Bayesian Nonparametrics (BNP)

Chapter 7 by David B. Dunson

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Textbook

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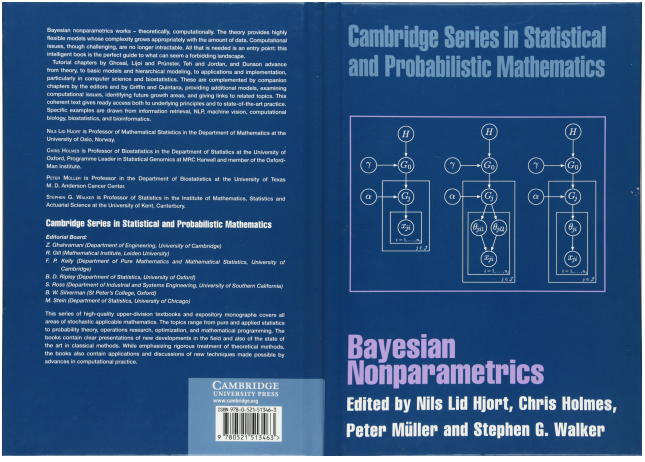


Preliminary

Real Analysis: Lebesgue measure

Probability Theory: Borel σ-field, probability space $(\Omega, \mathcal{F}, P)$

Stochastic Process: e.g., Chinese restaurant process, Indian buffet process

Experimental Design: linear mixed model

Generalized Linear Model: generalized estimating equations, generalized least squares

Bayesian Statistics: exchangeability, mixtures of g-priors, Markov chain Monte Carlo

Machine Learning: Gaussian mixture model

Nonparametric Bayes

Bayes' theorem: Posterior $\propto$ Likelihood $\times$ Prior

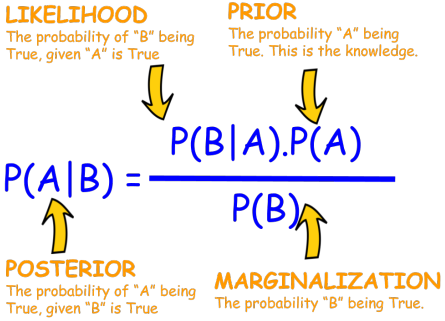


Image: Towards Data Science

Bayesian parametric model: A - parameters; B - data.
Bayesian nonparametric model: unbounded / growing / **infinite** number of parameters.

Repeated-Measures Design

- Longitudinal data: collected on the same subjects over time.
- Clustered data: measurements made on all units within a cluster in eq. 1 & 2.

  Intraclass correlation $\rho = \dfrac{g}{g+\sigma^2}$

  - proportion of total variance due to within-cluster variance;
  - how strongly units within same cluster resemble each other.

$$\mathcal{M}_1: \quad y_{ij} = \mu_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2), \tag{1}$$

where y_{ij} is the jth observation within subject i,
μ_i is a subject-specific mean,
ϵ_{ij} is an observation-specific residual,
σ is the within-subject standard deviation,
g is the between-subject variance on p. 7,
for $j = 1, \cdots, n_i$ and $i = 1, \cdots, n$.

Same Mixed Model but Different Designs

- Clustered data (Dobson & Barnett, 2018, p. 259; Dunson, 2010, p. 224)

  e.g., the income of the jth household in the randomly selected council area i
  e.g., the weight of the jth piglet in the randomly selected litter i

$$\begin{aligned} & \mathcal{M}_1\colon \quad y_{ij} = \mu + b_i + \epsilon_{ij} \\ \text{versus} \quad & \mathcal{M}_0\colon \quad y_{ij} = \mu + \epsilon_{ij} \end{aligned}$$

- One-way within-subject data (Rouder et al., 2012 & 2017)

  e.g., the response time of the ith subject for the jth Stroop task

$$\begin{aligned} & \mathcal{M}_1\colon \quad y_{ij} = \mu + b_i + t_j + \epsilon_{ij} \\ \text{versus} \quad & \mathcal{M}_0\colon \quad y_{ij} = \mu + b_i + \epsilon_{ij} \end{aligned}$$

Bayesian Parametric Modeling

$$\mathcal{M}_1: \quad y_{ij} = \mu_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2) \tag{1}$$

$$\mu_i \sim \mathcal{N}(\mu, g) \tag{2}$$

$$\text{or} \quad \mu_i = \underbrace{\mu}_{\text{fixed}} + \underbrace{b_i}_{\text{random effects}}, \qquad b_i \sim \mathcal{N}(0, g) \tag{3}$$

BNP Motivation

The normal distribution has light tails and does **not** allow some subjects to be very different from other subjects or to have groups of subjects that cluster close together. Hence, outlying subjects tend to have their means **over-shrunk** towards the population mean, and the data from such subjects may be overly-influential in estimation of the overall mean μ.

The t-distribution has heavier tails, but it still has a very restrictive unimodal and symmetric shape.

Bayesian Nonparametric Modeling

$$\mathcal{M}_1: \quad y_{ij} = \mu_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2) \tag{1}$$
$$\mu_i \sim P \tag{4}$$
$$P \sim DP(P_0, \alpha) \tag{5}$$

Choose a **Dirichlet process** (DP) prior for P.

P corresponds to a distribution function, or more formally, a random probability measure.

- (5) allows P to be an unknown distribution.
- P_0 is a fixed baseline probability measure (the expected value of the process), corresponding to one's best guess for P *a priori*,
  e.g., P_0 is a normal distribution with the normal-inverse gamma hyperpriors.
- α is the concentration parameter ($\alpha > 0$),
  characterizing prior precision and clustering (expressing confidence in P guess),
  e.g., $\alpha \sim$ Gamma$(1, 1)$, a gamma hyperprior.

Hierarchical Dirichlet Process

$$\mu_i \mid P_i \sim P_i$$

$$P_i \mid P_0, \alpha_i \sim \mathit{DP}(P_0, \alpha_i)$$

$$P_0 \mid H, \alpha \sim \mathit{DP}(H, \alpha)$$

mixed model $\neq$ mixture model

Dirichlet Distribution

The Dirichlet distribution, $\mathrm{Dir}(\boldsymbol{\alpha})$, is a multivariate generalization of the beta distribution for $K \geqslant 2$. The PDF is

$$f(x_1, \cdots, x_K; \alpha_1, \cdots, \alpha_K) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \prod_{i=1}^{K} x_i^{\alpha_i - 1}, \tag{6}$$

where $\mathrm{B}(\boldsymbol{\alpha}) = \dfrac{\prod_{i=1}^{K} \Gamma(\alpha_i)}{\Gamma(\alpha_0)}$,

$$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_K), \quad \alpha_0 = \sum_{i=1}^{K} \alpha_i \tag{7}$$

$$\sum_{i=1}^{K} x_i = 1 \text{ and } x_i \geqslant 0, \ \alpha_i > 0 \text{ for } i = 1, \cdots, K.$$

- If $\alpha_1 = \cdots = \alpha_K = \alpha$, the Dirichlet distribution will be symmetric.
- When $\alpha = 1$, it is a uniform distribution over the open standard $(K-1)$-simplex.

Relation to the Gamma Distribution

For K independently distributed gamma distributions,

$$Y_i \overset{\text{ind.}}{\sim} \text{Gamma}(\alpha_i, \beta) \quad \text{for } i = 1, \cdots, K,$$

it can be proved that the sum is also a gamma distribution,

$$Y = \sum_{i=1}^{K} Y_i \sim \text{Gamma}\left(\alpha_0, \beta\right). \tag{8}$$

Then, the K-dimensional Dirichlet distributed random vector is

$$\boldsymbol{X} = (X_1, \cdots, X_K) = \left(\frac{Y_1}{Y}, \cdots, \frac{Y_K}{Y}\right) \sim \text{Dir}(\boldsymbol{\alpha}). \tag{9}$$

$\text{Cov}(X_i, X_j) = \dfrac{-\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$ for $i \neq j$, not independent.

The marginal distributions are $X_i \sim \text{Beta}(\alpha_i, \alpha_0 - \alpha_i)$. Exponential family.

R Scripts

```
options(digits=3); set.seed(277)
# LaplacesDemon::rdirichlet()
(pis <- gtools::rdirichlet(n=3, #number of random vectors to generate
                           alpha=rep(1,5)))
#         [,1]   [,2]   [,3]    [,4]   [,5]
# [1,] 0.35982 0.2053 0.0148 0.07106 0.3490
# [2,] 0.00204 0.2274 0.6151 0.00293 0.1525
# [3,] 0.61949 0.0075 0.2448 0.06353 0.0647

rowSums(pis) # 1 1 1
gtools::rdirichlet(3, rep(1000,5))
#       [,1]  [,2]  [,3]  [,4]  [,5]
# [1,] 0.208 0.200 0.207 0.193 0.193
# [2,] 0.190 0.210 0.199 0.199 0.203
# [3,] 0.194 0.207 0.195 0.206 0.198
```

Dir($\boldsymbol{\alpha}$) is the canonical Bayesian distribution for the parameter estimates of a multinomial distribution.

Transitioning to the Dirichlet Process

For any finite partition $S_1, \cdots, S_K$ of the measurable set S $\left(\sum_{i=1}^{K} P_0(S_i) = 1\right)$,

$$P \sim DP(P_0, \alpha),$$

$$\Updownarrow$$

$$(P(S_1), \cdots, P(S_K)) \sim \text{Dir}(\alpha P_0(S_1), \cdots, \alpha P_0(S_K)).$$

Recall marginal betas.

Note that $\mathbb{E}[P(S_i)] = P_0(S_i)$ and $\text{Var}(P(S_i)) = \dfrac{P_0(S_i)(1 - P_0(S_i))}{\alpha + 1}$.

α determines the variance of the random probability measure.

Dirichlet Process

The infinite-dimensional generalization of the Dirichlet distribution is the Dirichlet process $DP(P_0, \alpha)$.

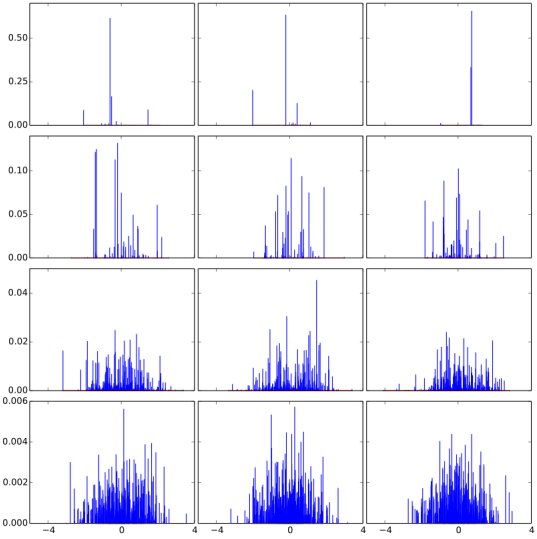
- Even if P_0 is continuous, the distributions drawn from the Dirichlet process are almost surely **discrete**.
- α specifies how strong this discretization is.
  - As $\alpha \to 0$, the realizations are all concentrated at a single value.
  - As $\alpha \to +\infty$, the realizations become continuous.

$DP(P_0, \alpha)$ is often used in Bayesian inference to describe the prior knowledge about the distribution of random variables — how likely it is that the random variables are distributed according to one or another particular distribution.

Example: $DP(\mathcal{N}(0,1),\alpha)$

$\alpha = 1,\ 10,\ 100,\ 1,000$ from top to bottom.

Each row contains three repetitions of the same process.



Source: Wikipedia

Stick-Breaking Representation

Question: How do we draw a random distribution P from the Dirichlet process?

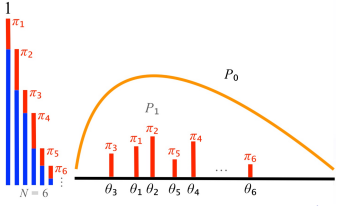
$P \sim DP(P_0, \alpha)$ is equivalent to assigning weights to point masses at *atoms* θ_h,

$$P = \underset{\text{PMF}}{p(\theta)} = \sum_{h=1}^{\infty} \pi_h \cdot \delta_{\theta_h}(\theta), \quad \theta_h \overset{\text{i.i.d.}}{\sim} P_0, \tag{10}$$

where $\pi_1 = V_1$ and $\pi_h = V_h \prod_{l<h}(1 - V_l)$ are probability weights that are formulated from a stick-breaking process with $V_h \overset{\text{i.i.d.}}{\sim} \text{Beta}(1, \alpha)$ for $h = 1, \cdots, \infty$,

and $\delta_{\theta_h}(\cdot)$ is the indicator function $\delta_{\theta_h}(\theta_h) = 1$, otherwise 0.

In practice, we choose some N to truncate the series.



R Scripts (continued)

```
stick_breaking_process <- function(N, alpha) {
  #' Input -
  #' N:       number of weights (stick-breaks)
  #' alpha:   concentration
  #' Output - a vector of weights
  V <- rbeta(N, 1, alpha)
  V * c(1, cumprod(1-V))[1:N]
}

N <- 1000; alpha <- 100; set.seed(277)
# Each random number from the base measure N(0,1) is
#  replicated a number of times corresponding to its weight.
draws <- rep(rnorm(N),
             round(stick_breaking_process(N, alpha) * 10000))

hist(draws, prob=T, col="white", yaxt="n", ylab="", xlab=expression(theta),
     main="Random Distribution From the Dirichlet Process")
lines(density(draws), col="red", lwd=2)
```

"The rich get richer" (Shelley, 1818): Frequently sampled values in the past are more likely to be selected again.

Spike-and-Slab

$$M_i \sim P \qquad \text{for } i = 1, \cdots, n$$
$$P \sim DP(P_0, \alpha)$$

Given observed data and a DP prior, we update our beliefs about the underlying distribution based on that data.
The posterior of P is

$$P \mid \boldsymbol{M} \sim DP\left(\frac{\alpha \overset{\text{PDF}}{P_0} + \overset{\text{PMF}}{\sum_{i=1}^{n} \delta_{M_i}}}{\alpha + n},\ \alpha + n \right). \tag{11}$$

The base probability measure for the posterior DP can involve a mix of the continuous distribution (the “slab") and the point masses (the “spikes”).

See also the Bayesian variable selection.

Thank you.

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