Exercise 9.50

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the uniform distribution over the interval (θ_1, θ_2). Show that $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$ and $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ are jointly sufficient for θ_1 and θ_2.

Exercise 9.64

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance 1.

a Show that the MVUE of μ^2 is $\widehat{\mu^2} = \bar{Y}^2 - 1/n$.

b Derive the variance of $\widehat{\mu^2}$.

Exercise 9.65

Let $Y_1, Y_2, \ldots, Y_n$ be independent Bernoulli random variables with

$$p(y_i \mid p) = p^{y_i}(1-p)^{1-y_i}, \quad y_i = 0, 1$$

That is, $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$. Find the MVUE of $p(1-p)$, which is a term in the variance of Y_i or $W = \sum_{i=1}^{n} Y_i$, by the following steps.

a Let

$$T = \begin{cases} 1, & \text{if } Y_1 = 1 \text{ and } Y_2 = 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that $E(T) = p(1-p)$.

b Show that

$$P(T = 1 \mid W = w) = \frac{w(n-w)}{n(n-1)}$$

c Show that

$$E(T \mid W) = \frac{n}{n-1}\left[\frac{W}{n}\left(1 - \frac{W}{n}\right)\right] = \frac{n}{n-1}\bar{Y}(1-\bar{Y})$$

and hence that $n\bar{Y}(1-\bar{Y})/(n-1)$ is the MVUE of $p(1-p)$.

Exercise 9.66

The likelihood function $L(y_1, y_2, \ldots, y_n \mid \theta)$ takes on different values depending on the arguments $(y_1, y_2, \ldots, y_n)$. A method for deriving a minimal sufficient statistic developed by Lehmann and Scheffé uses the ratio of the likelihoods evaluated at two points, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$:

$$\frac{L(x_1, x_2, \ldots, x_n \mid \theta)}{L(y_1, y_2, \ldots, y_n \mid \theta)}$$

Many times it is possible to find a function $g(x_1, x_2, \ldots, x_n)$ such that this ratio is free of the unknown parameter θ if and only if $g(x_1, x_2, \ldots, x_n) = g(y_1, y_2, \ldots, y_n)$. If such a function g can be found, then $g(Y_1, Y_2, \ldots, Y_n)$ is a minimal sufficient statistic for θ.

a Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a Bernoulli distribution (see Example 9.6 and Exercise 9.65) with p unknown.

- Show that
$$\frac{L(x_1, x_2, \ldots, x_n \mid p)}{L(y_1, y_2, \ldots, y_n \mid p)} = \left(\frac{p}{1-p}\right)^{\Sigma x_i - \Sigma y_i}$$
- Argue that for this ratio to be independent of p, we must have
$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 0 \quad \text{or} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$
- Using the method of Lehmann and Scheffé, what is a minimal sufficient statistic for p? How does this sufficient statistic compare to the sufficient statistic derived by using the factorization criterion?

b Consider the Weibull density $f(y \mid \theta) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{-y^2/\theta}, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$

- Show that
$$\frac{L(x_1, x_2, \ldots, x_n \mid \theta)}{L(y_1, y_2, \ldots, y_n \mid \theta)} = \left(\frac{x_1 x_2 \cdots x_n}{y_1 y_2 \cdots y_n}\right) \exp\left[-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2\right)\right].$$
- Argue that $\sum_{i=1}^{n} Y_i^2$ is a minimal sufficient statistic for θ.

Exercise 9.68

Suppose that a statistic U has a probability density function that is positive over the interval $a \le u \le b$ and suppose that the density depends on a parameter θ that can range over the interval $\alpha_1 \le \theta \le \alpha_2$. Suppose also that $g(u)$ is continuous for u in the interval $[a, b]$. If $E[g(U) \mid \theta] = 0$ for all θ in the interval $[\alpha_1, \alpha_2]$ implies that $g(u)$ is identically zero, then the family of density functions $\{f_U(u \mid \theta), \alpha_1 \le \theta \le \alpha_2\}$ is said to be complete.
Suppose that U is a sufficient statistic for θ, and $g_1(U)$ and $g_2(U)$ are both unbiased estimators of θ. Show that, if the family of density functions for U is complete, $g_1(U)$ must equal $g_2(U)$, and thus there is a unique function of U that is an unbiased estimator of θ.
Coupled with the Rao-Blackwell theorem, the property of completeness of $f_U(u \mid \theta)$, along with the sufficiency of U, assures us that there is a unique minimum-variance unbiased estimator (UMVUE) of θ.

Exercise 9.50

The uniform distribution on the interval (θ_1, θ_2) is

$$f(y \mid \theta_1, \theta_2) = \frac{1}{(\theta_2 - \theta_1)} I_{\theta_1,\theta_2}(y)$$

The likelihood function, using the same logic as in Ex. 9.49, is

$$L(\theta_1, \theta_2) = \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{\theta_1,\theta_2}(y_i) = \frac{1}{(\theta_2 - \theta_1)^n} I_{\theta_1,\theta_2}\left(y_{(1)}\right) I_{\theta_1,\theta_2}\left(y_{(n)}\right)$$

So, Theorem 9.4 is satisfied with $g\left(y_{(1)}, y_{(n)}, \theta_1, \theta_2\right) = \frac{1}{(\theta_2-\theta_1)^n} I_{\theta_1,\theta_2}\left(y_{(1)}\right) I_{\theta_1,\theta_2}\left(y_{(n)}\right)$ and $h(y) = 1$.

Exercise 9.64

a From Ex. 9.38, $\bar{Y}$ is sufficient for μ. Also, since $\sigma = 1, \bar{Y}$ has a normal distribution with mean μ and variance $1/n$. Thus, $E\left(\bar{Y}^2\right) = V(\bar{Y}) + [E(\bar{Y})]^2 = 1/n + \mu^2$. Therefore, the MVUE for μ^2 is $\bar{Y}^2 - 1/n$.

b $V\left(\bar{Y}^2 - 1/n\right) = V\left(\bar{Y}^2\right) = E\left(\bar{Y}^4\right) - \left[E\left(\bar{Y}^2\right)\right]^2 = E\left(\bar{Y}^4\right) - \left[1/n + \mu^2\right]^2$. It can be shown that $E\left(\bar{Y}^4\right) = \frac{3}{n^2} + \frac{6\mu^2}{n} + \mu^4$ (the mgf for $\bar{Y}$ can be used) so that

$$V\left(\bar{Y}^2 - 1/n\right) = \frac{3}{n^2} + \frac{6\mu^2}{n} + \mu^4 - -\left[1/n + \mu^2\right]^2 = \left(2 + 4n\mu^2\right)/n^2.$$

Exercise 9.65

a $E(T) = P(T = 1) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 1) P(Y_2 = 0) = p(1 - p)$.

b

$$\begin{aligned}
P(T = 1 \mid W = w) &= \frac{P(Y_1 = 1, Y_2 = 0, W = w)}{P(W = w)} = \frac{P\left(Y_1 = 1, Y_2 = 0, \sum_{i=3}^{n} Y_i = w - 1\right)}{P(W = w)} \\
&= \frac{P(Y_1 = 1) P(Y_2 = 0) P\left(\sum_{i=3}^{n} Y_i = w - 1\right)}{P(W = w)} = \frac{p(1-p)\binom{n-2}{w-1} p^{w-1}(1-p)^{n-(w-1)}}{\binom{n}{w} p^w (1-p)^{n-w}} \\
&= \frac{w(n-w)}{n(n-1)}
\end{aligned}$$

c $E(T \mid W) = P(T = 1 \mid W) = \frac{W}{n}\left(\frac{n-W}{n-1}\right) = \left(\frac{n}{n-1}\right)\frac{W}{n}\left(1 - \frac{W}{n}\right)$. Since T is unbiased by part (a) above and W is sufficient for p and so also for $p(1-p), n\bar{Y}(1-\bar{Y})/(n-1)$ is the MVUE for $p(1-p)$.

Exercise 9.66

a

- The ratio of the likelihoods is given by

$$\frac{L(\boldsymbol{x} \mid p)}{L(\boldsymbol{y} \mid p)} = \frac{p^{\sum x_i}(1-p)^{n-\Sigma x_i}}{p^{\sum y_i}(1-p)^{n-\Sigma y_i}} = \frac{p^{\sum x_i}(1-p)^{-\Sigma x_i}}{p^{\sum y_i}(1-p)^{-\Sigma y_i}} = \left(\frac{p}{1-p}\right)^{\sum x_i - \sum y_i}.$$

- If $\Sigma x_i = \Sigma y_i$, the ratio is 1 and free of p. Otherwise, it will not be free of p.
- From the above, it must be that $g\left(Y_1, \ldots, Y_n\right) = \sum_{i=1}^{n} Y_i$ is the minimal sufficient statistic for p. This is the same as the sufficient statistic derived by using the factorization theorem.

b

- The ratio of the likelihoods is given by

$$\frac{L(\boldsymbol{x} \mid \theta)}{L(\boldsymbol{y} \mid \theta)} = \frac{2^n \left(\prod_{i=1}^{n} x_i\right) \theta^{-n} \exp\left(-\sum_{i=1}^{n} x_i^2/\theta\right)}{2^n \left(\prod_{i=1}^{n} y_i\right) \theta^{-n} \exp\left(-\sum_{i=1}^{n} y_i^2/\theta\right)} = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} y_i} \exp\left[-\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2\right)\right]$$

- The above likelihood ratio will only be free of θ if $\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2$, so that $\sum_{i=1}^{n} Y_i^2$ is a minimal sufficient statistic for θ.

Exercise 9.68

For unbiased estimators $g_1(U)$ and $g_2(U)$, whose values only depend on the data through the sufficient statistic U, we have that $E\left[g_1(U) - g_2(U)\right] = 0$. Since the density for U is complete, $g_1(U) - g_2(U) \equiv 0$ by definition so that $g_1(U) = g_2(U)$. Therefore, there is only one unbiased estimator for θ based on U, and it must also be the MVUE.