distribution doesn't depend on θ, T is a pivotal quantity.

c Constants a and b can be found to satisfy $P(a < T < b) = 1-\alpha$ such that $P(T < a) = P(T > b) = \alpha/2$. Using the density function from part b , these are given by $a = (\alpha/2)^{1/(3n)}$ and $b = (1-\alpha/2)^{1/(3n)}$. So, we have

$$1-\alpha = P\left(a < Y_{(n)}/\theta < b\right) = P\left(Y_{(n)}/b < \theta < Y_{(n)}/a\right)$$

Thus, $\left(\frac{Y_{(n)}}{(1-\alpha/2)^{1/(3n)}}, \frac{Y_{(n)}}{(\alpha/2)^{1/(3n)}}\right)$ is a $(1-\alpha)100\%$C.I. for θ.

Exercise 9.108

a The MLE of θ is $\hat{\theta} = \bar{Y}$. By the invariance principle for MLEs, the MLE of $\bar{F}(t)$ is $\hat{\bar{F}}(t) = \exp(-t/\bar{Y})$. It is not unbiased.

b
- Let
$$V = \begin{cases} 1, & Y_1 > t \\ 0, & \text{elsewhere.} \end{cases}$$
$E(V) = P(Y_1 > t) = 1 - F(t) = \exp(-t/\theta)$. Thus, V is unbiased for $\exp(-t/\theta)$.
- $U = \sum_{i=1}^n Y_i$ is the minimal sufficient statistic for θ. Recall that U has a gamma distribution with shape parameter n and scale parameter θ. Also, $U - Y_1 = \sum_{i=2}^n Y_i$ is gamma with shape parameter $n-1$ and scale parameter θ, and since Y_1 and $U - Y_1$ are independent,
$$f(y_1, u - y_1) = \left(\frac{1}{\theta} e^{-y_1/\theta}\right) \frac{1}{\Gamma(n-1)\theta^{n-1}} (u - y_1)^{n-2} e^{-(u-y_1)/\theta}, 0 \le y_1 \le u < \infty.$$
Next, apply the transformation $z = u - y_1$ such that $u = z + y_1$ to get the joint distribution
$$f(y_1, u) = \frac{1}{\Gamma(n-1)\theta^n} (u - y_1)^{n-2} e^{-u/\theta}, 0 \le y_1 \le u < \infty$$
Now, we have
$$f(y_1 \mid u) = \frac{f(y_1, u)}{f(u)} = \left(\frac{n-1}{u^{n-1}}\right) (u - y_1)^{n-2}, 0 \le y_1 \le u < \infty$$
-
$$E(V \mid U) = P(Y_1 > t \mid U = u) = \int_t^u \left(\frac{n-1}{u^{n-1}}\right) (u - y_1)^{n-2} dy_1 = \int_t^u \left(\frac{n-1}{u}\right) \left(1 - \frac{y_1}{u}\right)^{n-2} dy_1$$
$$= -\left(1 - \frac{y_1}{u}\right)^{n-1} \Bigg|_t^u = \left(1 - \frac{t}{u}\right)^{n-1}$$
So, the MVUE is $\left(1 - \frac{t}{U}\right)^{n-1}$.

Exercise 9.109

Let $Y_1, Y_2, \ldots, Y_n$ represent the (independent) values drawn on each of the n draws. Then, the probability mass function for each Y_i is

$$P\left(Y_i = k\right) = \frac{1}{N}, k = 1, 2, \ldots, N.$$

a Since $\mu_1' = E(Y) = \sum_{k=1}^{N} kP(Y = k) = \sum_{k=1}^{N} k\frac{1}{N} = \frac{N(N+1)}{2N} = \frac{N+1}{2}$, the MOM estimator of N is $\frac{\hat{N}_1+1}{2} = \bar{Y}$ or $\hat{N}_1 = 2\bar{Y} - 1$.

$E(\hat{N}_1) = 2E(\bar{Y}) - 1 = 2\left(\frac{N+1}{2}\right) - 1 = N$, so $\hat{N}_1$ is unbiased.
Now, since $E\left(Y^2\right) = \sum_{k=1}^{N} k^2\frac{1}{N} = \frac{N(N+1)(2N+1)}{6N} = \frac{(N+1)(2N+1)}{6}$, we have that $V(Y) = \frac{(N+1)(N-1)}{12}$.
Thus, $V(\hat{N}_1) = 4V(\bar{Y}) = 4\left(\frac{(N+1)(N-1)}{12n}\right) = \frac{N^2-1}{3n}$.

b The likelihood is

$$L(N) = \frac{1}{N^n}\prod_{i=1}^{n} I\left(y_i \in \{1, 2, \ldots, N\}\right) = \frac{1}{N^n} I\left(y_{(n)} \leq N\right)$$

In order to maximize L, N should be chosen as small as possible subject to the constraint that $y_{(n)} \leq N$. Thus $\hat{N}_2 = Y_{(n)}$.
Since $P\left(\hat{N}_2 \leq k\right) = P\left(Y_{(n)} \leq k\right) = P\left(Y_1 \leq k\right)\cdots P\left(Y_n \leq k\right) = \left(\frac{k}{N}\right)^n$, so $P\left(\hat{N}_2 \leq k-1\right) = \left(\frac{k-1}{N}\right)^n$ and $P\left(\hat{N}_2 = k\right) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n = N^{-n}\left[k^n - (k-1)^n\right]$. So,

$$\begin{aligned} E\left(\hat{N}_2\right) &= N^{-n}\sum_{k=1}^{N} k\left[k^n - (k-1)^n\right] = N^{-n}\sum_{k=1}^{N}\left[k^{n+1} - (k-1)^{n+1} - (k-1)^n\right] \\ &= N^{-n}\left[N^{n+1} - \sum_{k=1}^{N}(k-1)^n\right] \end{aligned}$$

Consider $\sum_{k=1}^{N}(k-1)^n = 0^n + 1^n + 2^n + \ldots + (N-1)^n$. For large N, this is approximately the area beneath the curve $f(x) = x^n$ from $x = 0$ to $x = N$, or $\sum_{k=1}^{N}(k-1)^n \approx \int_0^N x^n dx = \frac{N^{n+1}}{n+1}$. Thus, $E\left(\hat{N}_2\right) \approx N^{-n}\left[N^{n+1} - \frac{N^{n+1}}{n+1}\right] = \frac{n}{n+1}N$ and $\hat{N}_3 = \frac{n+1}{n}\hat{N}_2 = \frac{n+1}{n}Y_{(n)}$ is approximately unbiased for N.

c $V\left(\hat{N}_2\right)$ is given, so $V\left(\hat{N}_3\right) = \left(\frac{n+1}{n}\right)^2 V\left(\hat{N}_2\right) = \frac{N^2}{n(n+2)}$.

d Note that, for $n > 1$,

$$\frac{V\left(\hat{N}_1\right)}{V\left(\hat{N}_3\right)} = \frac{n(n+2)}{3n}\frac{\left(N^2-1\right)}{N^2} = \frac{n+2}{3}\left(1 - \frac{1}{N^2}\right) > 1,$$

since for large N, $\left(1 - \frac{1}{N^2}\right) \approx 1$.