Exercise 7.104

Let Y_n be a binomial random variable with n trials and with success probability p. Suppose that n tends to infinity and p tends to zero in such a way that np remains fixed at $np = \lambda$. Prove that the distribution of Y_n converges to a Poisson distribution with mean λ.

Exercise 8.8

Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Consider the following five estimators of θ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = \min\left(Y_1, Y_2, Y_3\right), \quad \hat{\theta}_5 = \bar{Y}$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

Exercise 8.15

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population whose density is given by

$$f(y) = \begin{cases} 3\beta^3 y^{-4}, & \beta \leq y \\ 0, & \text{elsewhere} \end{cases}$$

where $\beta > 0$ is unknown. (This is one of the Pareto distributions introduced in Exercise 6.18.) Consider the estimator $\hat{\beta} = \min(Y_1, Y_2, \ldots, Y_n)$.

a Derive the bias of the estimator $\hat{\beta}$.

b Derive MSE($\hat{\beta}$).

Exercise 8.16

Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with parameters μ and σ^2.

a Show that $S = \sqrt{S^2}$ is a biased estimator of σ. [Hint: Recall the distribution of $(n-1)S^2/\sigma^2$]

b Adjust S to form an unbiased estimator of σ.

c Find an unbiased estimator of $\mu - z_\alpha \sigma$, the point that cuts off a lower-tail area of α under this normal curve.

Exercise 8.17

If Y has a binomial distribution with parameters n and p, then $\hat{p}_1 = Y/n$ is an unbiased estimator of p. Another estimator of p is $\hat{p}_2 = (Y+1)/(n+2)$.

a Derive the bias of $\hat{p}_2$.

b Derive $\text{MSE}\left(\hat{p}_1\right)$ and $\text{MSE}\left(\hat{p}_2\right)$.

c For what values of p is $\text{MSE}\left(\hat{p}_1\right) > \text{MSE}\left(\hat{p}_2\right)$?

Exercise 7.104

The mgf for Y_n is given by

$$m_{Y_n}(t) = \left[1 - p + pe^t\right]^n$$

Let $p = \lambda/n$ and this becomes

$$m_{Y_n}(t) = \left[1 - \frac{\lambda}{n} + \frac{\lambda}{n}e^t\right]^n = \left[1 + \frac{1}{n}\left(\lambda e^t - 1\right)\right]^n.$$

As $n \to \infty$, this is $\exp\left(\lambda e^t - 1\right)$, the mgf for the Poisson with mean λ.

Exercise 8.8

a Note that $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ and $\hat{\theta}_5$ are simple linear combinations of Y_1, Y_2, and Y_3. So, it is easily shown that all four of these estimators are unbiased. From Ex. 6.81 it was shown that $\hat{\theta}_4$ has an exponential distribution with mean $\theta/3$, so this estimator is biased.

b It is easily shown that $V(\hat{\theta}_1) = \theta^2, V(\hat{\theta}_2) = \theta^2/2, V(\hat{\theta}_3) = 5\theta^2/9$, and $V(\hat{\theta}_5) = \theta^2/9$, so the estimator $\hat{\theta}_5$ is unbiased and has the smallest variance.

Exercise 8.15

Using standard techniques, it can be shown that $E(Y) = (3/2)\beta, E\left(Y^2\right) = 3\beta^2$. Also it is easiliy shown that $Y_{(1)}$ follows the Pareto family with density function

$$g_{(1)}(y) = 3n\beta^{3n}y^{-(3n+1)}, y \geq \beta.$$

Thus, $E\left(Y_{(1)}\right) = \left(\frac{3n}{3n-1}\right)\beta$ and $E\left(Y_{(1)}^2\right) = \frac{3n}{3n-2}\beta^2$.

a With $\hat{\beta} = Y_{(1)}, B(\hat{\beta}) = \left(\frac{3n}{3n-1}\right)\beta - \beta = \left(\frac{1}{3n-1}\right)\beta$.

b Using the above, $\mathrm{MSE}(\hat{\beta}) = \mathrm{MSE}\left(Y_{(1)}\right) = E\left(Y_{(1)}^2\right) - 2\beta E\left(Y_{(1)}\right) + \beta^2 = \frac{2}{(3n-1)(3n-2)}\beta^2$.

Exercise 8.16

It is known that $(n-1)S^2/\sigma^2$ is chi-square with $n-1$ degrees of freedom.

a $E(S) = E\left\{\frac{\sigma}{\sqrt{n-1}}\left[\frac{[n-1)S^2}{\sigma^2}\right]^{1/2}\right\} = \frac{\sigma}{\sqrt{n-1}}\int_0^\infty v^{1/2}\frac{1}{\Gamma[(n-1)/2]2^{(n-1)/2}}v^{(n-3)/2}e^{-v/2}dv = \frac{\sigma}{\sqrt{n-1}}\frac{\sqrt{2}\Gamma(n/2)}{\Gamma[(n-1)/2]}$.

b The estimator $\hat{\sigma} = \frac{\sqrt{n-1}\Gamma(n-1)/2]}{\sqrt{2}\Gamma(n/2)}S$ is unbiased for σ.

c Since $E(\bar{Y}) = \mu$, the unbiased estimator of the quantity is $\bar{Y} - z_\alpha\hat{\sigma}$.

Exercise 8.17

It is given that $\hat{p}_1$ is unbiased, and since $E(Y) = np$, $E\left(\hat{p}_2\right) = (np+1)/(n+2)$.

a $B\left(\hat{p}_2\right) = (np+1)/(n+2) - p = (1-2p)/(n+2)$.

b Since $\hat{p}_1$ is unbiased, $\mathrm{MSE}\left(\hat{p}_1\right) = \mathrm{V}\left(\hat{p}_1\right) = p(1-p)/n$, $\mathrm{MSE}\left(\hat{p}_2\right) = V\left(\hat{p}_2\right) + B^2\left(\hat{p}_2\right) = \frac{np(1-p)+(1-2p)^2}{(n+2)^2}$.

c Considering the inequality

$$\frac{np(1-p)+(1-2p)^2}{(n+2)^2} < \frac{p(1-p)}{n}$$

this can be written as

$$(8n+4)p^2 - (8n+4)p + n < 0$$

Solving for p using the quadratic formula, we have

$$p = \frac{8n+4 \pm \sqrt{(8n+4)^2 - 4(8n+4)n}}{2(8n+4)} = \frac{1}{2} \pm \sqrt{\frac{n+1}{8n+4}}.$$

So $\mathrm{MSE}\left(\hat{p}_1\right) > \mathrm{MSE}\left(\hat{p}_2\right)$ for $\frac{1}{2} - \sqrt{\frac{n+1}{8n+4}} < p < \frac{1}{2} + \sqrt{\frac{n+1}{8n+4}}$.