Exercise 10.127

A merchant figures her weekly profit to be a function of three variables: retail sales (denoted by X), wholesale sales (denoted by Y), and overhead costs (denoted by W). The variables X, Y, and W are regarded as independent, normally distributed random variables with means μ_1, μ_2, and μ_3 and variances σ^2, $a\sigma^2$, and $b\sigma^2$, respectively, for known constants a and b but unknown σ^2. The merchant's expected profit per week is $\mu_1 + \mu_2 - \mu_3$. If the merchant has made independent observations of X, Y, and W for the past n weeks, construct a test of $H_0 : \mu_1 + \mu_2 - \mu_3 = k$ against the alternative $H_a : \mu_1 + \mu_2 - \mu_3 \neq k$, for a given constant k. You may specify $\alpha = .05$.

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Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function given by

$$f(y \mid \theta_1, \theta_2) = \begin{cases} \left(\frac{1}{\theta_1}\right) e^{-(y-\theta_2)/\theta_1}, & y > \theta_2 \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the likelihood ratio test for testing $H_0 : \theta_1 = \theta_{1,0}$ versus $H_a : \theta_1 > \theta_{1,0}$, with θ_2 unknown.

b Find the likelihood ratio test for testing $H_0 : \theta_2 = \theta_{2,0}$ versus $H_a : \theta_2 > \theta_{2,0}$, with θ_1 unknown.

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Let $P = X+Y-W$. Then, P has a normal distribution with mean $\mu_1+\mu_2-\mu_3$ and variance $(1+a+b)\sigma^2$. Further, $\bar{P} = \bar{X}+\bar{Y}-\bar{W}$ is normal with mean $\mu_1+\mu_2-\mu_3$ and variance $(1+a+b)\sigma^2/n$. Therefore,

$$Z = \frac{\bar{P}-(\mu_1+\mu_2-\mu_3)}{\sigma\sqrt{(1+a+b)/n}}$$

is standard normal. Next, the quantities

$$\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}{\sigma^2}, \frac{\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2}{a\sigma^2}, \frac{\sum_{i=1}^{n}\left(W_i-\bar{W}\right)^2}{b\sigma^2}$$

have independent chi-square distributions, each with $df = n-1$. We can build a random variable that follows a t-distribution (under H_0) by

$$T = \frac{\bar{P}-k}{S_p\sqrt{(1+a+b)/n}},$$

where $S_P^2 = \left(\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2 + \frac{1}{a}\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2 + \frac{1}{b}\sum_{i=1}^{n}\left(W_i-\bar{W}\right)^2\right)/(3n-3)$. For the test, we reject if $|t| > t_{.025}$, where $t.025$ is the upper .025 critical value from the t-distribution with $df = 3n-3$.

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a The likelihood function is $L(\Theta) = \theta_1^{-n}\exp\left[-\sum_{i=1}^{n}(y_i-\theta_2)/\theta_1\right]$. The MLE for θ_2 is $\hat{\theta}_2 = Y_{(1)}$. To find the MLE of θ_1, we maximize the log-likelihood function to obtain $\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i-\hat{\theta}_2\right)$. Under H_0, the MLEs for θ_1 and θ_2 are (respectively) $\theta_{1,0}$ and $\hat{\theta}_2 = Y_{(1)}$ as before. Thus, the LRT is

$$\lambda = \frac{L\left(\hat{\Omega}_0\right)}{L(\hat{\Omega})} = \left(\frac{\hat{\theta}_1}{\theta_{1,0}}\right)^n \exp\left[-\frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{\theta_{1,0}} + \frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{\hat{\theta}_1}\right]$$
$$= \left(\frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{n\theta_{1,0}}\right)^n \exp\left[-\frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{\theta_{1,0}} + n\right].$$

Values of $\lambda \le k$ reject the null hypothesis.

b The MLEs are $\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i-\hat{\theta}_2\right)$ and $\hat{\theta}_2 = Y_{(1)}$. Under H_0, the MLEs for θ_2 and θ_1 are (respectively) $\theta_{2,0}$ and $\hat{\theta}_{1,0} = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\theta_{2,0})$. Thus, the LRT is given by

$$\lambda = \frac{L\left(\hat{\Omega}_0\right)}{L(\hat{\Omega})} = \left(\frac{\hat{\theta}_1}{\hat{\theta}_{1,0}}\right)^n \exp\left[-\frac{\sum_{i=1}^{n}(y_i-\theta_{2,0})}{\hat{\theta}_{1,0}} + \frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{\hat{\theta}_1}\right] = \left[\frac{\sum_{i=1}^{n}\left(y_i-y_{(1)}\right)}{\sum_{i=1}^{n}(y_i-\theta_{2,0})}\right]^n.$$

Values of $\lambda \le k$ reject the null hypothesis.