INVESTIGATING NEUTRINO MASSES IN SUPERSYMMETRIC GUT EXTENSIONS OF THE STANDARD MODEL

by

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Abstract

The Standard Model of particle physics is the crowning achievement of modern science, providing a fundamental and complete description of the nature of matter, fields and their interactions. In 2012, the 5σ detection of the Higgs boson at CERN provided the last experimental verification for the Standard Model.

However, the Standard Model does not provide an appealing explanation for the phenomena of non-zero neutrino masses. In fact, the Standard Model is perfectly compatible with massless neutrinos. The discovery of non-zero neutrino mass via neutrino oscillations hence provides a window into Beyond Standard Model (BSM) physics, demanding new frameworks to explain how neutrino masses might fit in a unified framework.

In this thesis, we will explore models that explain the neutrino mass and mixing matrices. We first do so by extending the Standard Model framework to incorporate neutrino mass generation mechanisms. We then investigate the Left-Right Symmetric Model (LRSM), which unifies the left- and right-handed fields, as well the supersymmetric (SUSY) Grand Unified $SU(5)$, which unifies the gauge coupling constants at a high energy scale. Original contributions of this thesis includes model construction at the LRSM scale for neutrino masses, as well as model construction for the neutrino mass mixing matrix in the SUSY $SU(5)$ framework. Additionally, explicit calculations involving the irreducible representations of the $SU(5)$ group are presented.

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Chapter 1

Introduction

The Standard Model is the crowning achievement of modern physics. Its notable predictions include the 5σ detection of the Higgs boson [1], as well as the prediction of the W and Z bosons [2], and top and charm quarks [3, 4].

In this chapter, we will provide a brief history of the development of the Standard Model, focusing on the theme of unification – how seemingly disparate phenomena can be related via fundamental symmetries. We first start with the road to Quantum Electrodynamics. We will freely reference the textbooks [5, 6, 7].

1.1 The road to Quantum Electrodynamics

In the following sections, we will provide an overview of the key conceptual advances that have led to the formulation of Quantum Electrodynamics, with a focus on the Action Principle and the Minimal Coupling Principle, and the Path Integral formulation of Quantum Electrodynamics.

1.1.1 Classical Electrodynamics

Classical electrodynamics (CED) as formulated by Maxwell unified the electric and magnetic fields. With its re-writing into tensor form by Einstein in Special Relativity, we can describe the interaction of a free field and a free charged particle with the following Lagrangian density (using natural units, $c = \hbar = 1$):

$$S = \int dx^4 \mathcal{L}_{\text{CED}} = -\int dx^4 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e \int dx^4 J_\mu A^\mu - \frac{m}{2} \int d\tau (\dot{r} \cdot \dot{r}) \tag{1.1}$$

describing the motion of the free field, interaction term, and the free particle respectively.

By applying the Action Principle, one can compute the infinitesimal variation of the action, extremise it by setting the infinitesimal variation to zero, and obtain the Equations of Motion via the Euler-Lagrange equations. By using the Euler-Lagrange equation on A^ν and $\partial^\mu A_\nu$:

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{1.2}$$

which describes the motion of the field. Together with the dual field strength tensor:

$$F^*_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{1.3}$$

which is divergence-free due to the anti-symmetric properties of the Levi-Civita tensor and the commutability of the partial derivatives.

$$\partial^\mu F^*_{\mu\nu} = 0 \tag{1.4}$$

Together, these equations represent the Maxwell Field equations.

To obtain the Equations of Motion for the charged particle, we can obtain the Lorentz force equation with a slight rewriting of the interaction term:

$$S = \int dx^4 \mathcal{L}_{\text{CED}} = -\int dx^4 \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\int d\tau u^\mu(\tau)A^\mu(r(\tau)) - \frac{m}{2}\int d\tau(\dot{r}\cdot\dot{r}) \tag{1.5}$$

and applying the Euler-Lagrange equation to r and $\dot{r}$. This reproduces the Lorentz Force Law:

$$m\frac{d^2r^\mu}{d\tau^2} = eF^{\mu\nu}(r(\tau))u_\nu(\tau) \tag{1.6}$$

Hence, Classical Electrodynamics is a complete field theory of electromagnetic fields and the motion of charged particles under those fields.

However, it is widely recognised that classical electrodynamics led to unacceptable phenomenology. If the classical model of the atom required the electron to move in circular motion around the nuclei, the electromagnetic radiation produced by the electron would cause the orbit to lose energy. If classical electrodynamics was correct, we would not have any stable atoms. However, stable atoms are everywhere – a new description of particles was needed. This was one of the driving forces behind the development of Quantum Mechanics in the early 20th century.

1.1.2 Quantum Mechanics

The recipe of quantisation in Quantum Mechanics is to promote the position q and the momentum p variables into operators. To create new states, Dirac introduced the creation and annihilation operators with their commutation algebra, and Heisenberg related the time evolution of an operator to its commutator with the Hamiltonian [8].

However, Quantum Mechanics was built for non-relativistic particles: the wavefunction of Quantum Mechanics is not written as a tensor equation that is invariant to Lorentz transformations. To move from Quantum Mechanics to Quantum Field Theory, we need a few other upgrades, starting with upgrading wavefunctions into field operators.

1.1.3 Quantum Field Theory

For a scalar object ϕ, we can derive the Klein-Gordon equation using considerations from Quantum Mechanics and Special Relativity. The simple derivation is shown in Appendix A.1.

$$(-\Box + m^2)\phi(t, \vec{x}) = 0 \tag{1.7}$$

Treating it as a wavefunction equation is problematic as it leads to negative probability densities for ϕ, which is unphysical (also shown in Appendix A.2).

It turns out that the appropriate treatment is to upgrade the wavefunction ϕ into a field operator Φ that can be described using creation and annihilation operators using the approach of Second Canonical Quantisation.

The canonical quantisation of Quantum Mechanics poses practical problems for calculations. For an arbitrary N-particle system, we need to symmetrise and anti-symmetrise the N-particle wavefunction to respect the exchange symmetries between the bosons and the fermions.

Second Canonical Quantisation uses a change of basis such that creation and annihilation operators can be used to create N-particle states. With this machinery, we can write down the Hamiltonian and Lagrangian densities using the commutation relations of the creation and annihilation operators.

Explicitly, the Fourier decomposition of the field operator $\Phi(x)$ can be written as a sum of positive and negative frequency components:

$$\Phi(x) = \Phi^{(+)} + \Phi^{(-)} \tag{1.8}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{ik\cdot x} A_{\vec{k}} + \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik\cdot x} A^\dagger_{\vec{k}} \tag{1.9}$$

where $A_{\vec{k}}$ and $A^\dagger_{\vec{k}}$ are annihilation and creation operators.

The Hamiltonian operator can be expressed in terms of these creation and annihilation operators:

$$H = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \omega_{\vec{k}} \left(A^\dagger_{\vec{k}} A_{\vec{k}} + A_{\vec{k}} A^\dagger_{\vec{k}} \right) \tag{1.10}$$

which can be used to create N-particle energy states.

1.1.4 Path Integral perspective

However, the Second Canonical Quantisation approach poses challenges for calculating interesting physical quantities – such as scattering cross-sections via Green's Functions – in an efficient manner.

This is because we have to construct a new Green's Function for every new particle added to the system. Instead, we adopt the perspective of Feynman's path integral formulation, allowing us to reformulate QFT based on the Action Principle. In the path integral formulation, any N-particle Green's Function is obtained by repeated differentiation of a generating functional.

A discussion of how the path integral is constructed can be found in Section 5 of [6].

The matrix element $\langle x|e^{-iHt}|x'\rangle$ describes the probability amplitude of a state moving from x' at $t = 0$ to x at $t = t$. Considering infinitesimal motions, we can integrate over all possible position and momentum states – all possible paths. This connects the matrix element directly to the Action:

$$\langle x|e^{-iHt}|x'\rangle = \int_{x(0)=x'}^{x(t)=x} \mathcal{D}x'' \exp\left[i \int dx''^4 \mathcal{L}(x'') \right] \tag{1.11}$$

$$= \int_{x(0)=x'}^{x(t)=x} \mathcal{D}x'' \exp\left[iS(x'') \right] \tag{1.12}$$

(1.13)

in which $\int \mathcal{D}$ is a shorthand for integrating over all paths. Using this formulation, we can easily obtain any N-point Green's Functions by differentiating generating functionals.

The $S_{\beta\alpha}$-matrix, or the scattering matrix, gives the scattering amplitudes from state α to β – the S-matrix is the object that is used to calculate physical quantities of interest, such as scattering cross-sections. The S-matrix is connected to the Feynman's Green Function via Lehmann-Symanzik-Zimmermann (LSZ) reduction. By perturbing the Green's Function, and then applying Wick Factorisation, the resulting terms can be described graphically via the famous Feynman diagrams.

1.1.5 Quantum Electrodynamics and its successes

Quantum Electrodynamics is a description of the interaction between Dirac fields ψ (whose quanta are charged particles such as electrons), as well as the gauge field A_μ (whose quanta are photons).

The free Lagrangians of the Dirac field and the gauge field are given as:

$$\mathcal{L}^{\text{EM}} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \tag{1.14}$$

$$\mathcal{L}^{\text{Dirac}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{1.15}$$

However, we need to describe their interactions. We can take some inspiration from classical electrodynamics. In classical electrodynamics, the interaction term of the Lagrangian density is given by:

$$\mathcal{L} \supset eJ^\mu A_\mu \tag{1.16}$$

Similar to the current vector J_μ, we want a Dirac bilinear vector object that resembles a vector. We can use the object $\overline{\Psi}\gamma^\mu\Psi$ as a current vector that couples to the gauge field A_μ, with the coupling constant e in front. This should give a term like $e\overline{\Psi}\gamma^\mu\Psi A_\mu$

To confirm our guess, we apply the Minimum Coupling Principle, which claims that we can obtain all interaction terms by demanding local gauge invariance. This means that we can infer the required interaction terms by considering the transformation of the free fields under a local gauge transform and then working backwards to find the interaction term that would be required for local gauge invariance to hold.

In practice, all we have to do is to replace partial derivatives ∂_μ with the covariant derivative D_μ. For the gauge field A_μ with coupling e, the covariant derivative becomes:

$$D_\mu = \partial_\mu - ieA_\mu \tag{1.17}$$

Hence, the QED Lagrangian can be written as:

$$\begin{aligned} \mathcal{L}^{\text{QED}} &= -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \overline{\Psi}(i\not{D} - m)\Psi & (1.18) \\ &= -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + e\overline{\Psi}\gamma^\mu\Psi A_\mu & (1.19) \end{aligned}$$

The 1965 Nobel Prize in Physics was awarded to Feynman, Schwinger, and Tomonaga for their independent development of Quantum Electrodynamics. It was a renormalisable field theory that described electromagnetic interactions. Its real validation came with Schwinger's and Feynman's prediction of the anomaly in the magnetic moment of the electron. Up to one-loop:

$$g = 2\left(1 + \frac{\alpha}{2\pi}\right) = 2.0023228$$

which is very close to the current experimental value of 2.00231930436 [9], and is remarkable for a one-loop calculation. This result convincingly demonstrated the strength and validity of the Quantum Field Theory programme.

1.2 The puzzle of electroweak theory

However, some experimental observations did not fit into the QED framework. In the decay of the K^+ meson, it decays into $\pi^+\pi^0$ and also $\pi^+\pi^+\pi^-$. The intrinsic parity of the π meson is -1 due to its pseudoscalar properties, but the intrinsic parity of the two decay products is different. The three-particle system has a parity of -1, while the parent particle had a parity of $+1$ to begin with.

To solve this puzzle, Lee and Yang [10] suggested in 1956 that parity was not conserved in such decays. Such parity-violating decays would also appear in the β-decay of nuclei, which was experimentally observed by Wu et al. in 1956 [11].

QED preserves parity. To describe radioactive β-decay, we would need a new framework to describe parity-violating observations.

In QED, the photon-proton interaction in the Lagrangian was written as:

$$ej^{(em)}_{\mu}A^{\mu} = e[\overline{u}_p\gamma_\mu u_p]A^{\mu} \tag{1.20}$$

in which a proton-proton interaction can be described by two charged currents of the form $ej_\mu j^\mu$.

Taking inspiration from this, we can use two 'weak currents' to describe the nuclei decay $n \to p + e^- + \overline{\nu}$.

$$Gj^{(n\to p)}_{\mu}j^{\mu}_{(\nu\to e)} = G[\overline{u}_p\gamma_\mu u_n][\overline{u}_e\gamma^\mu u_\nu] \tag{1.21}$$

To make this parity violating, we would need to insert a term affected by parity transformations. It turns out that the weak current can be written as a vector minus axial-vector $(V - A)$ term of the form:

$$J^{(weak)}_{\mu} \sim (\overline{\nu}_e\gamma_\mu e - \overline{\nu}_e\gamma_\mu\gamma_5 e) \tag{1.22}$$

where we discussed the construction of the Dirac bilinears in Appendix A.3.

At this juncture, we have a passable description of weak interactions that allow us to compute cross-sections (and correctly! Refer to Chapter 9 of [12]). However, these phenomena lack a unifying explanation. We will explore this unifying explanation in the next chapter in the form of the Standard Model.

1.3 The road ahead

In Chapter 2, we will review the workings of the Standard Model, discussing its gauge group, electroweak unification, spontaneous symmetry breaking, the Higgs mechanism for mass generation, and the parameterisation of the neutrino mixing matrix.

Next, in Chapter 3, we discuss how neutrino masses are obtained, and show how a Majorana neutrino can lead to interesting physics. We introduce the Seesaw mechanism as an important model-building tool for neutrino masses as a natural explanation of light neutrino masses via the introduction of a heavy sterile right-handed neutrino.

In Chapter 4, we discuss where these Majorana masses might come from in the first place – we introduce Effective Field Theory, and argue that we can use the method of higher-dimensional effective operators as a model-building tool to obtain desired Lagrangian terms. An extended discussion of Effective Field Theory in Appendix H. We then show in Chapter 5 how these model-building tools are applied to explain neutrino masses at the Standard Model level.

Subsequently, we discuss the construction of neutrino mixing matrices in Chapter 6. We discuss how discrete symmetry groups can be used to obtain neutrino flavour mixing matrices. We provide an explicit example of how the A_4 discrete symmetry group can be used to obtain a neutrino mixing pattern that aligned closely with experimental data when it was initially proposed.

In Chapter 7, we make a discrete jump in theory-space and explore the Left-Right Symmetric Model (LRSM), in which left-handed particles and right-handed particles are treated on equal footing, with left and right electroweak unification at the LRSM energy scale. We discuss an LRSM tree-level diagram that could contribute to neutrino masses.

Additionally, to lay the groundwork for further investigation of higher theories, we review the construction of supersymmetric theories in Chapter 8 for completeness. This is to foreground the later discussion of Grand Unified Theories (GUTs), as well as supersymmetric GUTs (SUSY GUTs).

This culminates in Chapter 9, where we discuss a Grand Unified Theory with the $SU(5)$ gauge group. We discuss how the 15 fermions of the Standard Model can be packaged into irreducible representations of $SU(5)$, and how these irreducible representations and their tensor products are obtained. We discuss issues relating to proton decay, the Higgs sector, and reasons for supersymmetric grand unification. We also discuss an original D_4 discrete symmetry extension to SUSY $SU(5)$ GUT to generate neutrino masses and mixing matrices.

The final chapter will summarise our discussion, and highlight contemporary experimental efforts that aim to constrain model space. We briefly discuss possibilities for future work.

Original discussions in this thesis can be found in Chapter 5, which discusses extensions within the Left-Right Symmetric Model in Section 7.5.1. Original model-building is also found in Section 9.6, extending the SUSY $SU(5)$ framework to explain the neutrino masses and mixing matrices by introducing a D_4 discrete symmetry group. Furthermore, we explicitly compute the irreducible representations and tensor products of $SU(5)$ in Appendix I – this computation does not appear to be readily available in the literature.

Chapter 2

The Standard Model

For a pedagogical introduction to the construction of the Standard Model, one may refer to the following references [5, 6, 7]. Instead, we will provide a summary of its construction – gauge group, particle content, and the spontaneous-symmetry breaking mechanism.

2.1 Gauge group

The Standard Model is described by the tensor product $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, where $SU(3)_C$ describes the symmetry for the colour charges, $SU(2)_L$ the symmetry for the left-handed doublets, and $U(1)_Y$ the weak hypercharge.

Hence, its fields obey the following local gauge transformations.

$$SU(2)_L : \Psi \to U(x)\Psi \tag{2.1}$$

where $U(x)$ is a unitary matrix of the form.

$$U(x) = e^{i\vec{\rho}(x)\cdot\vec{\tau}} \tag{2.2}$$

where τ_i represents the Pauli matrices, which generates $SU(2)$ rotations. The Dirac adjoints denoted with the bar transform as :

$$\overline{\Psi} \to \overline{\Psi}U^{\dagger}(x) \tag{2.3}$$

For fields with $U(1)_Y$ charge assignments,

$$U(1)_Y : \Psi \to e^{i\theta(x)}\Psi \tag{2.4}$$

and the Dirac adjoints transform as:

$$\overline{\Psi} \to \overline{\Psi}e^{-i\theta(x)} \tag{2.5}$$

2.2 Particle content and Lagrangian sectors

For the fermions, we have the quarks and leptons. To simplify our description, we will ignore the fermion generations, and state their charge assignments under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

Explicitly, the $SU(2)_L \otimes U(1)_Y$ covariant derivative is given as:

$$D_\mu = \partial_\mu + ig\frac{\vec{\tau}}{2} \cdot \vec{W}_\mu + \frac{i}{2}g'YY_\mu \tag{2.6}$$

where $\vec{W}_\mu$ is the electroweak gauge field in the adjoint representation of $SU(2)_L$, and Y_μ the gauge field of $U(1)_Y$, and Y is the hypercharge number. For $SU(2)$ singlets, we would drop the second term, which is the $SU(2)_L$ term.

We can state the group multiplicities under $SU(3)_C \otimes SU(2)_L$ and hypercharge assignments under $U(1)_Y$ e.g. $(\dim[SU(3)], \dim[SU(2)], U(1)$ hypercharge).

$$q_L : \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim \left(3, 2, \frac{1}{6}\right) \tag{2.7}$$

$$\ell_L : \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \sim \left(1, 2, -\frac{1}{2}\right) \tag{2.8}$$

$$\phi : \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \sim \left(1, 2, -\frac{1}{2}\right) \tag{2.9}$$

$$u_R \sim \left(3, 1, \frac{2}{3}\right) \tag{2.10}$$

$$d_R \sim \left(3, 1, -\frac{1}{3}\right) \tag{2.11}$$

$$e_R \sim (1, 1, -1) \tag{2.12}$$

The Standard Model Lagrangian can be schematically divided into a few sectors. We have the quark sector with its gauge interactions that are described by terms of the form:

$$\begin{aligned}\mathcal{L} \supset\ & \overline{q}_L(i\not{D})q_L \\ & + \overline{u}_R(i\not{D})u_R \\ & + \overline{d}_R(i\not{D})d_R\end{aligned} \tag{2.13}$$

with the covariant derivative understood to take on different hypercharge values for different terms.

We have the lepton sector with its gauge interactions:

$$\begin{aligned}\mathcal{L} \supset\ & \bar{\ell}_L i\not{D}\ell_L \\ & + \bar{e}_R i\not{D} e_R\end{aligned} \tag{2.14}$$

We also have the sector for the free gauge fields:

$$\mathcal{L} \supset -\frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}Y^{\alpha\beta}Y_{\alpha\beta} \tag{2.15}$$

where $W^a_{\mu\nu}$ is understood to be the field strength tensor of the non-abelian gauge field W, with a indexing the three generators in $SU(2)_L$.

Lastly, we also have the Yukawa sector describing the coupling of the fermions and gauge bosons with the Higgs scalar field. The Higgs is represented as a $SU(2)_L$ doublet:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2.16}$$

consisting of a charged and neutral field. These coupling terms are responsible for giving mass to fermions and the W and Z bosons via the Higgs mechanism. We will describe the Higgs mechanism in the next section, with a focus on how this can break the gauge symmetry of the Lagrangian.

$$\mathcal{L} \supset -y_{(d)}\bar{q}_L\phi d_R - y_{(u)}\bar{q}_L\tilde{\phi}u_R - y_{(\ell)}\bar{\ell}_L\phi e_R + \text{h.c.} \tag{2.17}$$

which represent the Yukawa coupling terms of the down quarks, up quarks, and the leptons.

2.3 Higgs mechanism and Spontaneous Symmetry Breaking

The Lagrangian obeys symmetries. In Spontaneous Symmetry Breaking (SSB), the lowest energy physical vacuum does not obey the same symmetry, and hence the symmetry is broken. This is different from ‘Explicit Symmetry Breaking’ by the fact that symmetry breaking terms need not be explicitly inserted into the Lagrangian. Rather, Spontaneous Symmetry Breaking is dynamically generated, and is the consequence of adding a potential field of a certain kind – a Higgs field. This section follows Chapter 3.3 of [7].

To show how SSB works, we first consider a pedagogical Lagrangian with $U(1)$ global symmetry, and consider a complex scalar field ϕ which is assigned a scalar potential:

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - V(\phi^*\phi) \tag{2.18}$$

$$V = -\mu^2|\phi|^2 + \lambda|\phi|^4 \tag{2.19}$$

This scalar potential is also recognised as the 'Mexican-hat potential', shown in Figure 2.1. We can write explicitly the two degrees of freedom of the complex scalar

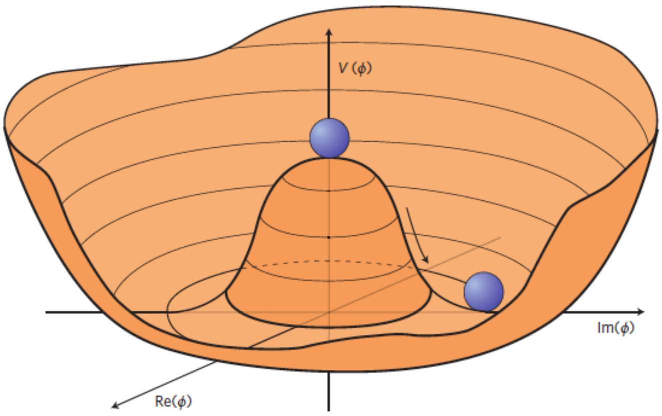


Figure 2.1: The 'Mexican-hat potential' of the Higgs field, extracted from [13]

field, ψ_1 and ψ_2 where $\phi = (\psi_1 + i\psi_2)/\sqrt{2}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \psi_1)(\partial^\mu \psi_1) + \frac{1}{2}(\partial_\mu \psi_2)(\partial^\mu \psi_2) - V(\psi_1^2 + \psi_2^2) \tag{2.20}$$

For $\mu^2 > 0$, the potential V has an infinite number of lowest energy states at $|\phi|^2 = \frac{\mu^2}{2\lambda^2}$. The vacuum state is no longer unique (as the energy minima are degenerate). Choosing an arbitrary point as the physical vacuum, we can choose without loss of generality for the real part of the field to be v and the imaginary part to be 0.

The resulting Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \psi_1)(\partial^\mu \psi_1) + \frac{1}{2}(\partial_\mu \psi_2)(\partial^\mu \psi_2) - \frac{1}{2}(2\lambda v^2)\psi_1^2 + \lambda v \psi_1(\psi_1^2 + \psi_2^2) - \frac{\lambda}{4}(\psi_1^2 + \psi_2^2)^2$$

The resulting Lagrangian no longer has the original $U(1)$ symmetry after spontaneous symmetry breaking. This also gives a mass term to one of the bosons - the original symmetry of the Lagrangian has been broken as the vacuum has a lower symmetry.

2.3.1 Electroweak Unification and gauge boson masses

We apply the same idea of Spontaneous Symmetry Breaking to the Standard Model $SU(2)_L \otimes U(1)_Y$ local gauge symmetry. After Spontaneous Symmetry Breaking, the electroweak group $SU(2)_L \otimes U(1)_Y$ is broken down into $U(1)_{em}$ via the Higgs boson. We show the main results here, though we do the full calculation for the gauge boson and Higgs boson masses in Appendix A.4.

At the electroweak energy scale, we have the gauge bosons W_μ^a and Y_μ. By construction, the Higgs doublet is given the vacuum energy value (vev):

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.21}$$

After symmetry breaking, the charged field of the Higgs doublet vanishes, and we obtain the charged gauge bosons:

$$W_\mu^\pm = \frac{W_\mu'^1 \mp i W_\mu'^2}{\sqrt{2}} \tag{2.22}$$

as well as the neutral photon A and Z^0 boson via a rotation

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu'^3 \\ Y_\mu \end{pmatrix} \tag{2.23}$$

where θ_W is the Weinberg angle, which relates the coupling constants of $SU(2)_L$ (g) and $U(1)_Y(g')$:

$$\tan\theta_W = \frac{g'}{g} \tag{2.24}$$

The original $SU(2) \otimes U(1)_Y$ symmetry of the Lagrangian is now broken after SSB into the more familiar $U(1)_{em}$ at low energies. This has given mass to the W and Z bosons and Higgs boson, and produced the massless photon. The charged field of the Higgs doublet vanishes after SSB, which means that the $U(1)_{em}$ symmetry is obeyed. In the next section, we will see how the Yukawa coupling terms give rise to lepton masses.

2.3.2 Dynamical generation of masses via Yukawa couplings

Consider the Yukawa term for the leptons:

$$\begin{aligned}\mathcal{L}_Y &= -G_e(\bar{\ell}\phi e_R + \bar{e}_R\phi^\dagger\ell) + \text{h.c.} &(2.25)\\ &= -G_e\left(\bar{e}_L\frac{1}{\sqrt{2}}(v+H)e_R + \bar{e}_R\frac{1}{\sqrt{2}}(v+H)e_L\right) + \text{h.c.} &(2.26)\\ &= \frac{G_e v}{\sqrt{2}}\bar{e}e - \frac{G_e}{\sqrt{2}}H\bar{e}e &(2.27)\end{aligned}$$

where we have used properties of the projection operators. This produces an electron mass term of:

$$m_e = \frac{G_e v}{\sqrt{2}} \tag{2.28}$$

Hence, we have completed our construction of the Standard Model – the Higgs mechanism explains the origin of fermion masses, as well as the $W^\pm$ and Z bosons. It also connects our observed $U(1)_{em}$ symmetry at low energy to a $SU(2)_L \otimes U(1)_Y$ gauge group at the electroweak scale, hence unifying the weak and electromagnetic force.

2.4 Going beyond the Standard Model

2.4.1 Open questions in the Standard Model

For all its achievements, the Standard Model has many open questions [14]. Firstly, QFT is built on flat spacetime. However, a treatment of general relativity would require consideration of curved spacetime. The Standard Model as-is does not produce a quantum model of gravity.

Secondly, the electroweak scale $\approx 10^2$ GeV and the fundamental Planck scale $\approx 10^{19}$ GeV, are very different. If we were to allow heavy ($\gg 10^2$ GeV) fermions or scalars to exist, the mass of the Higgs boson would require extreme fine-tuning.

Thirdly, cosmological observations have pointed towards the existence of large amounts of mass in the universe that does not emit light, which is termed 'dark matter' [15] that can only be observed through its gravitational effects. The Standard Model lacks a dark matter candidate whose only interaction is gravitational.

2.4.2 Neutrino masses as a window to beyond the Standard Model

This thesis is interested in the question of neutrino masses. Interestingly, the Standard Model does not come with an explanation for neutrino mass. In the Standard Model, the neutrinos are left-handed and massless, and the masslessness corresponds to lepton number conservation, which is an accidental symmetry of the SM, which makes it plausible that neutrinos have mass [16].

The observation of neutrino oscillations [17] indicates that their mass is finite – the calculations for the two-neutrino case are shown in Appendix B. If the smallness of the neutrino mass relative to the electroweak scale is taken to be unnatural – it requires a significant degree of fine-tuning on the order of 10^{12} – the question of neutrino masses points towards new physics.

This is especially significant given the lack of other promising leads to Beyond Standard Model (BSM) physics – no light supersymmetric particles have been found [18, 19], no proton decay has been observed [20], and no neutrinoless double-β decay has been observed [21]. The observation that neutrino mass is non-zero is the most promising sign of BSM physics.

2.4.3 The predictability argument and why we should have more models

As we build theories, it is worth asking how we might justify building more theories that try to explain the same set of phenomenology. With an unbounded number of potential models that fit existing data, it remains true that experimental verification has the final say on the validity of one theory over the other. However, without any groundbreaking experimental results, it is the job of the theorists to build models that can explain neutrino masses as long as these models can be experimentally falsified.

Hence, the ability to generate predictions is enough to excuse the creation of new models. Rather, theoretical constructions are meant to guide experimental searches, allowing for the possibility of discovering new physics.

2.5 The goal of the thesis, restated

If the history of physics is the history of unification, this will motivate constructions of higher and more complicated theories from which the Standard Model may be a low-energy realisation. Thus, it is the task of the theorist to build models that can explain neutrino masses within a broader and theoretically appealing framework.

In the following chapter, we will discuss various mechanisms one might use to explain neutrino masses within the context of the Standard Model. These mechanisms will be used in later chapters to build models.

Chapter 3

Strategies for obtaining neutrino masses

3.1 Overview

Neutrino oscillation experiments show that neutrino masses are non-zero [22], and the most recent upper bound on the mass of the heaviest neutrino is 0.8 eV [23]. We explain how neutrino oscillations imply neutrino mass in Appendix B.

In this chapter, we will first discuss the neutrino mass within the context of the Standard Model in Section 3.2, and provide the calculations to show how we can obtain neutrino masses within the confines of the Standard Model – this requires us to posit the existence of a sterile right-handed neutrino N_R that has not yet been detected. We will use the tools of field redefinitions and basis transformations to rotate into the mass basis to see the neutrino mass terms explicitly. However, we will find that this implies that the relevant Yukawa coupling is $\sim 10^{-12}$, which is very small even compared to the small electron mass Yukawa coupling $\sim 10^{-6}$, and much smaller than the quark Yukawa couplings.

To remedy this, we can introduce new physics in Section **??** by positing that neutrinos can be of Majorana nature, in which the particle is its own anti-particle. We introduce the Weyl spinors and properties of the conjugate charge operator, before using the same procedure to rotate into the mass basis and make the mass term manifest.

Subsequently, we consider the general case in Section 3.4 where we have Majorana light left-handed neutrinos ν_L, Majorana heavy right-handed sterile neutrinos N_R, and a Dirac mass term coupling the light left-handed and heavy right-handed neutrinos. This calculation will show that the lightness of the SM neutrinos is guaranteed by the heaviness of the heavy sterile N_R by the 'Seesaw mechanism' which provides a natural explanation of the light neutrinos.

The next chapter will address how one might explain Majorana terms in the Lagrangian using effective operators, and also re-derive the Seesaw mechanism using an Effective Field Theory approach, justifying this approach of obtaining neutrino masses.

We will reference Bilenky's monograph for the rest of this section [24].

3.2 Dirac Neutrino masses

In the Standard Model, electron masses have the form $m\bar{e}e$, in which $e = P_L e + P_R e = e_L + e_R$. [1] If neutrino masses are analogous to electron masses, we should expect a mass term that looks like $m\bar{\nu}\nu$, in which $\nu = \nu_L + \nu_R$. For the sake of argument, we will construct neutrino masses as Dirac masses (hence analogous to electron masses) to show that this implies an unnaturally small Yukawa coupling, demonstrating the need for new physics.

There is no right-handed neutrino in the Standard Model [25] – the neutrino is left-handed, while the anti-neutrino is right-handed. Hence, we posit the existence of the sterile right-handed neutrino N_R that is a singlet under $SU(2)_L$ and carries no hypercharge. Hence, we can construct a Yukawa interaction term involving the lepton doublet, Higgs boson, and the right-handed neutrino.

We will write ψ_L as the lepton doublet of $SU(2)_L$, and write N_R to represent the sterile right-handed neutrino. We can now consider a Dirac mass for the neutrino that is generated via the Higgs mechanism:

$$-\mathcal{L}_{\text{Dirac}} = \sqrt{2}(\overline{\psi_L} Y_\nu \tilde{H} N_R) + \text{h.c.} \tag{3.1}$$

[1] The projection operators have the properties $P_L + P_R = 1$, and $P_L^2 = P_L, P_R^2 = P_R$ as a consequence of the properties of the γ_5 matrices.

where $\tilde{H} \equiv i\tau_2 H^*$ is the conjugate Higgs doublet, and Y_ν represents the dimensionless Yukawa coupling constant matrix. Expanding:

$$\begin{aligned}
\sqrt{2}\begin{pmatrix}\overline{\nu_L} & \overline{e_L}\end{pmatrix} Y_\nu i \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\varphi^+ \\ \varphi^0\end{pmatrix} &= \sqrt{2}\begin{pmatrix}\overline{\nu_L} & \overline{e_L}\end{pmatrix} Y_\nu \begin{pmatrix}\varphi^0 \\ -\varphi^+\end{pmatrix} N_R + \text{h.c.} \quad \text{(expand)} \\
&= \sqrt{2}\begin{pmatrix}\overline{\nu_L} & \overline{e_L}\end{pmatrix} Y_\nu \begin{pmatrix}\frac{v+H}{\sqrt{2}} \\ 0\end{pmatrix} N_R + \text{h.c.} \quad \text{(SSB)} \\
&= \overline{\nu_L}(vY_\nu)N_R + \text{h.c.} \qquad (3.2)
\end{aligned}$$

We have dropped the term with the Higgs boson H in the last step as we are interested in the mass of the neutrinos and not the strength of the interaction. Now, the goal is to rotate from flavour into mass basis.

The dimensionless matrix of Yukawa constants Y_ν can be diagonalised using a bi-unitary transformation $Y_\nu = U\widehat{y}V^\dagger$. This is because Y_ν is complex but not symmetric. The bi-unitary transformation is described in more detail in Appendix C.

$$U^\dagger Y_\nu V = \widehat{y} \equiv \text{Diag}\{y_1, y_2, y_3\} \qquad (3.3)$$

We also identify M_D as the mass matrix in the flavour basis.

$$M_D = Y_\nu v = vV\widehat{y}U^\dagger \qquad (3.4)$$

Inserting the bi-unitary transformation, we will rotate from the flavour basis to the mass basis.

$$-\mathcal{L}_{\text{Dirac}} = \overline{\nu_L}Vv\widehat{y}U^\dagger N_R + \text{h.c.} \qquad (3.5)$$

We make the following field redefinitions.

$$\begin{aligned}
N_R' &= V^\dagger N_R \qquad (3.6) \\
\nu_L' &= U^\dagger \nu_L \implies \overline{\nu_L'} = \overline{\nu_L}U \qquad (3.7) \\
v\widehat{y} &= \widehat{M_\nu} \qquad (3.8)
\end{aligned}$$

where $\widehat{M_\nu}$ is the diagonal mass eigenvalue matrix. This lets us write in the mass basis:

$$-\mathcal{L}_{\text{Dirac}} = \overline{\nu_L'}\widehat{M_\nu}N_R' + \text{h.c.} \qquad (3.9)$$

We now define the four-component Dirac spinor:

$$\nu' = U^\dagger \nu_L + V^\dagger N_R = \nu'_L + N'_R$$

This expresses ν' as a sum of left and right components i.e. $\nu' = P_L\nu' + P_R\nu'$ in the mass basis. From the relations $U^\dagger \nu_L = \nu'_L$ and $V^\dagger N_R = N'_R$, we identify U and V as mixing matrices. In fact, U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix of the left-handed SM neutrinos, which rotates from the flavour into the mass basis.

We want to insert ν' into the Lagrangian term. Observe:

$$\begin{aligned}\overline{\nu'}\widehat{M_\nu}\nu' &= \overline{\nu'}\widehat{M_\nu}(P_L\nu' + P_R\nu') \\ &= \overline{\nu'}\widehat{M_\nu}(P_L^2\nu' + P_R^2\nu')\end{aligned}$$

Since $P_{L,R}$ is in a different vector space from $\widehat{M_\nu}$,

$$\overline{\nu'}\widehat{M_\nu}\nu' = \overline{\nu'}P_L\widehat{M_\nu}P_L\nu' + \overline{\nu'}P_R\widehat{M_\nu}P_R\nu'$$

To proceed, we make the explicit identification for a general $\overline{\psi}$:

$$\overline{\psi}P_L = \psi^\dagger\gamma^0\frac{1-\gamma_5}{2}$$

which works because γ_5 is real.

$$\begin{aligned}&= \psi^\dagger\frac{1+\gamma_5}{2}\gamma^0 \\ &= \left(\frac{1+\gamma_5}{2}\psi\right)^\dagger\gamma^0 \\ &= (P_R\psi)^\dagger\gamma^0 \\ &= \overline{\psi_R}\end{aligned}$$

The result follows for $L \leftrightarrow R$. We proceed and absorb the projection operators into the adjoint terms:

$$\begin{aligned}\overline{\nu'}\widehat{M_\nu}\nu' &= \overline{N'_R}\widehat{M_\nu}\nu'_L + \overline{\nu'_L}\widehat{M_\nu}\nu'_R \\ \overline{\nu'}\widehat{M_\nu}\nu' &= \overline{\nu'_L}\widehat{M_\nu}N'_R + \text{h.c.}\end{aligned}$$

Hence:

$$-\mathcal{L}_{\text{Dirac}} = \overline{\nu'}\widehat{M_\nu}\nu' + \text{h.c.}$$

We have absorbed the second term $\overline{N'_R}\widehat{M_\nu}\nu'_L$ into the higher correction term +h.c.. We are allowed to do this as the Lagrangian is required to be Hermitian (reality condition). The second term is simply the Hermitian conjugate of the first term, so the second term adds no information as we can reconstruct the second term from the first.

This rewriting shows that ν' is in the neutrino mass basis, and demonstrates that we can obtain a neutrino mass term using the framework of the Standard Model. All we had to do was introduce a hypothetical sterile neutrino N_R, allowing us to have neutrino masses just as we have electron masses. Now, we can check if this is a satisfying explanation. The upper bound on the heaviest neutrino mass m_3 is $\sim 10^{-1}$ eV [23]. With $v \sim 10^{11}$ eV,

$$\begin{aligned} vy_3 = m_3 &\lesssim 10^{-1} \\ y_3 &\lesssim 10^{-12} \end{aligned}$$

The heaviest fermions have Yukawa coupling constant y on the order of $\sim 10^2$, while the electron Yukawa coupling constant is $\sim 10^{-6}$. When compared to the required neutrino Yukawa coupling constant, this makes $y_3 \sim 10^{-12}$ appear uncharacteristically small.

Hence, that neutrino masses are small but non-zero is a mystery in the Standard Model – it would be simpler if we had massless neutrinos. This is our motivation to venture into physics beyond the Standard Model. To do so, we will consider another explanation in the next section. Instead of assuming that the neutrino mass mechanism is analogous to that of the electron, we will posit that neutrinos can be Majorana particles, which are their own anti-particles.

3.3 Majorana neutrino masses

3.3.1 Overview

Majorana particles are characterised by the condition $\chi^c = \pm\chi$ for a four-component spinor χ, where χ^c is the charge conjugated spinor [26]. The neutrino is the only known particle allowed to take on a Majorana nature as every other fermion has a known Dirac mass mechanism in the Standard Model. Properties of the charge conjugate matrix $\mathcal{C}$ are shown in Appendix J.1.

Our goal is to consider the consequences of Majorana neutrino. In the subsequent section, we then consider a model in which we have (1) Majorana left-handed light neutrinos, (2) Majorana right-handed heavy neutrinos, (3) a Dirac mass term that couples the left- and right-handed neutrinos.

3.3.2 Weyl spinorial properties and conservation of lepton number

Let us first consider the two-component Weyl spinor ψ_L which is obtained from the four-component Dirac spinor ψ as such:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3.10}$$

$$\begin{aligned} \psi_L &= \frac{1-\gamma_5}{2}\psi \\ &= P_L\psi \\ &= \begin{pmatrix} 0 \\ \eta \end{pmatrix} \end{aligned} \tag{3.11}$$

where P_L is the projection operator for left chirality. For consistency, we will denote the Weyl spinor ψ_L as ν_L to represent the neutrino. Using the Weyl representation of the γ_5 matrix, we note the following properties:

$$\gamma_5\nu_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \eta \end{pmatrix}$$

$$\gamma_5\nu_L = -\nu_L \tag{3.12}$$

$$\begin{aligned} \overline{\nu_L}\gamma_5 &= \nu_L^\dagger\gamma_0\gamma_5 \\ &= -\nu_L^\dagger\gamma_5\gamma_0 \\ &= -(\gamma_5\nu_L)^\dagger\gamma_0 \\ &= (\nu_L)^\dagger\gamma_0 \\ &= \overline{\nu_L} \end{aligned} \tag{3.13}$$

If we require that the total lepton number is conserved, it implies that the Lagrangian is invariant under a global $U(1)$ transformation $\nu_L \to e^{i\theta}\nu_L$. Consider a hypothetical mass term involving only the left-handed neutrinos

$$m_\nu \overline{\nu_L}\nu_L \tag{3.14}$$

which would preserve a global $U(1)$ symmetry for lepton number. However, from Eq 3.12 and 3.13, this mass term vanishes as such:

$$\begin{aligned}
\overline{\nu_L}\nu_L &= \overline{\nu_L}\gamma_5\nu_L \\
&= \overline{\nu_L}(\gamma_5\nu_L) \\
&= -\overline{\nu_L}\nu_L \\
&= 0
\end{aligned} \tag{3.15}$$

Hence, lepton number conservation implies that neutrinos are allowed to be massless if we did not introduce a heavy N_R. Now, let us assume that the total lepton number is violated through the introduction of Majorana neutrinos that satisfy the Majorana condition:

$$\chi^c = \pm\chi \tag{3.16}$$

for an arbitrary four-component spinor χ. Now, using Eqs J.2, J.5 and 3.13:

$$\begin{aligned}
\overline{\nu_L}\gamma_5 &= \overline{\nu_L} \\
\gamma_5^T\overline{\nu_L}^T &= \overline{\nu_L}^T \\
\mathcal{C}\gamma_5^T\mathcal{C}^{-1}\mathcal{C}\overline{\nu_L}^T &= \mathcal{C}\overline{\nu_L}^T \\
\gamma_5\nu_L^c &= \nu_L^c
\end{aligned} \tag{3.17}$$

Using the charge conjugated field, the mass term no longer vanishes, unlike Eq 3.15:

$$\overline{\nu_L}\nu_L^c = \overline{\nu_L}\gamma_5\nu_L^c = \overline{\nu_L}\nu_L^c \tag{3.18}$$

3.3.3 Majorana mass term in the Lagrangian

Using the fact that neutrinos have three flavours, we can write a general mass term. We will apply the same machinery as the previous section to transform the flavour basis into the mass basis using field-redefinitions and basis transformations.

$$-\mathcal{L}_{\text{Majorana}} = \frac{1}{2}\overline{\nu_L}_{l'}(M_L)_{l'l}(\nu_L)^c_l + \text{h.c.}$$

$$= \frac{1}{2}\overline{\nu_L} M_L \nu_L^c + \text{h.c.} \tag{3.19}$$

where $l', l = (e, \mu, \tau)$ represent flavour indices, and M_L is a general flavour-mixing matrix. We assume that M_L is symmetric and can be decomposed as:

$$M_L = U m U^T \tag{3.20}$$

where U is a unitary matrix and m is a diagonal matrix with mass eigenvalues. Using the following identifications,

$$\begin{aligned}
\overline{U^\dagger \nu_L} &= \overline{\nu_L} U \\
\mathcal{C} U^T \overline{\nu_L}^T &= \mathcal{C}(\overline{\nu_L} U)^T \\
&= \mathcal{C}(\overline{U^\dagger \nu_L})^T \\
&= (U^\dagger \nu_L)^c
\end{aligned}$$

we can now rewrite the Lagrangian:

$$\begin{aligned}
-\mathcal{L}_{\text{Majorana}} &= \frac{1}{2}\overline{\nu_L} M_L \nu_L^c + \text{h.c.} \\
&= \frac{1}{2}\overline{\nu_L} U m U^T \mathcal{C} \overline{\nu_L}^T + \text{h.c.}
\end{aligned}$$

U is in the mass vector space and commutes with $\mathcal{C}$

$$\begin{aligned}
-\mathcal{L}_{\text{Majorana}} &= \frac{1}{2}\overline{\nu_L} U m \mathcal{C} U^T \overline{\nu_L}^T + \text{h.c.} \\
&= \frac{1}{2}\overline{\nu_L} U m \mathcal{C}(\overline{\nu_L} U)^T + \text{h.c.} \\
&= \frac{1}{2}\overline{\nu_L} U m \mathcal{C}(\overline{U^\dagger \nu_L})^T + \text{h.c.} \\
&= \frac{1}{2}\overline{U^\dagger \nu_L} m (U^\dagger \nu_L)^c + \text{h.c.}
\end{aligned} \tag{3.21}$$

Let us choose the representation in the mass eigenstates:

$$\nu_{\text{mass}} = U^\dagger \nu_L + (U^\dagger \nu_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{3.22}$$

and when substituted back in, gives us the mass term:

$$= \frac{1}{2}\overline{\nu_{\text{mass}}} m \nu_{\text{mass}} + \text{h.c.} \tag{3.23}$$

Hence, we have rotated from the flavour into the mass basis, and arrived at the Majorana mass term of the Lagrangian in Eq 3.23 as a consequence of the properties of the Majorana neutrino. Observe that $(\nu_{\text{mass}})^c = \nu_{\text{mass}}$, so ν_{mass} obviously fulfils the Majorana condition.

To check that lepton number conservation is violated, consider the global phase transformation $\nu_L \to e^{i\theta}\nu_L$.

$$\begin{aligned}\overline{U^\dagger e^{i\theta}\nu_L} m (U^\dagger e^{i\theta}\nu_L)^c &= e^{-i\theta}\overline{U^\dagger\nu_L} m e^{-i\theta}(U^\dagger\nu_L)^c \\ &= e^{-2i\theta}\overline{U^\dagger\nu_L} m (U^\dagger\nu_L)^c\end{aligned}$$

As promised, the lepton number conservation is broken for massive Majorana neutrinos. Using the mass basis, rewrite the Majorana fermion field again to see the connection between the left and right-handed fields.

$$\nu^M = \nu_L^M + \nu_R^M$$

as we can write any field as a sum of left-handed and right-handed components. Recalling Eq 3.22, we can identify:

$$\begin{aligned}\nu_L^M &= U^\dagger \nu_L \\ \nu_R^M &= (U^\dagger \nu_L)^c\end{aligned}$$

which implies:

$$\nu_R^M = (\nu_L^M)^c \tag{3.24}$$

This connects the left-handed fields to the right-handed fields. While the Dirac mass term has independent left and right-handed fields, the Majorana fields are connected by Eq 3.24.

In the flavour basis, we can see the connection as a direct result of the Majorana condition:

$$\begin{aligned}(\nu_L)^c = \left(\frac{1-\gamma_5}{2}\nu\right)^c = \mathcal{C}\frac{1+\gamma_5^T}{2}\overline{\nu}^T &= \mathcal{C}\frac{1+\gamma_5^T}{2}\mathcal{C}^{-1}\mathcal{C}\overline{\nu}^T \\ &= \frac{1+\gamma_5}{2}\nu^c \\ &= \frac{1+\gamma_5}{2}\nu \qquad \text{(Majorana condition)}\end{aligned}$$

$$= \nu_R$$

This concludes the introduction of the Majorana mass term. We have seen its allowed form $m\overline{\psi}\psi^c$ and in the next section, we will see how we can put Dirac and Majorana masses together to explain the smallness of the left-handed neutrinos.

3.4 Dirac and Majorana masses

Just as we wrote the Majorana mass term for the left-handed neutrino flavour fields, we are allowed to do so for the right-handed neutrino flavour fields:

$$-\mathcal{L}_{\text{right-handed}} = \frac{1}{2}\overline{(N_R)^c}M_R N_R + \text{h.c.} \tag{3.25}$$

In general, the Yukawa terms in the Lagrangian that lead to neutrino mass are allowed to be a combination of both Dirac and Majorana masses. This general term is not invariant under the lepton number transformation and includes the left and right-handed Majorana fields. In the flavour basis:

$$-\mathcal{L}_{\text{D+M}} = \frac{1}{2}\overline{\nu_L}M_L(\nu_L)^c + \frac{1}{2}\overline{(N_R)^c}M_R N_R + \overline{\nu_L}M_D N_R + \text{h.c.} \tag{3.26}$$

where the first term is a Majorana term for the light left-handed neutrinos, the second term is a Majorana term for the heavy right-handed neutrinos, and the third term couples the left- and right-handed neutrinos using a Dirac mass term. $M_{L,R}$ refers to Majorana mass matrices which are complex and symmetric. M_D is the Dirac mass matrix and is a general complex matrix.

We can perform a field re-definition to contract the Dirac and Majorana Lagrangian. Let:

$$n_L = \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} \tag{3.27}$$

$$M_{D+M} = \begin{pmatrix} M_L & M_D \\ (M_D)^T & M_R \end{pmatrix} \tag{3.28}$$

where M_{D+M} is manifestly symmetric. We can now write:

$$-\mathcal{L}_{\text{D+M}} = \frac{1}{2}\overline{n_L}M_{D+M}(n_L)^c + \text{h.c.} \tag{3.29}$$

where we have used the relationship:

$$\overline{N_R^c} M_D^T (\nu_L)^c = \left[(N_R)^T \mathcal{C} M_D^T \mathcal{C} \overline{\nu_L}^T \right]^T = \overline{\nu_L} M_D N_R \tag{3.30}$$

Since M_{D+M} is symmetric, we can diagonalise $M_{D+M} = UmU^T$ using a unitary matrix U, using the mass basis. We use steps similar to Eq 3.21.

$$\begin{aligned} -\mathcal{L}_{\text{D+M}} &= \frac{1}{2} \overline{n_L} U m U^T (n_L)^c + \text{h.c.} \\ &= \frac{1}{2} \overline{U^\dagger n_L} m (U^\dagger n_L)^c + \text{h.c.} \end{aligned}$$

Again, perform the re-definitions to write the flavour fields into the mass basis

$$\begin{aligned} \nu_L^M &= U^\dagger n_L \\ \nu^M &= \nu_L^M + (\nu_L^M)^c \end{aligned}$$

which allows us to write

$$-\mathcal{L}_{\text{D+M}} = \frac{1}{2} \overline{\nu^M} m \nu^M + \text{h.c.} \tag{3.31}$$

As before, $(\nu^M)^c = \nu^M$, and the Majorana condition is fulfilled. Hence, we have written this combination of Dirac and Majorana terms into a Majorana field. Consequently:

$$n_L = \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix} = U \nu^M \tag{3.32}$$

Since U is a 6×6 unitary matrix, and since ν^M has 6 elements, this implies:

$$\nu_L = U \nu^M = (N_R)^c = U \nu^M \tag{3.33}$$

Hence, the flavour fields ν_L and N_R are both a mixture of the general Majorana field ν^M via a unitary transformation U. In the next section, we will proceed to show that this construction implies the Seesaw mechanism, in which the heaviness of the right-handed neutrinos leads to the lightness of the left-handed neutrinos.

3.5 Diagonalising the Seesaw Matrix with unitary matrices

Let us consider the block mass matrix $M = \begin{pmatrix} M_L & m_D \\ m_D^T & M_R \end{pmatrix}$ which appeared in Eq 3.29. In the Standard Model, the left-handed neutrinos are already assumed to be massless, and only obtain a mass through their interactions, so we set $M_L = 0$. We also assume that we are in the basis where M_R is diagonal and symmetric.

Since the block matrix M is symmetric, we can diagonalise it as $U^T M U = m$ where U is a unitary matrix. Let us choose a special U such that m becomes block diagonal.

$$U = \begin{pmatrix} 1 & a^\dagger \\ -a & 1 \end{pmatrix}, a \ll 1, U^\dagger U \simeq 1 \tag{3.34}$$

$$U^T M U = \begin{pmatrix} 1 & -a \\ a^\dagger & 1 \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \begin{pmatrix} 1 & a^\dagger \\ -a & 1 \end{pmatrix} \tag{3.35}$$

$$= \begin{pmatrix} -a^T m_D^T - m_D a + a^T M_R a & -a^T m_D^T a^T + m_D - a^T M_R \\ m_D^T - a m_D a - M_R a & m_D^T a^\dagger + a m_D + M_R \end{pmatrix} \tag{3.36}$$

We assume that $a \ll 1$. For the off-diagonal terms to be 0, we can use $a = M_R^{-1} m_D^T$, and suppress the terms that are linear or higher in a.

$$m \simeq \begin{pmatrix} -m_D M_R^{-1} m_D^T & 0 \\ 0 & M_R \end{pmatrix} \tag{3.37}$$

Let us substitute this back into Eq 3.30.

$$-\mathcal{L}_{\mathrm{D+M}} = \frac{1}{2} \overline{n_L} M_{D+M} (n_L)^c + \mathrm{h.c.} \tag{3.38}$$

where n_L is the block matrix

$$n_L = \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} \tag{3.39}$$

and let's use the diagonalisation we have just introduced, and absorbing the transformation matrices U using a field re-definition:

$$-\mathcal{L}_{\mathrm{D+M}} = \frac{1}{2} \overline{\nu_L} (-m_D M_R^{-1} m_D^T)(\nu_L)^c + \frac{1}{2} \overline{N_R^c} M_R N_R + \mathrm{h.c.} \tag{3.40}$$

Observe that the Majorana left-handed neutrinos have a mass term of the form $\frac{m_D^2}{M_R}$. Since we assume that the right-handed sterile neutrinos are at a higher mass scale, this provides a natural explanation for the lightness of the SM left-handed neutrinos. This is known as the Seesaw mechanism.

3.6 Summary

In this chapter, we have progressively introduced the Seesaw mechanism by introducing (1) a heavy sterile right-handed neutrino, and (2) Majorana neutrinos, which can provide a natural explanation for the lightness of the left-handed neutrinos. In the next chapter, we will discuss where these Majorana mass terms come from in the first place – if electron masses are produced via SSB and the Higgs mechanism, we want to find a mechanism that can explain the Majorana masses that allow the Seesaw mechanism to appear.

Chapter 4

Effective Field Theory

4.1 Overview

In this chapter, we will use arguments from Effective Field Theory to justify the introduction of Majorana masses in the previous chapter. Firstly, we discuss the Wilsonian perspective on renormalisability in Section 4.2, arguing that it is not required for physical predictions. Next, we apply the Wilsonian perspective by arguing that effective operators must appear as a consequence of introducing higher energy (heavier) fields in Section 4.3. We then use this same argument in Section 4.4 to show that the Seesaw mechanism is a consequence of an Effective Field Theory approach, which completes the discussion in the previous chapter. Lastly, we will use a simpler and alternative approach in Section 4.5 to think about constructing effective operators without explicitly using the path integral method. This will give us the tools to create new operators of arbitrary mass dimension when discussing extensions in later chapters.

4.2 Perspectives on renormalisability

A discussion of renormalisation is found in Matthew Schwartz's textbook [5]. In QED, ultraviolet divergences in the Feynman diagrams are removed by adding counterterms with renormalisation factors. In a renormalisable theory, all UV divergences can be cancelled with a finite number of counterterms.

Although renormalisability is a desirable property, loops (at all orders) are difficult to evaluate, and in perturbation theory, there exists an energy limit after which the theory breaks down. However, if one is making physical predictions, renormalisability is not required.

Following Wilson's classification scheme, operators with mass dimension $r = 4$ are marginal, $r < 4$ as relevant, and $r > 4$ as irrelevant. We are interested in non-renormalisable interactions with mass dimension $r > 4$, which are suppressed by factors of mass dimensions Λ^{d-4} such that the Lagrangian is dimensionally consistent.

For example, a dimension-5 operator with the form $\overline{\psi}\psi HH$ has a coefficient $1/\Lambda$, where Λ is a large cutoff scale. As long as Λ is sufficiently large, we can perform loop integrals as if $\Lambda = \infty$, and cutoff-dependent effects will be suppressed by powers of $1/\Lambda$.

Hence, we are allowed to add higher dimension operators as renormalisability is not a requirement for the field theory, and physical predictions can be made regardless. With this knowledge, we can now consider how we might obtain dimension-5 Weinberg operators that will lead to Majorana masses which will lead to the Seesaw mechanism.

4.3 Path integral justification for building effective operators

However, can we find a more motivated justification for the introduction of effective operators? This section will illustrate how heavier fields at a higher energy scale can give rise to effective operators at lower energies – these effective operators are terms of a perturbative series, and identifying high-dimensional operators is similar to naming the coefficients of a Taylor expansion. This is more theoretically motivated as compared to simply inserting arbitrary non-renormalisable high-dimension terms.

The general argument uses an Effective Field Theory approach via the Wilsonian perspective to renormalisation. The full discussion is found in Appendix H. For brevity, only the main idea is presented here.

Consider two real scalar fields $\phi(x)$ and $\Phi(x)$ with masses m_ϕ and M_Φ respectively, where $m_\phi \ll M_\Phi$. Schematically, the Lagrangian can be written as:

$$\mathcal{L}_{\text{full}}[\phi;\Phi] = \mathcal{L}_{\text{free}}[\phi] + \mathcal{L}_{\text{free}}[\Phi] + \mathcal{L}_{\text{int}}[\phi,\Phi] \tag{4.1}$$

where the $\mathcal{L}_{\text{free}}$ includes the kinetic and mass terms, and $\mathcal{L}_{\text{int}}$ is the interaction term. At a cutoff energy scale below M_Φ, we can make do with an effective field theory only involving ϕ [27, 28].

Consider the following path integral:

$$Z = \int \mathcal{D}\Phi\mathcal{D}\phi \exp\left(i\int \mathcal{L}_{\text{full}}[\phi, \Phi]\right)$$

In the low energy regime, we can set $\Phi = 0$ inside and factor out $\int \mathcal{D}\Phi$. This moves the Φ dependence outside.

$$Z = \int \mathcal{D}\Phi \int \mathcal{D}\phi \exp\left(i\int \mathcal{L}_{\text{full}}[\phi, \Phi = 0]\right) \tag{4.2}$$

The prefactor is just some number. Rewrite this prefactor as $e^{i\mathcal{O}[\phi]}$ and bring it inside:

$$Z = \int \mathcal{D}\phi \exp\left(i\int \mathcal{L}_{\text{full}}[\phi, \Phi = 0] + \mathcal{O}[\phi]\right) \tag{4.3}$$

such that the effective Lagrangian no longer depends on the heavy field Φ, and now only depends on the lighter field ϕ.

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{full}}[\phi, \Phi = 0] + \mathcal{O}[\phi] \tag{4.4}$$

From this perspective, we can introduce effective operators into a known Lagrangian to represent the effect of heavier fields and their interactions with low energy fields. This was accomplished by moving all the Φ dependence outside and re-defining that to be an effective operator.

More precisely, the effective operators arise from the perturbative series involving the full Lagrangian. In principle, effective operators of arbitrary dimensions can be computed by considering arbitrary high-energy fields and their interactions with known low-energy fields.

In summary, we are allowed to add higher-dimensional effective operators into a known Lagrangian e.g. the Standard Model Lagrangian. This is because we assume that heavy fields exist at a higher energy scale. Their interactions with known fields (e.g. the Standard Model fields) can be described by a perturbative series. By 'adding' effective operators, we are simply identifying relevant terms in the perturbative series that will lead to physical phenomena (e.g. neutrino masses). Hence, we will be adopting this approach for the rest of this thesis.

4.4 Effective Field Theory approach to re-obtain Type-I Seesaw model

Having discussed the Effective Field Theory approach, we will now demonstrate how the Type-I Seesaw model can be obtained via path integral considerations. Heavy right-handed neutrinos at a higher energy scale are introduced. Their interactions with the low energy fields (SM fields) can be described by effective operators, which eventually leads to the Seesaw mechanism. This argument follows Section 4.1.3 of [16].

Consider the type-I seesaw model, which extends the SM with three right-handed heavy Majorana neutrinos. The full Lagrangian of the model is written as the sum of a kinetic term, mass term, and Yukawa term.

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \overline{N_R} i\partial\!\!\!/ N_R - \left[\frac{1}{2}\overline{N_R^c} M_R N_R + \overline{\ell_L} Y_\nu \tilde{H} N_R + \text{h.c.}\right] \tag{4.5}$$

which has a kinetic term for the right-handed neutrino, a Majorana mass term, and a Yukawa coupling with the Higgs particle. The Majorana mass matrix M_R is symmetric. We will rewrite the Yukawa term to obtain a Gaussian path integral.

With the mass term, we can perform the same diagonalisation steps with a unitary matrix U and absorb U into Y_ν by a re-definition.

We will work henceforth in the mass basis. Let us define:

$$\widehat{N}_R \equiv U^T N_R \tag{4.6}$$

$$N \equiv U^T N_R + (U^T N_R)^c \tag{4.7}$$

which allows us to write:

$$\mathcal{L}_{\text{N}} = \frac{1}{2}\overline{N_i} i\partial\!\!\!/ N_i - \frac{1}{2} M_i \overline{N_i} N_i - \left[\overline{\ell_{\alpha L}} (\tilde{Y}_\nu)_{\alpha i} \tilde{H} N_i + \text{h.c.}\right] \tag{4.8}$$

where the index i labels the mass eigenstates. Also observe:

$$\begin{aligned}
\overline{N} &= \overline{\hat{N}_R} + \overline{\hat{N}_R}^c \\
&= \overline{\hat{N}_R} + \overline{\mathcal{C}\overline{\hat{N}_R}^T} \\
&= \overline{\hat{N}_R} + (\overline{N_R})^* \gamma^0 \mathcal{C}^T
\end{aligned}$$

$$
\begin{aligned}
&= \overline{\hat{N}_R} + (N_R^\dagger \gamma^0)^* \gamma^0 \mathcal{C}^T \\
&= \overline{\hat{N}_R} - \hat{N}_R{}^T \mathcal{C}^{-1}
\end{aligned}
$$

and also:

$$
\begin{aligned}
N^T &= \hat{N}_R{}^T + (\mathcal{C}(\overline{\hat{N}_R})^T)^T \\
&= \hat{N}_R{}^T \mathcal{C}^{-1}\mathcal{C} + \overline{\hat{N}_R}\mathcal{C}^T \\
&= \hat{N}_R{}^T \mathcal{C}^{-1}\mathcal{C} - \overline{\hat{N}_R}\mathcal{C} \\
&= -\overline{N}\mathcal{C}
\end{aligned}
$$

We will also need:

$$
\begin{aligned}
\left[\overline{\ell_L}\tilde{Y}_\nu \tilde{H} N\right]^T &= N^T \tilde{Y}_\nu^T \tilde{H}^T (\overline{\ell_L})^T \\
&= -\overline{N}\mathcal{C}\tilde{Y}_\nu^T \tilde{H}^T \overline{\ell_L}^T \\
&= -\overline{N}\tilde{Y}_\nu^T \tilde{H}^T \ell_L{}^c
\end{aligned}
\tag{4.9}
$$

We obtain:

$$
\mathcal{L}_{\mathrm{N}} = \frac{1}{2}\overline{N}KN - \frac{1}{2}\overline{N}\chi - \frac{1}{2}\overline{\chi}N \tag{4.10}
$$

where $K \equiv i\partial\!\!\!/ - \hat{M}$, $\chi = \tilde{Y}_\nu{}^T \tilde{H}^T \ell_L^c + \tilde{Y}_\nu^\dagger \tilde{H}^\dagger \ell_L$, and $\overline{\chi} = \chi^\dagger \gamma^0$

Let's put $\mathcal{L}_{\mathrm{N}}$ into the path integral:

$$
Z = \int [\mathcal{D}N][\mathcal{D}\overline{N}] \exp i\left(\frac{1}{2}\overline{N}KN - \frac{1}{2}\overline{N}\chi - \frac{1}{2}\overline{\chi}N\right) \tag{4.11}
$$

Noting that this resembles the generating functional for the free Dirac field, we can immediately write:

$$
Z = \exp\left\{-\frac{i}{2}\left(\int \overline{\chi}K^{-1}\chi\right)\right\} \tag{4.12}
$$

up to a normalisation constant. Expanding,

$$
= \exp\left\{-\frac{i}{2}\left(\overline{\ell_L}\tilde{Y}_\nu\tilde{H} + \overline{\ell_L^c}\tilde{Y}_\nu^*\tilde{H}^*\right)\widehat{M}^{-1}\left(\tilde{Y}_\nu{}^T\tilde{H}^T\ell_L^c + \tilde{Y}_\nu^\dagger\tilde{H}^\dagger\ell_L\right)\right\} \tag{4.13}
$$

Noting the inverse mass matrix, the result is that we obtain the seesaw mechanism for neutrino masses, in which the left-handed neutrino obtains a naturally small mass as a result of a heavy right-handed neutrino.

Although we have obtained the seesaw mechanism as desired, the path integral method is long and circuitous. Having accepted that we can build new physics via the introduction of higher-dimension operators without concern for renormalisability, we will discuss a simpler approach for building the dimension-5 Weinberg operator.

4.5 Using the effective Weinberg operators to skip the path integral

In the previous section, we had to do a lengthy and tedious calculation to connect the Effective Theory of the higher field with the Seesaw mechanism. With reference to the extended discussion on Effective Field Theory in Appendix H, our alternative approach is to assume that higher dimensional operators are already in the perturbative series of the full Lagrangian that includes heavier fields. Schematically, the effective Lagrangian looks like:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \frac{\mathcal{L}_{\text{d = 5}}}{\Lambda} + \frac{\mathcal{L}_{\text{d = 6}}}{\Lambda^2} + \cdots \tag{4.14}$$

where Λ represents the scale of this effective theory. In principle, non-renormalisable terms that satisfy Lorentz invariance and CPT invariance are allowed in the Lagrangian. We will show that we can find a dimension-5 term that leads to Majorana neutrino masses, and hence the Seesaw mechanism, which explains the lightness of the left-handed neutrinos.

This dimension-5 operator is called the Weinberg operator, and is built using SM fields. It is designed to produce neutrino mass terms after Spontaneous Symmetry Breaking (SSB).

4.5.1 Building block of Dimension-5 Weinberg Operators

First, consider the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ scalar $(\overline{\psi_L}\tilde{H})$, with the lepton doublet and the conjugated Higgs doublet. After SSB, we obtain:

$$(\overline{\psi_L}\tilde{H}) = \frac{v + H}{\sqrt{2}}\overline{\nu_L} \tag{4.15}$$

in a fashion similar to Eq 3.2. From the discussion on the Majorana masses, we know that Majorana mass terms of the form $m\overline{\psi}\psi^c$ are allowed, and this form implies the violation of lepton number. We can use this scalar to build a term that gives us Majorana masses after SSB.

4.5.2 Constructing the Weinberg operator

We can construct the following Weinberg operator:

$$-\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda}(\overline{\psi_L}\tilde{H})Y'\mathcal{C}(\overline{\psi_L}\tilde{H})^T + \text{h.c.} \tag{4.16}$$

where we have suppressed flavour indices and we identify $(\overline{\psi_L}\tilde{H})$ with the form $\overline{\psi}$ that produces the Majorana mass term. Note that since the Lagrangian is dimension-4, we need a $1/M$ term as a prefactor for the dimension-5 operator. This mass term is provided by Λ, which characterises a high-scale theory where lepton number violation is allowed.

After SSB, we obtain:

$$-\mathcal{L}_{\text{eff}} = \frac{v^2}{2\Lambda}\overline{\nu_L}'Y'\mathcal{C}\overline{\nu_L}'^T + \text{h.c.} \tag{4.17}$$

where we have dropped terms involving the Higgs boson H. We can rotate from the general basis to the flavour basis using the relation $\nu_L = U_L\nu_L'$, where U_L is a unitary matrix, and ν_L is the flavour eigenvector.

$$\begin{aligned}-\mathcal{L}_{\text{eff}} &= \frac{v^2}{2\Lambda}\overline{\nu_L}U_L^\dagger Y'\mathcal{C}(\overline{\nu_L}U_L^\dagger)^T + \text{h.c.} \\ &= \frac{v^2}{2\Lambda}\overline{\nu_L}U_L^\dagger Y'(U_L^\dagger)^T\mathcal{C}\overline{\nu_L}^T\end{aligned}$$

as U and $\mathcal{C}$ are in different vector spaces. We now redefine $Y = U_L^\dagger Y'(U_L^\dagger)^T$

$$-\mathcal{L}_{\text{eff}} = \frac{v^2}{2\Lambda}\overline{\nu_L}Y\mathcal{C}\overline{\nu_L}^T$$

Observe that Y must be a symmetric matrix, and hence we can use the bi-unitary transformation for a complex symmetric matrix to decompose it into the mass basis. We write $Y = U\widehat{y}U^T$ where $\widehat{y}$ represents the diagonal mass eigenvalues. After transforming from the flavour basis to the mass basis:

$$= \frac{v^2}{2\Lambda}\overline{\nu_{\text{mass}}}\widehat{y}\nu_{\text{mass}} \tag{4.18}$$

where ν_{mass} represents the neutrino field in the mass basis, and we have used the Majorana relation $\nu = \nu^c$. Hence, we have obtained the Majorana mass term from the Weinberg operator.

Observe that we can write:

$$m = \frac{v^2}{\Lambda}\widehat{y} = \frac{v}{\Lambda}(vy) \tag{4.19}$$

where $v\widehat{y}$ is the mass of the neutrino in low-scale theory, and the ratio $\frac{v}{\Lambda}$ connects the SM to the high-scale theory.

Hence, this represents a more efficient way of proposing effective operators to introduce into the Lagrangian. Instead of doing the full perturbative computation, we can simply identify operators which must be in the perturbative series anyway. We can then use the machinery of SSB and the Higgs mechanism to generate our desired neutrino mass terms.

4.6 Summary

In this chapter, we have introduced new tools for model building. Previously, we have seen how the Majorana mass term leads to a naturally light left-handed neutrino. After our discussion on Effective Field Theory, we have provided an effective operator that explains where the Majorana mass term comes from in the first place. This is justified as we are identifying terms of a perturbative series, which allows us to skip a lengthy computation – the introduction of a heavy field at a higher energy scale implies that at lower energies, the physics can be described by effective operators.

In the next chapter, we will use the tools we have collected to review other mass mechanisms within the framework of the Standard Model.

Chapter 5

Neutrino masses in extensions of the Standard Model

5.1 Overview

Having discussed the Seesaw mechanism in Chapter 3 and the Effective Field Theory approach in Chapter 4, we will now examine how these tools have been applied to various models, examining extensions of the Standard Model that obtain neutrino masses. First, we will examine tree-level neutrino masses for Majorana neutrinos. Next, we discuss one-loop radiative contributions to neutrino mass for Majorana neutrinos. These examples demonstrate the use of the effective operator as a model building tool, allowing us to introduce new particles and consider the Yukawa interactions to obtain neutrino masses.

5.2 Tree-level Majorana neutrino mass

Tree diagrams are Feynman diagrams without loop integrals - the mass term is manifest in the Lagrangian. In the previous section, we have seen how in the Standard Model the Dirac left-handed neutrino can obtain a tree-level neutrino mass via the Higgs mechanism. We then made the argument that the required Yukawa coupling for a Dirac left-handed neutrino would be unnaturally small, which allows us to consider alternative mass mechanisms, such as those involving Majorana neutrinos.

As Cai et al. write in their review paper [29], Majorana neutrinos have many favourable properties. They constitute the simplest spinorial representation, where a Dirac 4-component spinor is equivalent to two degenerate two-component Majorana spinors.

The primary tool for obtaining tree-level Majorana masses is the Weinberg operator, where we add non-renormalisable terms and new scalar particles that after Spontaneous Symmetry Breaking, turn into neutrino mass terms.

As shown in Chapter 6.7 of Fukugita [30], we can obtain tree-level neutrino mass with the Higgs boson via the dimension-5 Weinberg operator $y\bar{\ell}_L^c \ell_L \frac{\phi\phi}{M}$ with the Feynman diagram shown in Figure 5.1.

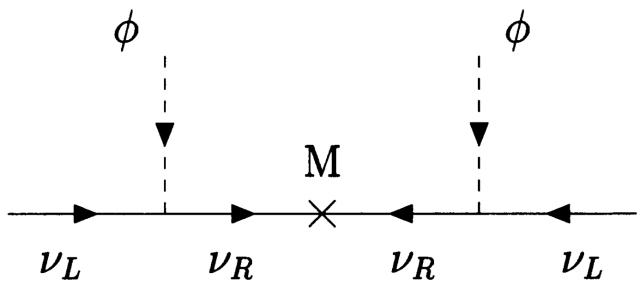


Figure 5.1: Feynman diagram of the dimension-5 Weinberg operator. Diagram extracted from [30]

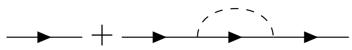
In general, we can construct any Weinberg operator of arbitrary mass dimension as long as the terms obey the Standard Model group transformation rules. The existence of the Weinberg operator is assumed to be a low-energy feature of a higher-energy theory. As Cai et al. write, there are a few options for the masses:

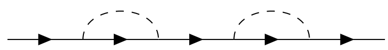
1. Tree-level operators with only new massive fermions and scalars
2. j-loop level using new heavy particles only
3. j-loop level using light SM particles and new heavy particles

For option (1), we will obtain the well-known seesaw models. With the interaction of lepton-Higgs couplings using heavy Majorana singlet leptons, we obtain a type-I seesaw mechanism. With a lepton pair and a Higgs pair of triplet heavy scalar bosons, we obtain a type-II seesaw. With heavy Majorana triplet fermions and Higgs scalars, we obtain a type-III seesaw [24].

5.3 Radiative generation of neutrino mass

Loop effects can give rise to observable effects. Following Schwartz [5], let us consider corrections to the 2-point function for a scalar particle:

$$iG^F =$$  (5.1)

$$+$$  $$+\cdots$$ (5.2)

The two-point function obtains corrections from an infinite number of graphs. We can interpret the previous computation as a mass correction to the neutrino propagator.

$$\begin{aligned} iG^F &= G_0^F + G_0^F(i\Sigma)G_0^F + G_0^F(i\Sigma)G_0^F(i\Sigma)G_0^F + \cdots \\ &= \frac{i}{p^2 - m^2}\left[1 + \left(\frac{-\Sigma}{p^2 - m^2}\right) + \left(\frac{-\Sigma}{p^2 - m^2}\right)^2 + \cdots\right] \\ &= \frac{i}{p^2 - m^2}\frac{1}{1 + \frac{\Sigma}{p^2 - m^2}} \\ &= \frac{i}{p^2 - m^2 + \Sigma} \end{aligned} \tag{5.3}$$

Hence, by computing the one-loop diagram, we obtain a mass term. We will review radiative models for Majorana neutrino mass, paying special attention to Ernest Ma's scotogenic model [31] as this is a template for radiative models.

5.4 Neutrino mass at one-loop

In this section, we explicitly show the calculation in Ma's scotogenic model [31] as this illustrates the steps required for all calculations involving the radiative generation of neutrino mass. To make this explicit, we follow the discussion found in [32].

Let us consider the Feynman diagram in Figure 5.2 before SSB, which generates neutrino masses by introducing the scalar doublet (η^+, η^0), along with the right-handed neutrino N_k, and demanding that the left-handed neutrino is a Majorana neutrino. After SSB, we require the charged scalar to vanish to preserve $U(1)_{EM}$ symmetry.

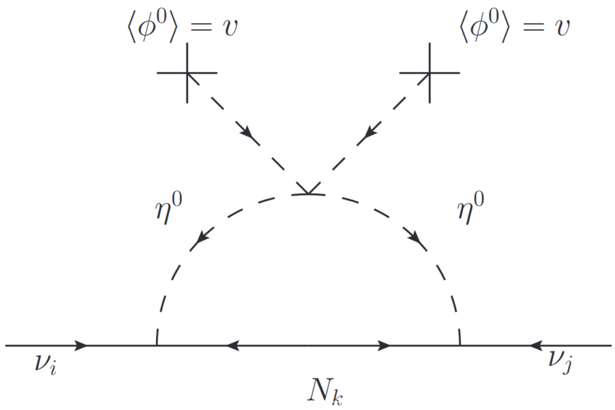


Figure 5.2: Feynman diagram for Ma's one-loop model after SSB, extracted from [32]

The loop function before SSB is:

$$-i\Sigma_{ij}^{(\nu)} = \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{(p-k)^2 - m_\eta^2}\right)^2 h_{ik} \left(\frac{\not{k} + m_N}{k^2 - m_N^2}\right)_k h_{jk}$$

where i, j, k denote generational indices, and h is a Yukawa coupling constant

By naive power counting, we see that this diagram is superficially convergent. After EWSB, the Higgs field ϕ^0 obtains a vev, and the Higgs particles drop out of the Feynman diagram, and so does the quartic interaction vertex. We can simplify the Feynman diagram into Figure 5.3.

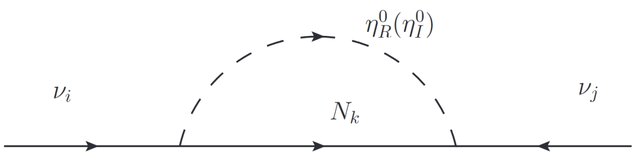


Figure 5.3: Simplified one-loop Feynman diagram, extracted from [32]

By power counting, although the integral in Eq 5.4 is logarithmically divergent after SSB, it was convergent before SSB, and SSB cannot physically create new divergences. We proceed with the calculation for the one-loop correction to the neutrino self-energy Σ_i where i indexes the heavy right-handed neutrino.

The trick is to represent the neutral component $\eta^0 = \eta^0_R + \eta^0_I$ by its real and imaginary components.

For an external momentum p:

$$-i\Sigma^{(\nu)}_{ij} = \int \frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2 - m^2_\eta} h_{ia}\left(\frac{\not{k} + M_{N_k}}{k^2 - m^2_{N_k}}\right)\delta_{ab}h_{bj}$$

Note that the results must be the same for zero external momentum, so we can set $p = 0$ to simplify the calculation. Also, since the 4-dimensional momentum integral can be converted into a spherical integral, the linear term $\not{k}$ must vanish due to spherical symmetry. Contracting the Yukawa matrices, we obtain:

$$-i\Sigma^{(\nu)}_{ij} = h_{ij}M_{N_k}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2_\eta}\left(\frac{1}{k^2 - M^2_{N_k}}\right)$$

To do this integral explicitly, we would typically use the Feynman parameterisation trick, and then perform a Wick rotation to evaluate the integral. Alternatively, the scalar one-loop integral has already been calculated by t'Hooft and Veltman in 1979 [33]. We will use their results and skip the lengthy calculation. Their calculation goes as:

$$\begin{aligned} B(k, m_1, m_2) &= \int d^n k \frac{1}{(k^2 - m_1^2)(k^2 - m_2^2)} \\ &= \Delta - i\pi^2 \int_0^1 dx \ln(-k^2x^2 + x(k^2 + m_2^2 - m_1^2 + m_1^2) \end{aligned}$$

where Δ is a term that includes the $1/\epsilon$ divergence that arises from dimensional regularisation. Note that m_1 and m_2 are interchangeable as we can make the substitution $x' = 1 - x$ in the integral. In our case, $k^2 = 0$ which simplifies this to:

$$B(0, m_1, m_2) = \Delta - i\pi^2\left(\frac{m_1^2}{m_2^2 - m_1^2}\ln\frac{m_2^2}{m_1^2} + \ln m_2^2 - 1\right) \tag{5.4}$$

where

$$\Delta = \frac{2}{\epsilon} - \ln(\pi) + \gamma_E$$

and γ_E is the Euler-Mascheroni constant, with $\epsilon = 4 - d$ the divergent part as $\epsilon \to 0$.

Conveniently, the divergent part Δ is independent of the parameters m_1, m_2. Since we are computing probability amplitudes, the factor of i in the second term squares into a minus sign. Subtracting the integrals, the infinite divergences cancel, and we obtain:

$$\begin{aligned} I_R - I_I &= h_{il}h_{lj}M_{N_k}\frac{i}{16\pi^2}[B_0(0, m_{\eta_R}^2, M_N^2) - B_0(0, m_{\eta_I}^2, M_N^2)] \\ &= i\frac{h_{ij}}{16\pi^2}M_{N_k}\left[\frac{m_{\eta_R}^2}{M_{N_k}^2 - m_{\eta_R}^2}\ln\frac{m_{\eta_R}^2}{M_{N_k}^2} - \frac{m_{\eta_I}^2}{M_{N_k}^2 - m_{\eta_I}^2}\ln\frac{m_{\eta_I}^2}{M_{N_k}^2}\right] \end{aligned} \tag{5.5}$$

Hence, by considering the one-loop contribution, we can obtain a contribution to the neutrino mass. This calculation is the prototype for many similar calculations in the literature [29].

5.5 Majorana neutrinos at more than one-loop

The six possible topologies of the Weinberg operator are also enumerated in [29]. These models are mostly similar to the Ma-model as well, with differences lying in particle content, newly introduced symmetry, and choice of charge assignments.

5.6 Summary

In this chapter, we have seen how the Weinberg operator was used to propose tree-level Majorana masses. We have also seen an example of a calculation for radiative neutrino masses that are generated at one-loop. Hence, we have demonstrated how the effective operators can be used. In the next chapter, we will turn to the question of the neutrino mixing matrices, and discuss the model-building techniques found there.

Chapter 6

Neutrino mixing in the Standard Model and extensions

6.1 Introduction

We have previously introduced the neutrino mixing matrix – the PMNS matrix – in Section 3.2. In this chapter, we will provide more detail on the neutrino mixing matrix. We have previously identified this as the PMNS matrix in Chapter 3. We first draw a parallel between the PMNS matrix and the Cabibbo-Kobayashi-Masakawa matrix in Section 6.2, which is close to identity and represents small mixing angles. In Section 6.4 we are interested in how the PMNS matrix is parameterised into 3 angles and 1 global phase. We then review why discrete symmetry groups are of interest in Section 6.5, and subsequently provide a concrete example of model building in Section 6.6. These tools for building the neutrino mixing matrix will be used again in Section 9.6.

6.2 Drawing an analogy to the quark Cabibbo-Kobayashi-Masakawa matrix

To understand the neutrino mixing matrix, we can first draw a connection with the quark mixing matrix, which is the Cabibbo-Kobayashi-Masakawa (CKM) matrix:

$$V_{CKM} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} 0.97373 & 0.2243 & 0.00382 \\ 0.221 & 0.975 & 0.0408 \\ 0.0086 & 0.0415 & 1.014 \end{pmatrix} \tag{6.1}$$

where we have taken the best-fit values from [34]. We reference Schwartz [5] and Griffiths [12] for this discussion.

The quarks obtain a Dirac mass from Yukawa terms in the Lagrangian. After SSB, the quark mass terms become:

$$-\mathcal{L}_{\text{mass}} = \frac{v}{\sqrt{2}}[\overline{d}_L Y_d d_R + \overline{u}_L Y_u u_R] + \text{h.c.} \tag{6.2}$$

where Y_u and Y_d are Yukawa coupling terms for the up- and down-quarks respectively. To diagonalise both of these matrices, we will use the bi-unitary transformation:

$$Y_d = U_d M_d K_d^\dagger, \qquad Y_u = U_u M_u K_u^\dagger \tag{6.3}$$

for unitary matrices U_d, U_u, K_u, K_d. It turns out that we can absorb K_u and K_d via field redefinitions such that the mixing effects are only given by the matrix:

$$V = U_u^\dagger U_d \tag{6.4}$$

which is the CKM matrix as previously identified. For ease of interpretation, we typically assume that U_u is the identity matrix, and all rotations are effectively carried by the down-quark mixing matrix. Hence, the up-quarks are assumed to already be in the mass basis, while the down-quarks are superpositions of down-quark mass eigenstates.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{6.5}$$

The CKM matrix is close to identity, which means that the mixing angles are small. We see later that the neutrino mixing matrix angles in the PMNS matrix are comparatively larger.

In the next section, we will use the same procedure to see how the PMNS matrix is obtained.

6.3 General model for neutrino mixing

We will build a general model for neutrino mixing, following Bilenky Chapter 6.3 of [24] for this calculation. The leptonic charged current is given by:

$$J^\mu_{CC} = 2 \sum_{i=e,\mu,\tau} \overline{\nu'_{L,i}} \gamma^\mu \ell'_{L,i} \tag{6.6}$$

After diagonalising the charged lepton and neutrino masses where $\ell'_L = U_e \ell_L, \nu'_L = U_\nu \nu_L$:

$$J^\mu_{CC} = 2 \sum_{i=e,\mu,\tau} \overline{\nu_{L,i}} U^\dagger_\nu U_e \gamma^\mu \ell_{L,i} \tag{6.7}$$

$$= 2\overline{\nu_{L,i}} (U^\dagger_e U_\nu)^\dagger_{ij} \gamma^\mu \ell_{L,j} \tag{6.8}$$

More specifically, U_e diagonalises the product $M_e M^\dagger_e$ to get squared masses. Consider the charged lepton mass term

$$-\mathcal{L}_\ell = \overline{\ell_{L,i}} (M_e)_{i,j} \ell_{R,j} + \text{h.c.} \tag{6.9}$$

where $i, j = \{e, \mu, \tau\}$. Hence,

$$U^\dagger_e M_e M^\dagger_e U_e = \text{diag}(m^2_e, m^2_\mu, m^2_\tau) \tag{6.10}$$

Hence, the PMNS neutrino mixing matrix can be described by two contributions [35].

$$U_{PMNS} = U^\dagger_e U_\nu \tag{6.11}$$

where U_e is from the diagonalisation of the charged lepton mass term, and U_ν is from the diagonalisation of the neutrino mass term. U_e is typically assumed to be identity for ease of interpretation, and also because charged lepton flavour violation has not yet been observed [36].

6.4 Standard Parameterisation of the neutrino mixing angles

To parameterise the neutrino mixing matrix for Dirac neutrinos, we will perform three Euler rotations around the mass eigenstates to obtain the flavour eigenstates. Schematically [17],

$$U^D = R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta 12) \tag{6.12}$$

where R_{ij} represents a rotation of axis i to j, parameterised by an angle θ_{ij}. We also introduce, in the second rotation, a complex phase δ. In matrix form, it is written as:

$$U^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\theta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

For Majorana neutrinos, $U^M = U^D S^M$, where S^M is a phase matrix characterised by two Majorana phases:

$$S^M = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1) \tag{6.14}$$

This is the standard parameterisation used by the Particle Data Group [37].

6.5 Neutrino mixing pattern via discrete symmetry groups

There are a few motivations for introducing discrete symmetry groups [35]. Firstly, the PMNS matrix is non-trivial and unlike the CKM matrix, quite far from unity. Secondly, we assume that there is a preference for an underlying explanation for the neutrino mixing angles, as opposed to assuming that it is simply random. Thirdly, introducing additional continuous symmetry groups would not represent a conceptual departure from the Standard Model - we would again rely on a Higgs-like scalar particle to break the continuous symmetry.

6.5.1 Why discrete symmetry groups

As reviewed by Petcov [38], the three neutrino mixing framework includes two large mixing angles $\theta_{12} \simeq 33°, \theta_{23} \simeq 45°$ and a small mixing angle $\theta_{13} \simeq 8.4°$. These values can be explained by extended SM with a discrete flavour symmetry corresponding a high-energy scale. This is because discrete symmetry groups can describe symmetries about large rotation angles. This was relevant as the neutrino PMNS matrix mixing angles were found to be much larger than the quark CKM mixing angles [39], the latter of which are much closer to 0.

Before the Daya Bay data showing that $\theta_{13} \neq 0$, discrete groups such as A_4 could produce the tri-bimaximal mixing matrix, discussed by Harrison et al. in 2002 [40], and looked to be a good approximation of the neutrino mixing matrix.

Other discrete symmetry groups within the literature include S_3, S_4, D_4 [41, 42, 40, 43, 44].

The tri-bimaximal mixing matrix is written as:

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{6.15}$$

which predicts $\theta_{12} = \sin^{-1} \simeq 35.3°$ (hence 'tri-maximal', as the second mass eigenstate comprises of an equal mixture of the 3 lepton flavours), $\theta_{23} = \sin^{-1}(\frac{1}{\sqrt{2}}) \simeq 45°$ (the 'bi-maximal' part, where the third mass eigenstate comprises of an equal mixture of 2 lepton flavours), and $\theta_{13} = 0$.

A complete example of the A_4 implementation is found in the next section.

6.6 The A_4 group and neutrino mixing

In this section, we will illustrate how the A_4 discrete symmetry group may be used to model neutrino mixing, following its application in 2006 by authors Altarelli and Feruglio [45]. We refer to this as the AF model.

6.6.1 Group representation

All possible permutations among N objects $x_i, i = 1, \cdots, N$ form a group S_N with order $N!$, which is known as the symmetric group. All even permutations among S_N form the alternating group A_4 with order $N!/2$.

The A_4 group is the symmetry of a tetrahedron, with order 12. We choose an algebraic definition such that the closed algebra of the generators $S = (14)(23)$ and $T = (123)$ (in cycle notation) is defined as the A_4 group, and $S^2 = T^3 = (ST)^3 = I$. The conjugacy classes are written as:

$$\begin{aligned}
&C_1 : I, \quad (h = 1) \\
&C_2 : T, ST, TS, STS, \quad (h = 3) \\
&C_3 : T^2, ST^2, T^2S, TST, \quad (h = 3) \\
&C_4 : S, TST^2, T^2ST, \quad (h = 2)
\end{aligned} \tag{6.16}$$

From the orthogonality relations, we can deduce the character table, where $\omega = e^{i2\pi/3}$

Class	C_1	C_2	C_3	C_4
χ^1	1	1	1	1
$\chi^{1'}$	1	ω	ω^2	1
$\chi^{1''}$	1	ω^2	ω	1
χ^3	3	0	0	-1

Table 6.1: A_4 character table

The three-dimensional unitary representations will be relevant for Lagrangian building. By representing the generators, this will allow us to obtain the product rules and hence choose an A_4 invariant Lagrangian. We choose a basis where T is diagonal, writing:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{6.17}$$

The full calculation is presented in Appendix E. We consider the tensor product of the two triplets $a \otimes b, a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$. When we enforce invariance under S and T, we obtain the following singlets:

$$\begin{aligned} 1 &\equiv (ab) = (a_1b_1 + a_2b_3 + a_3b_2) \\ 1' &\equiv (ab)' = (a_1b_2 + a_2b_1 + a_3b_3) \\ 1'' &\equiv (ab)'' = (a_1b_3 + a_2b_2 + a_3b_1) \end{aligned} \tag{6.18}$$

This tensor product also allows us to build two new triplets - one symmetric and one anti-symmetric.

$$\begin{aligned} 3_S &\equiv (ab)_S = \frac{1}{3}(2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_1b_3 - a_3b_1) \\ 3_A &\equiv (ab)_A = \frac{1}{2}(a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1) \end{aligned} \tag{6.19}$$

The product rules can be summarised as such [46]:

$$1 \otimes 1 = 1, \quad 1' \otimes 1'' = 1, \quad 1' \otimes 1' = 1'' \tag{6.20}$$

$$1^{(')('')} \otimes 3 = 3, \tag{6.21}$$

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A \tag{6.22}$$

Lepton	$SU(2)_L$	A_4
ℓ	2	3
e^c	1	1
μ^c	1	1”
τ^c	1	1’
Scalar		
h_u	2	1
h_d	2	1
φ	1	3
φ'	1	3
ξ	1	1

Table 6.2: A_4 character assignments

6.6.2 Lagrangian Building

We now possess the required tools to build Yukawa couplings in the Lagrangian. We now assign A_4 characters to different fields, as shown in the Table 6.2. In the AF model, we have two SM Higgs doublets (singlets under A_4). To break the flavour symmetry, we also have two A_4 triplets φ and φ', and A_4 singlet ξ, making for a total of 3 flavon particles. We assume that the flavon energy scale is significantly higher than the electroweak scale as it represents new physics.

This allows us to build the following Yukawa couplings using Weinberg operators. To show the A_4 structure, we deliberately leave out Higgs doublets and factors of $1/\Lambda$ for clarity. We also leave out writing adjoints as $SU(2)_L$ and Lorentz invariance are assumed.

$$-\mathcal{L}_{\mathrm{Y}} = y_e e^c(\varphi\ell) + y_\mu \mu^c(\varphi\ell)' + y_\tau \tau^c(\varphi\ell)'' + x_a\xi(\ell\ell)x_b(\varphi'\ell\ell) \tag{6.23}$$

where we can choose different singlet representations of each triplet tensor product such that the overall term transforms as the A_4 trivial singlet, achieving A_4 invariance. We can do this as we have previously worked out the product decomposition for the triplet-triplet tensor product.

For instance, $(\varphi\ell) \sim 1$, $(\varphi\ell)' \sim 1'$, and $(\varphi\ell)'' \sim 1''$. Written explicitly,

$$(\varphi\ell) = \varphi_1\ell_1 + \varphi_2\ell_3 + \varphi_3\ell_2$$

$$(\varphi\ell)' = \varphi_1\ell_2 + \varphi_2\ell_1 + \varphi_3\ell_3$$
$$(\varphi\ell)'' = \varphi_1\ell_3 + \varphi_2\ell_2 + \varphi_3\ell_1$$
$$(\ell\ell) = \ell_1\ell_1 + \ell_2\ell_3 + \ell_3\ell_2$$

The $(\varphi'\ell\ell)$ term is a tensor product of 3 triplets. Following the product decomposition rules, we can represent the $\ell\ell$ product as the symmetric triplet 3_S, leading to the term $(3 \otimes 3_S) \sim 1$.

Upon symmetry breaking at the flavon energy scale (the scalar particle associated with the discrete symmetry), we assign the VEVs of the fields to be:

$$\langle\xi\rangle = v_3, \quad \langle\varphi\rangle = (v_1, 0, 0), \quad \langle\varphi'\rangle = (v_2, v_2, v_2) \tag{6.24}$$

$$h_u = \begin{pmatrix} v_u \\ 0 \end{pmatrix}, \quad h_d = \begin{pmatrix} 0 \\ v_d \end{pmatrix} \tag{6.25}$$

Now including explicitly the previously ignored Higgs doublets, the Lagrangian after symmetry breaking becomes:

$$\begin{aligned} -\mathcal{L}_{\mathrm{Y}} = & \frac{v_d v_1}{\Lambda}\left(y_e e^c e + y_\mu \mu^c \mu + y_\tau \tau^c \tau\right) + \frac{x_a v_3 v_u^2}{\Lambda^2}(\nu_1\nu_1 + \nu_2\nu_3 + \nu_3\nu_2) \\ & + \frac{x_b v_2 v_u^2}{\Lambda^2}\frac{1}{3}\nu_i \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}_{ij} \nu_j \end{aligned} \tag{6.26}$$

The charged lepton mass matrix is manifestly

$$M_l = \frac{v_d v_1}{\Lambda}\begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \tag{6.27}$$

and the fermion masses can be read off directly. Defining $a = 2x_a v_3/\Lambda, d = 2x_b v'/\Lambda$ to simplify the presentation, the symmetric neutrino mass matrix is then

$$M_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} a + \frac{2d}{3} & -\frac{d}{3} & -\frac{d}{3} \\ \cdot & \frac{2d}{3} & a - \frac{d}{3} \\ \cdot & \cdot & \frac{2d}{3} \end{pmatrix} \tag{6.28}$$

We perform a change of basis to rotate from the flavour basis to the mass basis. Miraculously,

$$U_{\mathrm{TBM}}^T M_\nu U_{\mathrm{TBM}} = \frac{v_u^2}{\Lambda}\mathrm{diag}(a + d, a, -a + d) \tag{6.29}$$

which implements the tri-bimaximal neutrino mixing pattern for us. Here, the masses are complex and the Majorana phases are still attached. This also leads to the sum rule $2m_2 + m_3 = m_1$. These are hypothetical constraints relating to neutrino masses which can be experimentally falsified.

6.6.3 Challenges with TBM perspective

Given that the TBM is no longer an exact prediction of the neutrino mixing matrix, there are two options: (1) abandon the TBM, or (2) rescue it through ad-hoc arguments.

The latter option is briefly discussed in Section 6.5 of Bilenky [24]. The PMNS mixing matrix is given by $U_{PMNS} = U_l^\dagger U_\nu$, and the leptonic mixing matrix U_l is typically assumed to be the identity matrix in many models. To explain a non-zero θ_{13} angle, we include another rotation matrix parameterised by α. For example,

$$U = U_{TBM} R_{23}(\alpha) \tag{6.30}$$

where the matrix $R_{23}(\alpha)$ has the form

$$R_{23}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{pmatrix} \tag{6.31}$$

where α is fitted to the observed value of θ_{13}. The introduction of this rotation matrix adds another constraint on different terms and produces a prediction in the neutrino mixing matrix, which are called 'sum rules'. For this specific set-up,

$$\sin^2\theta_{23} = \frac{1}{2} - \sqrt{2}\sin\theta_{13}\cos\delta + \mathcal{O}(\sin^2\theta_{13}) \tag{6.32}$$

$$\sin^2\theta_{12} = \frac{1}{3} - \frac{2}{3}\sin^2\theta_{13} + \mathcal{O}(\sin^4\theta_{13}) \tag{6.33}$$

which function as falsifiable predictions that can be ruled out via experiment.

6.7 Froggatt-Nielsen Mechanism

In this section, we introduce the Froggatt-Nielsen mechanism, which attempts to explain apparent mass hierarchies. Looking at the relationships between quark masses,

$$\frac{m_u}{m_c} \simeq \frac{m_c}{m_t} \simeq \lambda^4$$
$$\frac{m_d}{m_s} \simeq \frac{m_s}{m_b} \simeq \lambda^2$$

where $\lambda \equiv \sin\theta_C \approx 0.225$, and θ_C is the Cabbibo angle. These relations hold to some degree of accuracy [47].

Assuming that these relations point to meaningful physics, we can attempt to explain the apparent hierarchy in the quark mass spectra. It was suggested in 1979 by Froggatt and Nielsen [48] to use a $U(1)_X$ symmetry group to assign charges to different fermion generations. The X of the $U(1)_X$ refers to an arbitrary label for the $U(1)$ charge.

This symmetry is broken at some high energy flavour scale Λ_F by a scalar particle ϕ (called a 'flavon') that takes on a non-zero vev.

We can hence construct Weinberg operators with arbitrarily high dimensions. For instance, in analogy to the dimension-5 Weinberg operator $\ell^c \ell H \frac{H}{\Lambda}$, we can build upwards with terms such as $\ell^c \ell H (\frac{\phi}{\Lambda_F})^r$ with r being any integer that would keep the term invariant under a $U(1)$. All we have to do is to assign charges to the different generations, which will couple to powers of ϕ/Λ. This is implemented in the literature in a paper by Altarelli and Feruglio in using the A_4 discrete symmetry group to model neutrino mass mixing [45].

One might observe that the charged leptons do not have a similar hierarchical relationship, where $\frac{m_e}{m_\mu} \approx 10^{-3}, \frac{m_\mu}{m_\tau} \approx 10^{-2}$. Furthermore, neutrino oscillation experiments [49, 50, 51] allow for either Normal Ordering (NO) or Inverted Ordering (IO), neither of which feature a strong hierarchy. This suggests that while the Froggatt-Nielsen mechanism can be used as a model-building tool, it may not always be well-motivated.

To conclude, while such a mechanism is attractive for implementing a mass hierarchy, it remains dubious whether it is needed at all. This is because not all fermions feature a strong mass hierarchy. Furthermore, it is equally dubious whether such a mechanism will produce a falsifiable prediction.

6.8 Summary

In this chapter, we have discussed the construction of the neutrino mixing matrix. The θ_{23} and θ_{12} angles are much larger than those found in the CKM matrix, which motivates attempts to build models that explain the largeness of these angles. We have seen how the discrete symmetry group can be used as a model building tool in the Lagrangian with the specific example of the A_4 group. We have also used effective operators and the Higgs mechanism to put it all together. Lastly, we have looked at the Froggatt-Nielsen mechanism as a way of implementing a mass hierarchy, which again uses Weinberg operators.

At this juncture, we have more or less exhausted the options available within the Standard Model gauge group – variation in models typically have to do with the choice of the discrete symmetry group, the dimension of the effective operator, the type of new hypothetical particles and their charge assignments. This thesis is a progressive treatment and seeks to proceed to higher theories. Having reviewed and provided clear examples of the key model building tools, we are in a good position to study a higher theory.

In the next section, we will study the Left-Right Symmetry Model (LRSM), which has a larger gauge group of $SU(2)_R \otimes SU(2)_L \otimes U(1)_X$ – its main appeal has to do with treating the left- and right-handed fields on equal footing, with gauge coupling unification at a high energy. This is because, within the Standard Model, the right-handed fields e_R, d_R, u_R are assigned as $SU(2)_L$ singlets, which is unsatisfying from a model building perspective as there is no reason why one chirality is privileged over the other.

Chapter 7

Left-Right Symmetric Models

7.1 Introduction

Within the Standard Model, the left- and right-chiral fields are treated differently. The left-handed neutrino and electron are packaged as a doublet under the $SU(2)_L$ gauge group, while the right-handed electron is a singlet under the $SU(2)_L$ gauge group.

The weak interaction is known to not preserve parity, which is the discrete transformation between left- and right-chiral fields as shown by the classic experiments Wu et al. in 1957 [11, 52].

To continue with the unification project, we can consider a new gauge group in which the left- and right-chiral fields are treated on equal footing, where

$$f_L \overset{\mathcal{P}}{\leftrightarrow} f_R \tag{7.1}$$

under the parity transformation, where f_L, f_R are arbitrary fields.

Historically, this was suggested by Pati and Salam in 1974 [53] who introduced the gauge group $SU(2)_L \times SU(2)_R \times SU(4)$. They suggested that quarks carry four colours, red, green, and blue, with the fourth colour representing a lepton. By extending $SU(3)_C$ to $SU(4')$ of the four colours, this unifies the leptons and quarks of the Standard Model.

This idea of expanding the gauge group was further extended by Mohapatra and Senjanović [54, 55], introducing a left-right symmetric model with the gauge group

$SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \times U(1)_{B-L}$, where the hypercharge is associated with the conservation of $B - L$ number in what is known as the left-right symmetric model (LRSM).

It was shown in Mohapatra and Pati in 1975 [56] that in a left-right symmetric model, the renormalised coupling constants g_L and g_R are equal up to finite radiative corrections as long as the symmetry breaking is achieved via mass terms in the Higgs potential. Since all mass differences can be neglected at sufficiently high energy, they argue that parity nonconservation will disappear in this class of models, which represents a pathway to gauge coupling unification.

With the direct product of two $SU(2)$ groups, we will obtain two gauge bosons, W_L and W_R, which mediate the weak force.

7.1.1 Particle content

Following [57], the quantum number assignment of the LRSM for the gauge groups $SU(2)_L$, $SU(2)_R$ and $U(1)_{B-L}$ can be listed as such:

$$\begin{aligned}
\ell_L &= \ell_L(\mathbf{2}, \mathbf{1}, -1) \\
Q_L &= Q_L(\mathbf{2}, \mathbf{1}, \frac{1}{3}) \\
\ell_R &= \ell_R(\mathbf{1}, \mathbf{2}, -1) \\
Q_R &= Q_R(\mathbf{1}, \mathbf{2}, \frac{1}{3})
\end{aligned}$$

where $\ell_{L,R}$ and $Q_{L,R}$ are the lepton and quark doublets:

$$L_{L,R} = \begin{pmatrix} \nu_{L,R} \\ e_{L,R} \end{pmatrix}, \qquad Q_{L,R} = \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \tag{7.2}$$

The gauge bosons are then $\vec{W}_{\mu L}, \vec{W}_{\mu R}$ and B_μ, with gauge couplings g_L, g_R and g' respectively. As mentioned above, we can assume that at high energies, $g_L = g_R = g$.

The covariant derivative can be written as:

$$D_\mu = \partial_\mu - ig(\vec{W}_{\mu L} \cdot \vec{T}_{\mu L} + \vec{W}_{\mu R} \cdot \vec{T}_{\mu R}) - ig' B_\mu \frac{Y}{2} \tag{7.3}$$

where $\vec{T}_{\mu L}$ and $\vec{T}_{\mu R}$ are the generators of $SU(2)_L$ and $SU(2)_R$ respectively.

The $U(1)_{em}$ charge is given by the modified Gell-Mann Nishijima formula:

$$Q_{em} = I_{3L} + I_{3R} + \frac{B-L}{2} \tag{7.4}$$

7.1.2 Symmetry breaking and a new scalar potential

While the fermion and gauge content of the LRSM are familiar, we will introduce a new scalar potential to break the LRSM gauge symmetry into the Standard Model symmetry, and then into $U(1)_{em}$ symmetry at low energies. In the Standard Model, a scalar Higgs doublet is used to break $SU(2)_L \otimes U(1)_Y$ into $U(1)_{em}$. Since we have a different gauge structure, we need a Higgs particle with gauge assignments such that it can interact with both left- and right-handed particles.

We choose to introduce a Higgs bi-doublet Φ and the $SU(2)_{L,R}$ triplets $\Delta_{L,R}$:

$$\Phi = \begin{bmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & -\phi_2^{0*} \end{bmatrix}, \qquad \Delta_{L,R} = \begin{bmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{bmatrix}_{L,R} \tag{7.5}$$

with the quantum number assignments:

$$\Phi : (\mathbf{2}, \mathbf{2}, 0), \qquad \Delta_L : (\mathbf{3}, \mathbf{1}, 2), \qquad \Delta_R : (\mathbf{1}, \mathbf{3}, 2) \tag{7.6}$$

In the first symmetry breaking, the triplet scalars Δ_L and Δ_R break left-right symmetry into the Standard Model:

$$SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SU(2)_L \times U(1)_Y$$

The two columns of the bi-doublet Φ represent doublets of $SU(2)_L$, while the two rows form two doublets of $SU(2)_R$.

After the second symmetry breaking in which $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is broken, the bi-doublet acts similarly to the SM Higgs mechanism, providing masses for the quarks and charged leptons.

The full scalar potential contains 17 gauge invariant terms. Some components of Φ and $\Delta_{L,R}$ obtain non-zero VEVs, while others do not. The desired VEV alignment is [58]:

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \kappa_1 e^{i\theta_2} & 0 \\ 0 & \kappa_2 e^{i\theta_2}, \end{pmatrix}, \langle\Delta_L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ v_L e^{i\theta_L} & 0 \end{pmatrix}, \langle\Delta_R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ v_R e^{i\theta_R} & 0 \end{pmatrix} \tag{7.7}$$

Two of the phases, which we choose as θ_1 and θ_R, can be removed by a $SU(2)$ gauge transformation. After a re-definition, we obtain:

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2 e^{i\theta_2},\end{pmatrix}, \langle\Delta_L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ v_L e^{i\theta_L} & 0\end{pmatrix}, \langle\Delta_R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ v_R & 0\end{pmatrix} \tag{7.8}$$

We require that the vevs obey the hierarchy $v_L \ll \kappa_{1,2} \ll v_R$ to agree with phenomenology, namely tiny neutrino masses and heavy right-handed gauge boson masses. Also, v_L is allowed to vanish, similar to how neutrino masses are assumed to be 0 at the Standard Model level, and then dynamically generated via the Seesaw mechanism.

For arbitrary parameters of the scalar potential, we may not necessarily obtain the desired vev alignment. A comprehensive study of parameter space is provided by other authors [59, 60].

7.1.3 Gauge transformation rules of the new scalar particles

The new particles have the following gauge transformation rules under the LRSM [59]:

$$SU(2)_L \otimes SU(2)_R : \quad \Phi \to U_L \Phi U_R^\dagger, \quad \Delta_L \to U_L \Delta_L U_L^\dagger, \quad \Delta_R \to U_R \Delta_R U_R^\dagger, \tag{7.9}$$

$$U(1)_{B-L} : \quad \Phi \to \Phi, \Delta_L \to e^{i\theta_{B-L}}\Delta_L, \Delta_R \to e^{i\theta_{B-L}}\Delta_R \tag{7.10}$$

where $U_L \in SU(2)_L, U_R \in SU(2)_R$ represent transformation matrices under their respective gauge group. We also define the conjugate bi-doublet:

$$\tilde{\Phi} = \sigma_2 \Phi^* \sigma_2 \tag{7.11}$$

$$= \begin{bmatrix} -\phi_2^0 & \phi_1^{*+} \\ \phi_2^{*-} & -\phi_1^{0*} \end{bmatrix} \tag{7.12}$$

Under the parity transformation, the scalars transform as such:

$$\mathcal{P} : \quad \Phi \to \Phi^\dagger, \quad \Delta_L \leftrightarrow \Delta_R \tag{7.13}$$

7.1.4 Gauge boson mass spectrum

In the electroweak symmetry breaking scheme, expanding the kinetic term involving the Higgs doublet $(D_\mu\phi)^\dagger(D^\mu\phi)$ leads to the neutral gauge boson masses after a matrix diagonalisation. We expect that a similar procedure will be required in the Left-Right symmetric case.

A full derivation of the gauge boson masses can be found in a 2015 paper from Kokado and Saito [61]. The charged boson masses are given as:

$$
\begin{aligned}
M_{W_L}^2 &\simeq g_L^2 \frac{\kappa_1^2 + \kappa_2^2}{4} \\
M_{W_R}^2 &\simeq \frac{1}{2} g_R^2 v_R^2
\end{aligned}
\tag{7.14}
$$

where the approximation comes from ignoring terms of order v^2/v_R which is small by construction. This means that W_R is much heavier than the electroweak scale and hence out of experimental reach, while W_L is light and at the electroweak scale, which is compatible with phenomenology.

7.2 Vacuum expectation value Seesaw relation

A seesaw relation of vevs in the LRSM [62] can be derived, and is discussed in Appendix G.2.1. We will simply state the result:

$$
v_L \propto \frac{v^2}{v_R} \tag{7.15}
$$

Hence, large v_R implies that v_L will be suppressed, corresponding to tiny neutrino masses. This seesaw mechanism exists due to the properties of the scalar potential. In contrast, the Standard Model seesaw mechanism works as Dirac and Majorana neutrino mass terms are present in the Lagrangian.

Hence, this seesaw relation operates on the level of the vevs. We still need to examine what happens to the Yukawa sector after spontaneous symmetry breaking, and we do so in the next section.

7.3 Generation of neutrino mass

In this section, we will walk through the two symmetry breaking steps, showing how fermion masses are obtained within the LRSM.

7.3.1 First Symmetry Breaking for lepton masses

The Yukawa couplings for the lepton doublet are shown below:

$$\mathcal{L}_{\Phi}^{L} = \overline{\ell_L}(y_1\Phi + y_2\Phi^c)\ell_R + \text{h.c.}$$
$$\mathcal{L}_{\Delta}^{L} = -\frac{1}{2}y_3\overline{\ell_L^c}\Delta_L\ell_L - \frac{1}{2}y_4\overline{\ell_R^c}\Delta_R\ell_R + \text{h.c.}$$

We will work on the $\mathcal{L}_{\Delta}^{L}$ term in the second line. We first consider how the left-handed lepton doublet transforms under a $SU(2)_L$ symmetry to deduce the explicit form of the conjugated lepton doublet ℓ_L^c. Let us consider an infinitesimal $SU(2)_L$ transformation:

$$\begin{aligned}
\ell_L &= \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
\ell_L &\to U\ell_L = \ell_L' \\
&= e^{i\vec{\sigma}\cdot\vec{\theta}}\ell_L \\
&\approx \left[1 + \begin{pmatrix} i\theta_3 & i\theta_1 + \theta_2 \\ i\theta_1 - \theta_2 & -i\theta_3 \end{pmatrix}\right]\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad (7.16)
\end{aligned}$$

$$(7.17)$$

Working with each term separately,

$$\left.\begin{aligned}
\nu_L' &= (1 + i\theta_3)\nu_L + (i\theta_1 + \theta_2)e_L \\
e_L' &= (i\theta_1 - \theta_2)\nu_L + (1 - i\theta_3)e_L
\end{aligned}\right\}$$

We know that $i\sigma_2\psi^*_{(\nu,e),L} = \psi^c_{(\nu,e),L}$, so we take the complex conjugate:

$$\left.\begin{aligned}
\nu_L^{*'} &= (1 - i\theta_3)\nu_L^* + (-i\theta_1 + \theta_2)e_L^* \\
e_L^{*'} &= (-i\theta_1 - \theta_2)\nu_L^* + (1 + i\theta_3)e_L^*
\end{aligned}\right\}$$

$$\left.\begin{aligned}
i\sigma_2\nu_L^{*'} &= (1 - i\theta_3)i\sigma_2\nu_L^* + (-i\theta_1 + \theta_2)i\sigma_2 e_L^* \\
i\sigma_2 e_L^{*'} &= (-i\theta_1 - \theta_2)i\sigma_2\nu_L^* + (1 + i\theta_3)i\sigma_2 e_L^*
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\nu_L^{c'} &= (1 - i\theta_3)\nu_L^c + (i\theta_1 - \theta_2)(-e_L^c) \\
-e_L^{c'} &= (i\theta_1 + \theta_2)\nu_L^c + (1 + i\theta_3)(-e_L^c)
\end{aligned}\right\} \qquad (7.18)$$

Comparing, we see that $-e_L^c$ transforms like ν_L, and ν_L^c transforms like e_L. This gives us:

$$\ell_L^c = \begin{pmatrix} -e_L^c \\ \nu_L^c \end{pmatrix} \qquad (7.19)$$

$$\overline{\ell_L^c} = \begin{pmatrix} -\overline{e_L^c} & \overline{\nu_L^c} \end{pmatrix} \tag{7.20}$$

which will be used in the following calculations. Hence,

$$\begin{aligned}
-\mathcal{L}_\Delta^L &= \frac{1}{2} y_3 \overline{\ell_L^c} \Delta_L \ell_L + (L \leftrightarrow R) + \text{h.c.} \\
&= \frac{y_3}{2}\left(\Delta_L^0 \overline{\nu_L^c}\nu_L - \frac{\Delta_L^+}{\sqrt{2}}\overline{e_L^c}\nu_L - \frac{\Delta_L^+}{\sqrt{2}}\overline{\nu_L^c} e_L - \Delta_L^{++}\overline{e_L^c} e_L\right) + (L \leftrightarrow R) + \text{h.c.}
\end{aligned}$$

We allow the neutrinos to be of Majorana type, such that $\nu^c = \nu$

$$= \frac{y_3}{2}\left(\Delta_L^0 \overline{\nu_L}\nu_L - \frac{\Delta_L^+}{\sqrt{2}}\overline{e_L^c}\nu_L - \frac{\Delta_L^+}{\sqrt{2}}\overline{\nu_L} e_L - \Delta_L^{++}\overline{e_L^c} e_L\right) + (L \leftrightarrow R) + \text{h.c.}$$

For the first symmetry breaking, we impose the vevs $\Delta_L^0 = v_L e^{i\theta_L}, \Delta_R^0 = v_R$. The complex phase of v_L can be absorbed into the Yukawa coupling constants. This gives us a Majorana mass term. We need not worry about the charged scalar terms, one of which includes e_L^c, as the charged scalar terms must vanish after the second symmetry breaking, when $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is broken into $U(1)_{em}$ to preserve $U(1)_{em}$ symmetry.

Hence, the Majorana terms involving the left- and right-handed neutrinos $\frac{y_3}{2}\Delta_L^0\overline{\nu_L}\nu_L + (L \leftrightarrow R)$ is the only relevant term here. We will see in the later section how this can combine with a Dirac mass term, which leads immediately to the Seesaw mechanism.

7.3.2 Second Symmetry Breaking for quark and charged lepton masses

Quark and charged lepton masses are generated by the existing Higgs doublet via Yukawa couplings in the second symmetry breaking. For the quark sector:

$$\mathcal{L}_\Phi^Q = \overline{q_L}(y_1^Q \Phi + y_2^Q \Phi^c) q_R + \text{h.c.} \tag{7.21}$$

which obeys the $SU(2)_L \times SU(2)_R$ gauge transformation rules.

The bi-doublets allow the left- and right-handed lepton doublets to form the terms:

$$-\mathcal{L}_\Phi^L = -\overline{\ell_L}(y_1 \Phi + y_2 \Phi^c)\ell_R + \text{h.c.} \tag{7.22}$$

$$-\mathcal{L}_\Delta^L = \frac{1}{2}\frac{M_L}{v_L}\overline{\ell_L^c}\Delta_L \ell_L + \frac{1}{2}\frac{M_R^*}{v_R}\overline{\ell_R^c}\Delta_R \ell_R + \text{h.c.} \tag{7.23}$$

Note that the $\mathcal{L}_\Phi^L$ and $\mathcal{L}_\Phi^Q$ terms represent Dirac particles, while the $\mathcal{L}_\Delta^L$ term represents the Majorana neutrinos

Let us now consider the vevs of the Φ bi-doublet, and show the anticipated Seesaw mechanism. We will follow Kokado and Saito in this section [61].

$$-\mathcal{L}_\Phi^L = -\overline{\ell_L}(y_1\Phi + y_2\Phi^c)\ell_R + \text{h.c.} \tag{7.24}$$

where the bi-doublet and its conjugate are:

$$\begin{aligned} \Phi &= \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix} \qquad \Phi^c = \begin{pmatrix} -\phi_2^0 & \phi_2^- \\ \phi_1^+ & -\phi_1^0 \end{pmatrix} \\ \Phi &\to \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 e^{i\theta_2} \end{pmatrix} \qquad \Phi^c \to \begin{pmatrix} -\kappa_2 e^{-i\theta_2} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{7.25}$$

Implementing the vev and absorbing phases into the Yukawa couplings:

$$-\mathcal{L}_\Phi^L = \overline{\nu_L}(y_1\kappa_1 + y_2\kappa_2)\nu_R + \overline{e_L}(y_2\kappa_1 + y_1\kappa_2)e_R$$

At this juncture, we may remark that the Dirac electron mass term is about 10^6 times that of the Dirac neutrino mass term, which requires fine-tuning in y_1, y_2. However, this is not the final mass relationship as we still have to include the Majorana neutrino masses and implement the seesaw mechanism. Hence, the electron mass is determined by the electroweak scale, while the neutrino masses is determined by the Seesaw mechanism – no such fine-tuning is required. For convenience, define:

$$y_1\kappa_1 + y_2\kappa_2 = m_D, \quad y_2\kappa_1 + y_1\kappa_2 = m_D', \quad y_3 v_L = m_L, \quad y_4 v_R = m_R$$

The relevant terms of the Lagrangian are:

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2}\overline{\nu_L} m_L \nu_L^c + \overline{\nu_L} m_D \nu_R + \frac{1}{2}\overline{\nu_R^c} m_R + +\text{h.c.} \tag{7.26}$$

which we recognise from Section 3.5. As we have previously done the working to diagonalise the mass matrix, we can immediately state the result:

$$m_{\nu_L} \approx m_L - \frac{m_D^2}{m_R}$$

$$m_{\nu_R} \approx m_R$$

which gives us the anticipated seesaw relationship. We also have excess degrees of freedom to recover the mass of the electron as well as the neutrino.

Note that the charged scalars cannot acquire a non-vanishing vacuum expectation value after SSB. This is because $U(1)_{em}$ is the surviving symmetry at the electroweak scale; any finite vev of a charged scalar will violate $U(1)_{em}$ invariance.

Hence, we have shown how the seesaw mechanism can also arise in the LRSM due to the seesaw mechanism of the vevs, and also due to the presence of Majorana and Dirac neutrino masses in the Lagrangian. In the next section, we will consider how extensions of the LRSM try to build models for neutrino masses.

7.4 Proposed methods for generation of neutrino masses using the LRSM

Tree-level and radiative models for neutrino mass using the left-right symmetric gauge group have been proposed by various authors, via the introduction of vector-like quarks and charged leptons [63, 64].

In addition to proposing new fermions, these models typically require discrete symmetry groups to protect the Yukawa sector as well. A survey of the relevant literature is found in [65].

In this section, we will propose methods for the generation of neutrino masses without further modifying the fermion or the scalar sector. To do so, we work out the primitive interaction vertices and demonstrate the construction explicitly.

As previously mentioned, the new scalar particles obey the following gauge transformations:

$$\begin{aligned} SU(2)_L \otimes SU(2)_R: &\quad \Phi \to U_L \Phi U_R^\dagger, \quad \Delta_L \to U_L \Delta_L U_L^\dagger, \quad \Delta_R \to U_R \Delta_R U_R^\dagger, \\ U(1)_{B-L}: &\quad \Phi \to \Phi, \Delta_L \to e^{i\theta_{B-L}} \Delta_L, \Delta_R \to e^{i\theta_{B-L}} \Delta_R \end{aligned}$$

The $B-L$ numbers of Φ and $\Delta_{L,R}$ are 2 and 0 respectively, while the lepton doublet $\ell_{L/R}$ has $B-L$ number of -1.

We will need the transformation of the adjoint conjugate under $U(1)$:

$$\overline{\nu^c} \to \overline{\nu^c} = (\nu^{c\prime})^\dagger \gamma^0 \tag{7.27}$$

$$= (\mathcal{C}\overline{\nu'}^T)^\dagger \gamma^0 \tag{7.28}$$

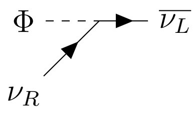
$$= \overline{\nu'}^{*}\mathcal{C}^{\dagger}\gamma^{0} \tag{7.29}$$

The field ν transforms as $\nu \to e^{i\theta_{B-L}}\nu$, and we know that the Dirac adjoint will transform as $\overline{\nu} \to e^{-i\theta_{B-L}}\overline{\nu}$:

$$\overline{\nu^c} = (e^{-i\theta_{B-L}})^{*}(\overline{\nu})^{*}\mathcal{C}^{\dagger}\gamma^{0} \tag{7.30}$$

$$= e^{+i\theta_{B-L}}\overline{\nu^c} \tag{7.31}$$

We can construct the following primitive vertex $\overline{\nu_L}\Phi\nu_R$:



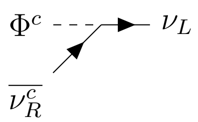
which is invariant under the LRSM gauge transformations:

$$\overline{\nu_L}\Phi\nu_R \to \overline{\nu_L}U_L^{\dagger}U_L\Phi U_R^{\dagger}U_R\nu_R = \overline{\nu_L}\Phi\nu_R \tag{7.32}$$

Moreover, the hypercharge of the bi-doublet Φ is 0; the adjoint ν_L has a hypercharge of $+1$ and ν_R has a hypercharge of -1, so the hypercharges cancel out and the term is invariant under $U(1)_{B-L}$.

The conjugate of the Higgs bi-doublet Φ^c can equivalently transform as $\Phi^c \to U_R\Phi^c U_L^{\dagger}$ under $SU(2)_L \times SU(2)_R$. This is allowed as the scalar potential is constructed with the pattern $\Phi^{\dagger}\Phi$. To see this, we can consider how $\Phi^{\dagger}\Phi$ transforms under infinitesimal transformations of $SU(2)_L \times SU(2)_R$. We will observe that the matrix multiplication structure will ensure that a $\Phi^{c\dagger}\Phi$ is also invariant under $SU(2)_L \times SU(2)_R$ transformations.

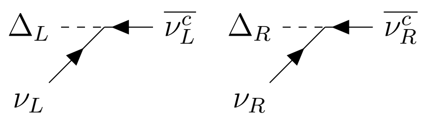
Hence, we can construct the following vertex:



with the following $SU(2)_L \times SU(2)_R$ transformation:

$$\overline{\nu_R^c}\Phi\nu_L \to \overline{\nu_R^c}U_R^{\dagger}U_R\Phi U_L^{\dagger}U_L\nu_L = \overline{\nu_R^c}\Phi\nu_L \tag{7.33}$$

Also, we can construct the following two primitive vertices $\overline{\nu_L^c}\Delta_L\nu_L$ and $\overline{\nu_R^c}\Delta_R\nu_R$.



Under the LRSM gauge transformations:

$$SU(2)_L \times SU(2)_R : \overline{\nu_L^c}\Delta_L\nu_L \rightarrow \overline{\nu_L^c}U_L^\dagger U_L \Delta_L U_L^\dagger U_L \nu_L = \overline{\nu_L^c}\Delta_L\nu_L \tag{7.34}$$

$$U(1)_{B-L} : \overline{\nu_L^c}\Delta_L\nu_L \rightarrow e^{-i\theta_{B-L}}\overline{\nu_L^c}\Delta_L e^{i\theta_{B-L}}\nu_L = \overline{\nu_L^c}\Delta_L\nu_L \tag{7.35}$$

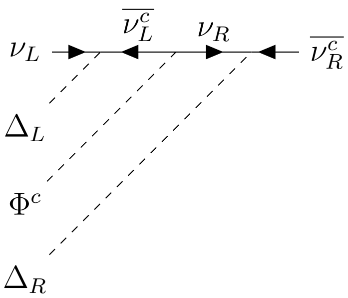
Hence, these primitive vertices are invariant under the LRSM gauge transformations.

7.5 Higher dimensional operators in the LRSM context

In this section, we will consider a dimension-6 Feynman diagram that leads to neutrino mass at tree-level. This is justified by our previous discussion of Effective Field Theory in Chapter 4 and Appendix H.

7.5.1 Dimension-6 tree-level diagram

Consider the following Feynman diagram, with external momenta lines from $v_L, v_R, \Delta_L, \Delta_R, \Phi$, which makes this a dimension-6 operator.



As this is a tree-level diagram, the effective operator in the Lagrangian can be written as $\frac{1}{\Lambda^2}\nu_L\Delta_L\Phi^c\Delta_R\overline{\nu_R^c}$. After the chosen vev alignments, this term unfortunately vanishes:

$$\frac{1}{\Lambda^2}\nu_L\Delta_L\Phi^c\Delta_R\overline{\nu_R^c} \rightarrow \frac{1}{2\Lambda^2}\nu_L \begin{pmatrix} 0 & 0 \\ v_L e^{i\theta_L} & 0 \end{pmatrix} \begin{pmatrix} -\kappa_2 e^{i\theta_2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix} \overline{\nu_R^c} \tag{7.36}$$

$$= 0 \tag{7.37}$$

and does not provide a contribution to the seesaw matrix. That being said, we now have the tools to construct mass terms in the LRSM of arbitrary dimension.

In the next section, we will consider an example of how the neutrino mixing matrix can be explained in the Left-Right Symmetric Model. This will follow the template of Section 6.5.

7.6 Discrete symmetry groups and the mixing matrix

In this section, we will review the use of the D_4 discrete symmetry group in the LRSM as proposed by Bonilla et. al in 2020 [66]. To reproduce the structure of the neutrino mixing matrix as observed in phenomenology, D_4 and Z_2 symmetry groups restrict the scalar potential. Similar to Section 6.6, we assume that the LRSM is a lower energy realisation of a more fundamental theory, possibly $SO(10)$ Grand Unified Theory at an energy scale Λ_F. Below this fundamental energy scale Λ_F, the flavon symmetries are broken by the scalar flavon particles η and χ. Hence, we are using the Higgs mechanism again and also introducing effective Yukawa coupling terms to achieve the desired model and neutrino mixing matrix.

For this model, we will consider the lepton doublets $\ell_{L/R}$, the LRSM scalars Φ, Δ_L and Δ_R, as well as the new flavon scalars η and χ. Their charge assignments are reproduced in Table 7.1. More details of how the D_4 product rules are obtained can be found in Appendix F.

	$\ell_{L_{D(S)}}$	$\ell_{R_{D(S)}}$	Δ_L	Δ_R	Φ	η	χ
$SU(2)_L$	2	1	3	1	2	1	1
$SU(2)_R$	1	2	1	3	2	1	1
$U(1)_{B-L}$	-1	-1	2	2	0	0	0
D_4	$2 \oplus 1$	$2 \oplus 1$	1	1	1	2	1
Z_2	1	1	1	1	-1	-1	-1

Table 7.1: A_4 charge assignments in the model by Altarelli and Feruglio

They assign two generations of ℓ_L and ℓ_R into a D_4 doublet, and then the other generation into a D_4 singlet. Here, a choice is made to package the first and second generations into the doublet i.e. $\ell_{L_D} = (\ell_{L_1}, \ell_{L_2})$ and $\ell_{L_S} = \ell_{L_3}$.

Schematically, the relevant terms of the Lagrangian are shown below:

$$\begin{aligned}\mathcal{L}_Y \supset\, & \overline{\ell}_{L_D}\left(y_1\frac{\chi}{\Lambda_F}\Phi + \tilde{y}_1\frac{\chi}{\Lambda_F}\tilde{\Phi}\right)\ell_{R_D} + \overline{\ell}_{L_D}\left(y_2\frac{\eta}{\Lambda_F}\Phi + \tilde{y}_2\frac{\eta}{\Lambda_F}\tilde{\Phi}\right)\ell_{R_s} \\ & + \overline{\ell}_{L_S}\left(y_3\frac{\eta}{\Lambda_F}\Phi + \tilde{y}_3\frac{\eta}{\Lambda_F}\tilde{\Phi}\right)\ell_{R_D} + \overline{\ell}_{L_S}\left(y_4\frac{\chi}{\Lambda_F}\Phi + \tilde{y}_4\frac{\chi}{\Lambda_F}\tilde{\Phi}\right)\ell_{R_S} \\ & + \frac{Y_{L_1}}{2}\overline{\ell^c_{L_D}}\Delta_L\ell_{L_D} + \frac{Y_{L_2}}{2}\overline{\ell^c_{L_S}}\Delta_L\ell_{L_S} + \frac{Y_{R_1}}{2}\overline{\ell^c_{R_D}}\Delta_R\ell_{R_D} + \frac{Y_{R_2}}{2}\overline{\ell^c_{R_S}}\Delta_L\ell_{R_S} + \text{h.c.}\end{aligned} \tag{7.38}$$

The relevant D_4 product rule is the decomposition of the tensor product of two doublets into four singlets:

$$2 \otimes 2 = \otimes 1 \oplus 1' \oplus 1'' \oplus 1''' \tag{7.39}$$

We assign products of two doublets to give a singlet 1. This allows us to have fermion-scalar-fermion couplings in which the generation mixing is restricted by the D_4 symmetry. It is possible that in general, two doublets can be assigned to $1', 1'', 1'''$, but this will lead to terms of dimension 8 and above, which will be suppressed by large powers of Λ_F and hence can be ignored.

Hence, for the product of two doublets $(x_1, x_2) \sim 2$ and $(y_1, y_2) \sim 2$, we obtain $x_1y_2 + x_2y_1 \sim 1$ as the tensor product.

As a consequence of the restrictions imposed by the D_4 symmetry group, we choose $\langle\chi\rangle = v_\chi$ and the vev alignment $\langle\eta\rangle \sim (v_\eta, 0)$, we obtain the following matrices for the charged leptons:

$$M_L = \frac{1}{\sqrt{2}\Lambda_F}\begin{pmatrix} 0 & (y_1v_2 + \tilde{y}_1v_1)v_\chi & 0 \\ (y_1v_2 + \tilde{y}_1v_1)v_\chi & 0 & (y_2v_2 + \tilde{y}_2v_1)v_\eta \\ 0 & (y_3v_2 + \tilde{y}_3v_1)v_\eta & (y_4v_2 + \tilde{y}_4v_1)v_\chi \end{pmatrix} \tag{7.40}$$

The relevant mass matrices for the neutrinos are as follows:

$$Y_{L(R)} = \begin{pmatrix} 0 & Y_{L_1(R_1)} & 0 \\ Y_{L_1(R_1)} & 0 & 0 \\ 0 & 0 & Y_{L_2(R_2)} \end{pmatrix} \tag{7.41}$$

where $m_L = \sqrt{2}Y_L v_L$ and $m_R = \sqrt{2}Y_R v_R$, and

$$m_D = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & (y_1 v_1 + \tilde{y}_1 v_2)v_\chi & 0 \\ (y_1 v_1 + \tilde{y}_1 v_2)v_\chi & 0 & (y_2 v_1 + \tilde{y}_2 v_2)v_\eta \\ 0 & (y_3 v_1 + \tilde{y}_3 v_2)v_\eta & (y_4 v_1 + \tilde{y}_4 v_2)v_\chi \end{pmatrix} \tag{7.42}$$

After spontaneous symmetry breaking, we can immediately use the seesaw relationship to write:

$$M_\nu^{\text{light}} = m_L - m_D m_R^{-1} m_D^T$$

which leads to a light neutrino via the seesaw mechanism, assuming that $v_R \gg v_L, v_1, v_2$

The neutrino mixing matrix is then obtained by diagonalising a matrix of the form

$$M_\nu = \begin{pmatrix} 0 & a & 0 \\ a & d & b \\ 0 & b & c \end{pmatrix} \tag{7.43}$$

whose allowed parameter space is studied in more detail in [66]. Hence, we have shown explicitly how the same model building tools can be used to construct neutrino mass mixing matrices in a higher theory, as well as to re-obtain the seesaw matrix in the Left-Right Symmetric Model.

7.7 Summary

To summarise, we have discussed a realisation of the left-right symmetric gauge group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, whose Lagrangian is invariant under the parity transformation. This treats the left- and right-chiral fields on equal footing and the gauge coupling constants $g_L = g_R = g$ are unified at sufficiently high energies. This is an appealing perspective as this points to a more fundamental gauge unification at higher energies.

In this chapter, we have discussed the particle content of the LRSM, its symmetry-breaking pathway, and how light neutrino masses are naturally realised in the LRSM.

We have also shown a calculation for a high-dimension tree-level operator, and discussed how the same model building tools for the neutrino mixing matrix can be used to build a model for neutrino mixing in the LRSM.

For the rest of the thesis, we seek to go up the ladder in energy scale and consider Grand Unified Theories. Before we can do this, we want to review Supersymmetry (SUSY) in the next chapter to lay some groundwork, before discussing $SU(5)$ Grand Unified Theory (GUT) and SUSY $SU(5)$ GUT in Chapter 9.

Chapter 8

Supersymmetric Theories

In this chapter, we will briefly review $\mathcal{N} = 1$ supersymmetry (SUSY) by introducing superspace and superfields, symmetry breaking, and the Minimal Supersymmetric Standard Model. $\mathcal{N} = 1$ refers to the number of supersymmetric transformations allowed between fermions and bosons, which we discuss as this theory contains the Standard Model as a special case.

Throughout this chapter, we will be drawing heavily from references [67, 68, 69, 70, 71].

8.1 Why supersymmetry?

Symmetries are present in nature, and symmetry principles are the basis of much of the Standard Model. This is easier to see by considering Noether's Theorem, which states that a conserved quantity is equivalent to a symmetry of the Lagrangian. For example, time-reversal symmetry is equivalent to energy conservation, and space-translation symmetry is equivalent to momentum conservation.

We know that particles in electrodynamics and QFT are representations of the Lorentz group, which is a $SO(3,1)$ symmetry with a space metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{8.1}$$

invariant under Lorentz transformations of the coordinates.

A second type of symmetries is internal symmetries, which can be global symmetries or local gauge symmetries. For example, the $U(1)_{em}$ interaction is a continuous local symmetry whose associated conserved quantity is electric charge.

It may be appealing to combine these two different types of symmetries into a larger symmetry group that would contain both $SO(3,1)$ and the Standard Model gauge groups in another. However, the Coleman-Mandula theorem was a no-go result that killed this project [72].

Theorem 1 (Coleman-Mandula theorem)**.** *The most general bosonic symmetry of scattering amplitudes is a direct product of the Poincare and internal symmetries.*

$$G = G_{Poincare} \times G_{internal}$$

This stated that there is no non-trivial way to combine space-time symmetries.

For completeness, the Poincare algebra is as follows:

$$[M^{\alpha\beta}, M^{\mu\nu}] = i(\eta^{\alpha\mu}M^{\beta\nu} - \eta^{\beta\mu}M^{\alpha\nu}) - i(\eta^{\alpha\nu}M^{\beta\mu} - \eta^{\beta\nu}M^{\alpha\mu}) \tag{8.2}$$

$$[P^{\mu}, M^{\alpha\beta}] = i(\eta^{\mu\beta}P^{\alpha} - \eta^{\mu\alpha}P^{\beta}) \tag{8.3}$$

$$[P^{\mu}, P^{\nu}] = 0 \tag{8.4}$$

where P^{μ} generates spacetime translations, and $M^{\mu\nu}$ generates the Lorentz transformation.

Fortunately, there is a loophole thanks to Haag, Lopuskanski and Sohnius [73], who considered the case of fermionic symmetry generators instead of only bosonic symmetry generators. This was done so by extending the Poincare algebra to include transformations that turn bosons into fermions and fermions into bosons. We can extend this algebra into a super-Poincare algebra with the following steps. The following equation shows the *supersymmetry transformation*, where the generator Q transforms:

$$Q|\text{fermion}\rangle = |\text{boson}\rangle \tag{8.5}$$

and vice versa. This changes the spin of a particle, and hence its space-time properties. This also means that each one-particle state has a superpartner.

Theorem 2 (Haag, Lopuskanski, and Sohnius)**.** *The most general symmetry of the S-matrix is a direct product of super-Poincare and internal symmetries.*

$$G = G_{super\text{-}Poincare} \times G_{internal}$$

We will discuss how the super-Poincare algebra is constructed in a later section.

Supersymmetry happens to be the unique extension of the Lorentz group as a symmetry of the S-matrix, which makes supersymmetry an appealing project. Aside from its theoretical appeal, there are more practical reasons to entertain supersymmetry. This is discussed in the next section.

8.2 SUSY and the hierarchy problem

One unsolved problem in the Standard Model has to do with quadratic radiative corrections to the scalar Higgs mass, which would seem to point to an unnatural amount of fine-tuning on the order of 1 in 10^{38} as the Higgs potential would be extremely sensitive to new physics in any extension of the Standard Model.

Consider the Higgs potential for a complex scalar H with the Mexican hat potential:

$$V = m_H^2|H|^2 + \lambda|H|^4 \tag{8.6}$$

The Standard Model requires a non-zero vev at the minimum of the potential, and experimentally we know that $\langle H\rangle$ is approximately 174 GeV.

However, radiative corrections for hypothetical heavier particles impart quadratic corrections to the Higgs mass. Consider a Dirac fermion f with mass m_f. If the Higgs field couples to f with a Lagrangian term $\lambda_f H\overline{f}f$, we would obtain a correction in the Feynman diagram:

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{\mathrm{UV}}^2 \tag{8.7}$$

where Λ_{UV} is an ultraviolet momentum cutoff used to regulate the loop integral, representing the energy scale at which new physics enters.

Furthermore, there may be heavy scalars that exist. For a heavy complex scalar S with mass m_S that couples to the Higgs with a Lagrangian term $-\lambda_S|H|^2|S|^2$, we obtain a radiative correction:

$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}\left[\Lambda_{UV}^2 - 2m_S^2\ln(\Lambda_{\mathrm{UV}}/m_S)\right] \tag{8.8}$$

Hence, m_H^2 is extremely sensitive to the masses of the heaviest particles that the Higgs couples to. If there exist heavy particles, it is difficult to understand why the

Higgs mass is so small compared to a high cut-off, assumed to be the Planck cut-off. This would imply fine-tuning to the order of 1 in 10^{38}.

This problem arises even without direct coupling between the Standard Model Higgs boson and the unknown heavy particle. As long as an arbitrary particle has some gauge interaction with the Standard Model Higgs e.g. through electroweak interactions, or a new high energy gauge force, we will have quadratic corrections to the scalar mass.

Fortunately, supersymmetry can cancel out the quadratic contributions to the Higgs mass. If each of the quarks and leptons in the standard model is accompanied by two complex scalars with $\Lambda_S = |\lambda_f|^2$, then the quadratic λ_{UV}^2 contributions will neatly cancel. This cancellation is unavoidable once we assume that fermions and bosons are related by a supersymmetry transformation. This motivates the supersymmetry discussion.

8.3 Spinor representation and notation

$SO(3,1)$ generates two copies of the $SU(2)$ algebra.

$$SO(3,1) = SU(2) \otimes SU(2) \tag{8.9}$$

which allows us to label representations of $SO(3,1)$ using two $SU(2)$ 'charges'.

In the following material, undotted indices correspond to left-handed spinors ψ_α, while dotted indices correspond to right-handed spinors $\chi_{\dot{\alpha}}$.

The reason why they have two different indices (undotted and dotted) is that we have here two different objects that transform differently under a Lorentz rotation.

The undotted object $\psi^\alpha\psi_\alpha$ is Lorentz invariant, while the other object $(-i\sigma^2\overline{\eta})^T\overline{\eta}$ is Lorentz invariant. These two objects have different forms. We recast [6]:

$$(-i\sigma^2\overline{\eta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\eta}^{\dot{1}} \\ \overline{\eta}^{\dot{2}} \end{pmatrix} = \eta_{\dot{\alpha}} \tag{8.10}$$

This means that we can break down the four-component Dirac spinor into 2 two-component complex anti-commuting objects:

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} \tag{8.11}$$

with two types of spinor indices, $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$.

The field ξ is a left-handed Weyl spinor, while $\chi^\dagger$ is a right-handed Weyl spinor, since

$$P_L \Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad P_R \Psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} \tag{8.12}$$

The Hermitian conjugate of a left-handed Weyl spinor gives a right-handed Weyl spinor, and vice-versa.

$$(\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot{\alpha}} \tag{8.13}$$

$$(\psi^{\dagger\dot{\alpha}})^\dagger = \psi^\alpha \tag{8.14}$$

By convention, left-handed Weyl spinors do not carry daggers while right-handed Weyl spinors carry daggers.

The height of the indices is raised and lowered using the antisymmetric symbol:

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0 \tag{8.15}$$

with raising and lowering operations following:

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \quad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \quad \chi^\dagger_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta},\chi^{\dagger\dot{\beta}}}, \quad \chi^{\dagger\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\beta}} \tag{8.16}$$

8.4 Superspace and Superfields

We jump directly to discussing superspace and superfields. This is the most natural framework to formulate SUSY and will lead to the Wess-Zumino Lagrangian, which is a supersymmetric description of a matter field. This will illuminate the key ideas of the supersymmetric approach.

In QFT, we label spatial coordinates with the four-vector x^μ consisting of 4 real numbers. Incidentally, we have also encountered anti-commuting numbers as Grassmann coordinates in the discussion of fermionic path integrals.

The basic idea of $\mathcal{N}=1$ supersymmetry is to enlarge the space-time coordinate x^μ with two pairs of anti-commuting Grassmann coordinates $\theta_\alpha, \overline{\theta}_{\dot{\alpha}}$.

The coordinates x^μ are associated with the momentum operator P^μ, while the coordinates $\theta_\alpha, \overline{\theta}_{\dot{\alpha}}$ are associated with the supersymmetry generators $Q_\alpha, \overline{Q}_{\dot{\alpha}}$. The commutation relations of the generators are stated below:

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\sigma_{\alpha\dot{\beta}}P \tag{8.17}$$

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \tag{8.18}$$

$$[P, Q_\alpha] = [P, \overline{Q}_{\dot{\alpha}}] = 0 \tag{8.19}$$

where $\alpha, \dot{\alpha}$ are spinor indices, $\overline{Q}_{\dot{\alpha}}$ is the Hermitean conjugate of Q_α.

Hence, we have an eight-coordinate superspace labelled by $(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}})$, which makes supersymmetry properties manifest. The Grassmann numbers anti-commutes with fermionic objects and commutes with bosonic objects. A generic element of the super-Poincare group can be written as:

$$G(x, \theta, \overline{\theta}, \omega) = \exp(ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\overline{\theta}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}} + \frac{i\omega^{\mu\nu}}{2}M_{\mu\nu}) \tag{8.20}$$

Now that we have our super-coordinates, an arbitrary superfield can hence be written as the following Taylor expansion:

$$Y(x, \theta, \overline{\theta}) = f(x) + \theta\psi(x) + \overline{\theta}\chi(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) \tag{8.21}$$

$$+ \theta\theta\overline{\theta\lambda}(x) + \overline{\theta\theta}\theta\rho(x) + \theta\theta\overline{\theta\theta}d(x) \tag{8.22}$$

where each entry above is a field. Due to the properties of Grassmannian numbers, there are no higher-order terms in factors of Grassmannian variables as they simply vanish.

The goal is to construct supersymmetric Lagrangians out of superfields. To do so, we need to define derivatives with respect to the Grassmannian coordinates:

$$\frac{\partial}{\partial\theta^\alpha}(\theta^\beta) = \delta^\beta_\alpha, \quad \frac{\partial}{\partial\theta^\alpha}(\theta^\dagger_\beta) = 0, \quad \frac{\partial}{\partial\theta^\dagger_\alpha}(\theta^\dagger_{\dot{\beta}}) = \delta^{\dot{\alpha}}_{\dot{\beta}}, \quad \frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}}(\theta^\beta) = 0 \tag{8.23}$$

To integrate over superspace, we also define:

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}, \qquad d^2\theta^\dagger = -\frac{1}{4}d\theta^\dagger_{\dot{\alpha}}d\theta^\dagger_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \tag{8.24}$$

such that

$$\int d^2\theta\theta\theta = 1, \qquad \int d^2\theta^\dagger\theta^\dagger\theta^\dagger = 1 \tag{8.25}$$

How does a superfield transform under supersymmetry transformations? We understand the supersymmetry generators $Q_\alpha, \overline{Q}_{\dot{\alpha}}$ as differential operators, in the same way we use the differential operator to generate translation in Quantum Mechanics.

Consider an arbitrary generator $\mathcal{P}_\mu$ that generates a translation, with infinitesimal parameter a^μ on a field $\phi(x)$, and the generator is represented as a differential operator in field space by P_μ. This is defined as:

$$\phi(x+a) = e^{-ia\mathcal{P}}\phi(x)e^{ia\mathcal{P}} = \phi(x) - ia^\mu[\mathcal{P}_\mu, \phi(x)] + \dots \tag{8.26}$$

where on the right-hand side we have expanded the exponential terms. If we Taylor expand the left-hand side, we will obtain:

$$\phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x) + \dots \tag{8.27}$$

Equating the two:

$$[\phi(x), \mathcal{P}_\mu] = -\partial_\mu\phi(x) \equiv P_\mu\phi(x) \tag{8.28}$$

where P_μ is the representation of the generator $\mathcal{P}_\mu$ which generates translations.

Hence, this induces a change in the field as:

$$\delta_a\phi \equiv \phi(x+a) - \phi(x) = ia^\mu P_\mu\phi \tag{8.29}$$

We want to do the same thing for a superfield. A supersymmetry transformation is a transformation in superspace. Consider a superspace translation on a superfield $Y(x, \theta, \overline{\theta})$ by a quantity $(\epsilon_\alpha, \overline{\epsilon}_{\dot{\alpha}})$. This is written as:

$$Y(x+\delta x, \theta+\delta\theta, \overline{\theta}+\delta\overline{\theta}) = e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}Y(x,\theta,\overline{\theta})e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})} \tag{8.30}$$

$$= e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}e^{-i(x\mathcal{P}+\theta Q+\overline{\theta}\overline{Q})}Y(0,0,0)e^{i(x\mathcal{P}+\theta Q+\overline{\theta}\overline{Q})}e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})} \tag{8.31}$$

We can apply the Baker-Campbell-Hausdorff formula, and eventually obtain the result:

$$\delta x^\mu = i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta} \tag{8.32}$$

$$\delta\theta^\alpha = \epsilon^\alpha \tag{8.33}$$

$$\delta\overline{\theta}^{\dot{\alpha}} = \overline{\epsilon}^{\dot{\alpha}} \tag{8.34}$$

Notice that δx^μ is non-zero, in which two supersymmetry transformations generate a space-time translation. We can now represent the supersymmetry generators as differential operators. Skipping the steps, this gives us:

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \tag{8.35}$$

$$\overline{Q}_{\dot{\alpha}} = +i\overline{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{8.36}$$

These operators hence close the supersymmetry algebra, since:

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\sigma_{\alpha\dot{\beta}}P \tag{8.37}$$

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \tag{8.38}$$

8.5 Chiral Superfields

We can construct covariant derivatives defined as:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \tag{8.39}$$

$$\overline{D}_{\dot{\alpha}} = \overline{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \tag{8.40}$$

It turns out that $D_\alpha Y = 0$ is a supersymmetric invariant constraint we can impose on an arbitrary superfield to reduce its number of components to obtain a superfield that describes matter fields.

A chiral superfield Φ is a superfield such that

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{8.41}$$

and an anti-chiral superfield Ψ is such that:

$$D_\alpha\Psi = 0 \tag{8.42}$$

With the applied constraint, a left-chiral superfield is described by:

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y) \tag{8.43}$$

where F is an auxiliary degree of freedom. Taylor expanding this about x, we obtain:

$$\Phi(x,\theta,\overline{\theta}) = \phi(x)+\sqrt{2}\theta\psi(x)+i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\phi(x)-\theta\theta F(x)-\frac{i}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\overline{\theta}-\frac{1}{4}\theta\theta\overline{\theta\theta}\Box\phi(x) \tag{8.44}$$

We can compute the supersymmetric variation of the different field components of the chiral superfield Φ. This is done so by computing:

$$\delta_{\epsilon,\overline{\epsilon}}\Phi(y;\theta) = (i\epsilon Q + i\overline{\epsilon}\overline{Q})\Phi(y;\theta) \tag{8.45}$$

by expressing $Q_{\alpha}, \overline{Q}_{\dot{\alpha}}$ in superspace coordinates as we have done so earlier. We eventually obtain the supersymmetric variations for a chiral superfield:

$$\delta\phi = \sqrt{2}\epsilon\phi \tag{8.46}$$

$$\delta\psi_{\alpha} = \sqrt{2}i(\sigma^{\mu}\overline{\epsilon})_{\alpha}\partial_{\mu}\phi - \sqrt{2}\epsilon_{\alpha}F \tag{8.47}$$

$$\delta F = i\sqrt{2}\partial_{\mu}\psi\sigma^{\mu}\overline{\epsilon} \tag{8.48}$$

where F is an auxiliary degree of freedom that is used to impose invariance of the Lagrangian to supertranslations.

8.6 Superspace Lagrangians and minimal supersymmetry

The goal is to construct a supersymmetric invariant action to describe the interactions between chiral superfields. Consider a single chiral superfield Φ:

$$\int d^2\theta d^2\overline{\theta\Phi}\Phi \tag{8.49}$$

which is a supersymmetric Lagrangian. It is a real scalar object, and has the right physical dimensions. Expanding the components of the superfield, we obtain:

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta\Phi}\Phi = \int \partial_{\mu}\overline{\phi}\partial^{\mu}\phi + \frac{i}{2}(\partial_{\mu}\psi\sigma^{\mu}\overline{\psi}\psi\sigma^{\mu}\partial_{\mu}\overline{\psi}) + \overline{F}F \tag{8.50}$$

which is the kinetic term describing the degrees of freedom of a chiral superfield. As anticipated, the F field is an auxiliary field and is non-propagating. After integrating F out, its equation of motion is $F = 0$, and supersymmetry is only realised on-shell.

Hence, by studying superfields in superspace, and imposing a constraint to obtain chiral superfields to describe matter fields, we arrive at the Wess-Zumino Lagrangian.

In the following section, we will also cover vector superfields as this is the other ingredient for building the Minimally Supersymmetric Standard Model (MSSM).

8.7 Vector Superfields

To have gauge interactions, we need a constraint that will give us a vector field v_μ in the Lagrangian. In the previous discussion on the Taylor expansion of the super-field, we saw that the term v_μ has a coefficient $\theta\sigma^\mu\overline{\theta}$.

It turns out that the correct trick is to impose the reality condition on the superfield such that $Y \to \overline{Y}$, it follows that $v^\mu \to \overline{v}^\mu$, such that the vector component survives as an independent degree of freedom that is also real.

Hence, a real (vector) superfield V is a superfield such that:

$$V = \overline{V} \tag{8.51}$$

The expansion of V is shown below:

$$V(x,\theta,\overline{\theta}) = C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \theta\sigma^\mu\overline{\theta}v_\mu + \frac{i}{2}\theta\theta(M(x) + iN(x)) \tag{8.52}$$

$$- \frac{i}{2}\overline{\theta\theta}(M(x) - iN(x)) + i\theta\theta\overline{\theta}\left(\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\right) \tag{8.53}$$

$$- i\overline{\theta\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right) + \frac{1}{2}\theta\theta\overline{\theta\theta}\left(D(x) - \frac{1}{2}\partial^2 C(x)\right) \tag{8.54}$$

This superfield has 8 bosonic and 8 fermionic degrees of freedom. After introducing the supersymmetric gauge transformation, and imposing on-shell, we will be left with 2 bosonic and 2 fermionic degrees of freedom.

For a chiral superfield Φ, observe that combinations $\Phi + \overline{\Phi}, i(\Phi - \overline{\Phi}), \Phi\overline{\Phi}$ are all real vector fields.

Hence, under the following transformation on V,

$$V \to V + \Phi + \overline{\Phi} \tag{8.55}$$

we obtain a superfield V that is still real. By looking at the components of Φ, we can write down the supersymmetric version of a gauge transformation:

$$C \to C + 2\mathrm{Re}\phi \tag{8.56}$$

$$\chi \to \chi - i\sqrt{2}\phi \tag{8.57}$$

$$M \to M - 2\text{Im}F \tag{8.58}$$

$$N \to N + 2\text{Re}F \tag{8.59}$$

$$D \to D \tag{8.60}$$

$$\lambda \to \lambda \tag{8.61}$$

$$v^\mu \to v^\mu - 2\partial_\mu \text{Im}\phi \tag{8.62}$$

with the components of Φ being (ϕ, ψ, F)

If we make an educated choice for Φ, such that:

$$\text{Re}\phi = -\frac{C}{2}, \quad \psi = -\frac{i}{\sqrt{2}}\chi, \quad \text{Re}F = -\frac{N}{2}, \quad \text{Im}F = \frac{M}{2} \tag{8.63}$$

we can set $C = M = N = \chi = 0$. This choice is the Wess-Zumino gauge, and the vector superfield can simply be written as:

$$V_{WZ} = \theta\sigma^\mu\overline{\theta}v_\mu(x) + i\theta\theta\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D(x) \tag{8.64}$$

At present, we have 4 bosonic and 4 fermionic degrees of freedom off-shell. Since D is an auxiliary field, imposing the equations of motion for D, spinor λ and vector v will give us 2 bosonic and 2 fermionic degrees of freedom off-shell.

Hence, we have the tools required to construct the Minimally Supersymmetric Standard Model, which is done elsewhere in the literature. The next section will discuss symmetry breaking in the context of supersymmetry as it is quite different from the Standard Model case.

8.8 Soft supersymmetry breaking

We can implement explicit supersymmetry breaking by adding soft terms into the superpotential: operators with mass dimension less than 4. This is because giving a non-vanishing vev to F-terms or D-terms does not give any satisfactory phenomenology – breaking supersymmetry within the Minimally Supersymmetric Standard Model (MSSM) predicts a massless photino (fermion counterpart to the photon) which has not been observed [74, 75].

To avoid this problem, we can implement an explicit Lagrangian breaking sector by adding quadratic and cubic scalar couplings, which would be irrelevant at high energy scales [67]:

$$\mathcal{L}_{\text{soft}} = m_\lambda \lambda\lambda - m^2 \overline{\phi}\phi + b\phi\phi + a\phi^3 + \text{h.c.} \tag{8.65}$$

where λ represents gauginos and ϕ any possible scalar of the MSSM. The first two terms provide masses for gauginos and scalar particles respectively; the third term couples the up and down scalar Higgs, and the fourth term corresponds to cubic combinations of MSSM scalars.

Where do these terms come from? To break MSSM, we assume that there is a fully renormalisable theory, where supersymmetry is broken spontaneously in a hidden sector. Using the Effective Field Theory perspective, this generates effective couplings, which are the soft supersymmetry breaking terms. Hence, introducing soft supersymmetry terms is an attempt to connect the MSSM with an unknown and hidden SUSY breaking sector by adding an effective Lagrangian term.

8.9 Summary

In conclusion, we have reviewed a few key ideas in supersymmetry for reasons of completeness. An interested reader may refer to the cited references for a more complete treatment of the topic. The ingredients reviewed can be used to construct the Minimally Supersymmetric Standard Model (MSSM).

In the subsequent chapter, we will look at further extensions of supersymmetric theory by building towards a Grand Unified Theory using the $SU(5)$ group, so as to unite the gauge couplings at a fundamental high energy scale.

Chapter 9

Supersymmetric Grand Unified Theories

9.1 Introduction

Having reviewed the construction of supersymmetric (SUSY) theories in Chapter 8, we will proceed by considering $SU(5)$ Grand Unified Theory (GUT), and subsequently SUSY $SU(5)$ GUT. To do so, we will first introduce the history of $SU(5)$ in Section 9.2 and its particle content in Section 9.3. Subsequently, we discuss problems in $SU(5)$ in Section 9.4 to do with Higgs doublet-triplet splitting, proton decay, and gauge coupling unification.

We then construct SUSY $SU(5)$ GUT in Section 9.5. This framework is used for a key piece of original work in this thesis, which is an extension of SUSY $SU(5)$ GUT with a D_4 discrete symmetry group in Section 9.6.

9.2 The history of the unification project

Unification is a common theme in the history of physics, with the unification of the electric and magnetic forces under Maxwell, space and time under Einstein, and the electroweak force under the Glashow-Salam-Weinberg model.

In this review section, we will reference [76, 77, 78]. To move further beyond the physics of the standard model, we can ask if there is a larger gauge group that can accommodate all of the Standard Model and also unify the gauge coupling constants.

As previously discussed in the LRSM section, the Pati and Salam group was a major step in this direction:

$$SU(4)_C \times SU(2)_L \times SU(2)_R \tag{9.1}$$

featuring lepton-quark unification, with the lepton number becoming the fourth colour. This was an important step towards thinking about Left-Right Symmetric Models that featured gauge coupling unification for $SU(2)_L$ and $SU(2)_R$ at sufficiently high energies.

Interestingly, the Standard Model gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ can be obtained by the decomposition of $SU(5)$, which is the smallest group that contains the Standard Model gauge group. This was first introduced by Georgi and Glashow [79] and has several incredibly appealing properties – we will discuss these properties in the next section.

9.3 Unbroken SU(5)

The $SU(5)$ gauge group can explain the quantisation of electric charge, why the proton and electron have exactly equal and opposite charges, as well as the unification of all coupling constants at a high energy scale [77], which makes $SU(5)$ appealing from a model-building perspective. This also represents another step in the unification project, unifying the electroweak with the strong force.

We will now discuss its construction and particle content. The irreducible representations of the Standard Model gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ can be written as [8]:

$$\left(3, 2, \frac{1}{6}\right) \oplus \left(\overline{3}, 1, -\frac{2}{3}\right) \oplus \left(\overline{3}, 1, \frac{1}{3}\right) \oplus \left(1, 2, -\frac{1}{2}\right) \oplus (1, 1, 1) \tag{9.2}$$

These represent the quark doublet, anti-up and anti-down quarks, the lepton doublet, and the positron respectively. These are the 15 fermions of one full generation in the Standard Model particles. Note that in $SU(5)$ we need not distinguish between left- or right-handed fields.

Intriguingly, all of their hypercharges add up to 0:

$$3 \cdot 2 \cdot \frac{1}{6} + 3 \cdot 1 \cdot \left(-\frac{2}{3}\right) + 3 \cdot 1 \cdot \frac{1}{3} + 1 \cdot 2 \cdot \left(-\frac{1}{2}\right) + 1 \cdot 1 \cdot 1 = 0 \tag{9.3}$$

which points to a hidden unification. Even more interestingly, we can accommodate the 15 fermions in $SU(5)$ using only two irreducible representations. We choose these to be the irreducible representations $\overline{5} \oplus 10$. The particle content of $\psi_{\overline{5}}$, the five-dimensional multiplet is shown as [74]:

$$\psi_{\overline{5}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ -e \\ \nu \end{pmatrix} \tag{9.4}$$

where the charge conjugate symbol of a particle ψ^c represents its anti-particle. On the other hand, the other 10 particles can be neatly packaged into a 5×5 anti-symmetric matrix. ψ_{10} is written explicitly as:

$$\psi_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u^1 & d^1 \\ -u_3^c & 0 & u_1^c & u^2 & d^2 \\ u_2^c & -u_1^c & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & e^c \\ -d^1 & -d^2 & -d^3 & e^c & 0 \end{pmatrix} \tag{9.5}$$

where the upper numerical indices are used to identify the quark doublet, and the lower numerical indices to identify the anti-up quarks. Upon decomposition into $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, the charge conjugated particles in $SU(5)$ will become the familiar right-handed particles that are singlets of $SU(2)_L$.

As previously mentioned, $SU(5)$ is the minimal gauge group that decomposes into $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. We write the decomposition of the representations explicitly:

$$5 : \left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, \frac{1}{2}\right) \tag{9.6}$$

$$10 : \left(3, 2, \frac{1}{6}\right) \oplus \left(\overline{3}, 1, -\frac{2}{3}\right) \oplus (1, 1, 1) \tag{9.7}$$

We take the complex conjugate of the 5 representation and see that $(\overline{3}, 1, \frac{1}{3})$ is an irreducible representation of $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. Hence, $\psi_{\overline{5}}$ contains the fields (d^c, ν, e), which are the anti-down quarks and lepton doublet. ψ_{10} contains (q, u^c, e^c), which are the quark doublets, anti-up quarks, and the anti-electron. The reason why down quarks are packaged in $\psi_{\overline{5}}$ is because the alternative (up quarks) does not lead to a traceless generator of $U(1)$ [78]. By construction, these $\psi_{\overline{5}}$ and ψ_{10} are left-handed by construction – there is no'right-handed' inside the $SU(5)$ gauge group.

9.3.1 Product decompositions in $SU(5)$

The group $SU(5)$ has rank 4, and hence has 4 quantum numbers to label p, q, r, s to label the representations D^{pqrs}, following [80].

In Appendix I, we compute the dimensions of the irreducible representations up to quantum numbers of 3. We also derive explicitly the product decomposition rules of $SU(5)$ using the method of Young tableaux. These tensor products will be used to construct Yukawa coupling terms for the rest of this chapter.

9.3.2 Gauge sector

The gauge bosons transform according to the adjoint representation of $SU(5)$, which means that we have $N^2 - 1 = 24$ gauge bosons for $N = 5$ of $SU(5)$. Their $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ decomposition is shown as below [74]:

$$A_{24} = (8, 1, 0) \oplus (1, 3, 0) \oplus \left(3, 2, \frac{5}{3}\right) \oplus \left(\overline{3}, 2, -\frac{5}{3}\right) \oplus (1, 1, 0) \tag{9.8}$$

where the first two terms represent the 8 gluons of $SU(3)_C$ and the 3 $\vec{W}$ bosons of $SU(2)_L$. The last term represents the $U(1)_Y$ hypercharge. The third term represents a doublet of X and Y bosons with hypercharges $\frac{4}{3}$ and $\frac{1}{3}$ respectively. The X and Y bosons also carry colour charges. The fourth term is the complex conjugate of the third, representing the $X^\dagger$ and $Y^\dagger$ bosons with hypercharges $-\frac{4}{3}$ and $-\frac{1}{3}$ respectively. These X and Y bosons are involved in proton decay, an important prediction of $SU(5)$ – this is discussed later in Section 9.4.2.

From our discussion in Appendix I, the following gauge-fermion couplings are allowed:

$$\psi_5 A_{24} \psi_{\overline{5}}, \psi_{\overline{10}} A_{24} \psi_{10} \tag{9.9}$$

which allows us to build the kinetic terms of the $SU(5)$ Lagrangian. However, we are more interested in the Yukawa sector in this thesis. Hence, we will examine the Higgs sector in the next section.

9.3.3 Higgs sector

There are two parts to the Higgs sector. Firstly, we choose to introduce a 24-dimensional representation Σ_{24} of $SU(5)$. This couples to a fermion-fermion term of irreducible representation $5 \otimes \overline{5}$ (i.e. $\psi_5 \psi_{\overline{5}}$) and $10 \otimes \overline{10}$ (i.e. $\psi_{10}\psi_{\overline{10}}$) as this is allowed by the $SU(5)$ product decomposition, shown in Eq I.9. Σ_{24} is used to break $SU(5) \to SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, which is another application of the Higgs mechanism.

The other Higgs multiplet is in the fundamental representation of $SU(5)$, consisting of 5 particles:

$$H_5 = \begin{pmatrix} T \\ H_u \end{pmatrix}, \quad H_{\overline{5}} = \begin{pmatrix} \overline{T} \\ H_d' \end{pmatrix} \tag{9.10}$$

that transform as 5 and $\overline{5}$ respectively, where H_u and H_d are two different Higgs doublets that couple to up and down quarks respectively. T represents a scalar triplet that has colour charges. We will see that this triplet T is slightly problematic in Sections 9.4.1 and 9.4.2.

This concludes the construction of minimal $SU(5)$, where we have shown its particle content and their decompositions into $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. For a group theory-motivated construction, one may refer to Part IX.2 of [8].

9.4 Problems with minimal $SU(5)$

Having introduced the particle content of minimal $SU(5)$, we will now discuss problems relating to its phenomenology and construction.

9.4.1 The Double-Triplet Problem

The heavy triplet T in the Higgs H_5 is identified as having colour charges. Firstly, this triplet provides a pathway for proton decay and is required to be heavy (at the GUT scale $\sim 10^{16}$ GeV) to be compatible with present experimental limits on the rate of proton decay. Secondly, the mass of the Higgs doublets is on the electroweak scale ($v \sim 10^2$ GeV). This presents a hierarchy problem in trying to explain why the colour triplets are significantly heavier than the electroweak scale Higgs doublets [74, 81, 82].

The first problem of proton decay is not a real problem if the proton decay rate is sufficiently low, or if the choice of theory allows for proton decay lifetimes that are outside the bounds of current experimental searches. We will instead look at the second problem, which has to do with explaining why particles in the same H_5 representation can have masses on vastly different scales.

A solution discussed by [83, 84] proposes to introduce 50- and 75-dimensional Higgs representations to avoid the Doublet-Tripet problem. This means that we no longer need the Σ_{24} Higgs multiplet.

In order to mix H_{50} and $H_5, H_{\overline{5}}$, we will also need a 75-dimensional representation H_{75} to break $SU(5)$ in place of Σ_{24}. This is known as the Missing Partner mechanism. This is shown to be allowed in Eq I.29 in Appendix I. When the H_{50} is decomposed into $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, there is no Higgs doublet field in the H_{50} and the $H_{\overline{50}}$. Consequently, the Higgs doublets cannot couple with the heavy Higgs particles, hence remaining light. Further details of the Missing Partner Mechanism are discussed in Section 9.1 of [74] and in [83, 84].

9.4.2 Proton decay

Proton decay is a key prediction of $SU(5)$ GUT. The heavy gauge bosons X and Y, as well as the heavy colour Higgs triplet T all carry colour charges. Hence, they can form interaction vertices with quarks that lead to proton decay by allowing quarks to couple to leptons. The pathway is shown in Fig 9.1 for the decay channel:

$$p^+ \to e^+ + \pi^0 \tag{9.11}$$

We will briefly discuss the mechanism and predictions for proton decay in minimal

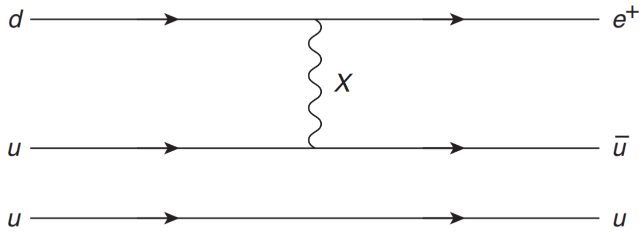
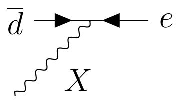


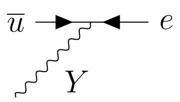
Figure 9.1: Proton decay diagram for the decay mode $p^+ \to e^+ + \pi^0$, extracted from Chapter 16.1 of [76].

$SU(5)$ GUT. Dimension-6 operators of the form $qqq\ell$ involve the exchange of heavy X, Y gauge bosons. Alternatively, they involve colour T triplet bosons. This leads to a finite proton lifetime. The relevant terms of the gauge Lagrangian involving X and Y bosons can be written as:

$$\mathcal{L}_{X,Y} \supseteq g_X X_\mu (\overline{d}\gamma^\mu e^c) - g_Y Y_\mu (\overline{u}\gamma^\mu e) + \text{h.c.} \tag{9.12}$$

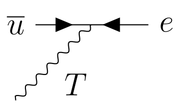
where Dirac adjoints are shown as bars. These are obtained from hypercharge considerations. This allows for the following interaction vertices that couple quarks and leptons:





where we recall that switching the direction of the fermionic propagator switches the charge conjugation of the electron.

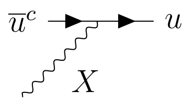
The Higgs colour triplets T have hypercharge $\frac{1}{3}$, which can be inferred from the hypercharge generator of $SU(5)$ – since the Higgs particles have hypercharge $-\frac{1}{2}$, we require that $3 \cdot \frac{1}{3} + 2 \cdot -\frac{1}{2} = 0$. Hence, we can form the interaction vertex:



How do the scalars couple to the quarks? We will use the example of the X boson. The relevant term in the Lagrangian is:

$$\mathcal{L}_X \supseteq g_X X_\mu \overline{u}^c \gamma^\mu u + \text{h.c.} \tag{9.13}$$

which gives the interaction vertex:



Put together, this allows a pathway for quarks to decay into lighter particles, hence allowing for proton decay as illustrated in Fig 9.1.

Proton decay represents a key prediction where experiments on the lifetime of the proton can help constrain GUT models. However, minimal $SU(5)$ has notably been ruled out – minimal $SU(5)$ predicts proton lifetimes on the order of $\tau_p \simeq 10^{30}$ years [20]. For instance, the Super-Kamiokande collaboration reports strong experimental lower bounds on proton decay lifetimes for a few channels, typically with $\tau > 10^{34}$ years [85, 86], which is incompatible with the GUT energy scale required for gauge coupling unification for minimal $SU(5)$.

This highlights the difficulty of finding experimental evidence for $SU(5)$: ruling out proton decay does not imply that $SU(5)$ must be falsified since $SU(5)$ can be modified to allow a proton decay lifetime that is outside of experimental bounds. A positive detection would provide strong experimental support for $SU(5)$ (and is likely to involve a Nobel Prize), but present experimental searches have turned up empty-handed.

9.4.3 Gauge coupling constants do not actually unify

The goal of $SU(5)$ is to have a unified gauge group that breaks down into the Standard Model and answers questions relating to neutrino mass. An additional goal of the GUT project is for gauge coupling unification at a high energy scale.

Embarrassingly, when we extrapolate low-energy data on the coupling constants to the GUT scale, the three gauge coupling constants (the strong, weak, and electromagnetic forces) do not unify as a consequence of the RG running [74, 76], shown in Fig 9.2.

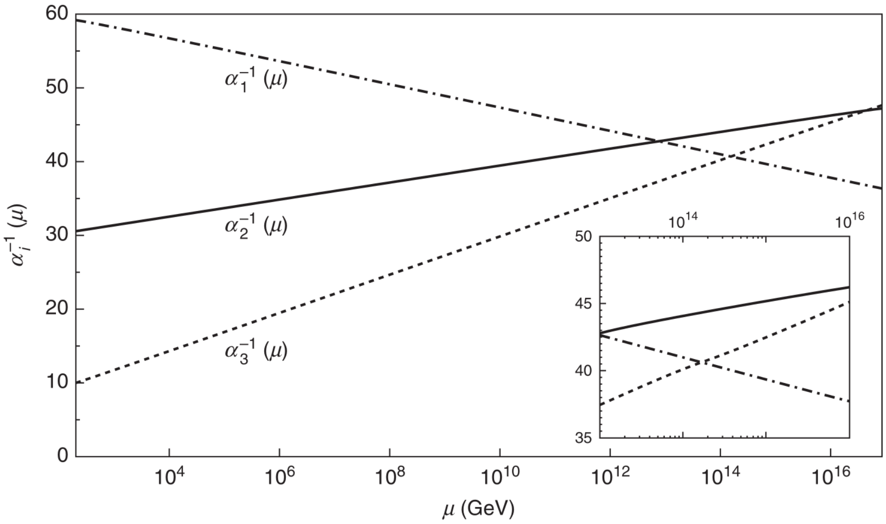


Figure 9.2: Running coupling constants in minimal $SU(5)$, extracted from [76], originally from [87]

Hence, we have discussed several issues that appear in $SU(5)$ theory. The lack of gauge coupling unification is problematic – that was the original promise of $SU(5)$ theory. This will motivate the construction of SUSY $SU(5)$ in the next section.

9.5 SUSY SU(5)

9.5.1 Why supersymmetric grand unification

As previously discussed in Section 8.1 there are several advantages of using a supersymmetric framework, which includes fixing the hierarchy problem of the Higgs boson mass due to radiative corrections from heavier particles.

Furthermore, in supersymmetric grand unification, the proton lifetime is $\tau_p \simeq 10^{35\pm1}$ years [20] which is just out of experimental reach and still renders SUSY $SU(5)$ a viable theory.

More importantly, it turns out that supersymmetric GUTs produce gauge unification. In the supersymmetric framework, superpartners contribute to the running of the coupling constants. The extrapolation shows that the gauge-coupling constants unify at a scale of $\Lambda \sim 10^{16}$ GeV [87], as shown in Fig 9.3.

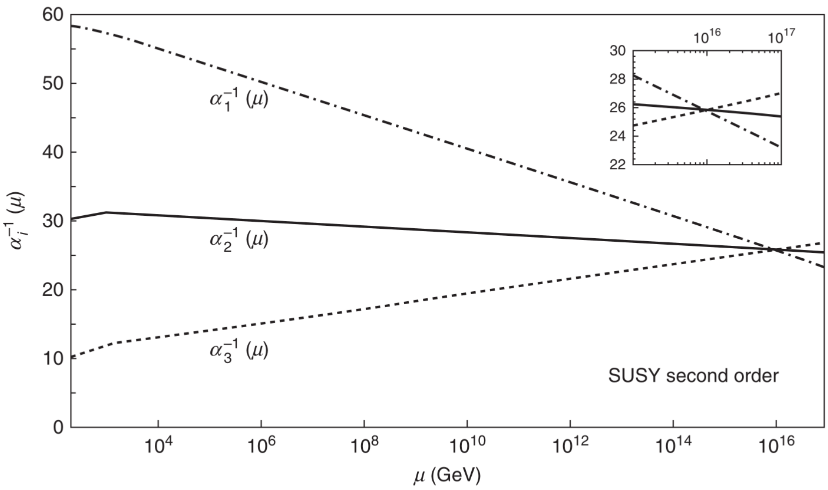


Figure 9.3: Running coupling constants in SUSY $SU(5)$, extracted from [76], originally from [87]

9.5.2 Supersymmetric construction

The fermion content and the Higgs particles can be packaged into chiral superfields, while the gauge bosons will have to be packaged into vector superfields.

Typically, supersymmetric fields will have a hat on top. For simplicity, we will suppress the hat in the notation.

For the superpotential breaking, we will have to introduce soft-breaking terms of the form:

$$\mathrm{Tr}\Phi + \mathrm{Tr}\Phi^2 + \mathrm{Tr}\Phi^3 \subset \mathcal{L}$$

where Φ represents the scalar Higgs bosons. Superpotential breaking was briefly discussed in Section 8.8.

9.5.3 Particle content

A paper by Altarelli, Feruglio, and Masina in 2001 provides an example of a SUSY SU(5) GUT model, using a Froggatt-Nielsen mechanism to generate the mass hierarchy and particle mixings [88]. We will adopt their particle content as a starting point.

This is because these authors introduce the 50- and 75-dimensional representation of a Higgs scalar to avoid the doublet-triplet splitting problem via the Missing Partner Mechanism as previously discussed in Section 9.4.1. Following our discussion in Appendix I, Eq. I.29 gives us the relevant Yukawa couplings:

$$H_{75}H_{50}H_5, H_{75}H_{\overline{50}}H_{\overline{5}} \tag{9.14}$$

9.5.4 Mass hierarchy and mixing mechanism

In addition to the fermion content of $SU(5)$, we introduce the sterile right-handed neutrino Ψ_1, in the trivial representation of $SU(5)$. Since minimal $SU(5)$ consists entirely of left-handed fields by construction, we have to manually add in the right-handed neutrino. This makes for a total of 16 fermions in one generation.

The relevant Yukawa sector of the superpotential has the form:

$$-\mathcal{L}_{\text{Yukawa}} = \Psi_{10}G_u\Psi_{10}H_5+\Psi_{10}G_d\Psi_{\overline{5}}H_{\overline{5}}+\Psi_{\overline{5}}G_\nu\Psi_1H_5+M\Psi_1G_M\Psi_1+\Psi_{10}G_{\overline{50}}\Psi_{10}H_{\overline{50}}+\text{h.c.} \tag{9.15}$$

where the various G terms represent Yukawa coupling matrices [88]. The first term $\Psi_{10}G_u\Psi_{10}H_5$ is related to the up-quark mass, and its $10 \otimes 10 \otimes 5$ tensor product is allowed by Eq I.16 and I.9.

The second term $\Psi_{10}G_d\Psi_{\overline{5}}H_{\overline{5}}$ is related to the down quark mass, and its $10 \otimes \overline{5} \otimes \overline{5}$ tensor product is allowed by Eq I.12 and I.9.

The third term $\Psi_{\overline{5}}G_\nu\Psi_1H_5$ is related to the Dirac neutrino mass, whose tensor product is allowed by Eq I.9.

The fourth term $M\Psi_1G_M\Psi_1$ is related to a Majorana neutrino mass for the new right-handed neutrino, whose tensor product is trivially allowed.

The last term $\Psi_{10}G_{\overline{50}}\Psi_{10}H_{\overline{50}}$ involves the quark masses, whose tensor product is allowed by Eq I.16 I.26.

Note that this Yukawa sector does not have coupling terms for $5 \otimes \overline{5}$ as this would give a term of mass dimension 3. The choice of Higgs representation also means that $10 \otimes \overline{10}$ is not included for the same reason.

9.6 Extending SUSY $SU(5)$ with a D_4 symmetry group

Altarelli and Feruglio use a Froggatt-Nielsen mechanism to generate a mass hierarchy in [88]. For our extension, we will use a D_4 discrete symmetry group to generate the neutrino mass mixing matrices, which also happens to generate the electron and quark mass mixing matrices. The D_4 algebra is elucidated in Appendix F. We will need the fact that the tensor product of two D_4 doublets produces a singlet: $(x_1, x_2) \otimes (y_1, y_2) \to 2 \otimes 2 = 1 \oplus 1' \oplus 1'' \oplus 1'''$, and the singlet is represented by $x_1 y_2 + x_2 y_1$

We follow the strategy of Bonilla et al. [66] and assign the first two generations (electron and muon generation) of $\Psi_{\overline{5}}$ and Ψ_1 as a D_4 doublet, while leaving the third generation as a singlet. We introduce a flavon D_4 doublet η and a singlet χ. The flavon scalars χ and η will acquire non-zero vev at a high-energy flavon scale Λ_F which will break the D_4 symmetry.

The particle content is shown in Table 9.1, where the parenthetical superscripts refer to the generations.

Particle	D_4 charge
$\Psi_1^{(1,2)}$	2
$\Psi_1^{(3)}$	1
$\Psi_{\overline{5}}^{(1,2)}$	2
$\Psi_{\overline{5}}^{(3)}$	1
$\Psi_{10}^{(1,2,3)}$	1
η	2
χ	1

Table 9.1: D_4 charge assignments for extending SUSY $SU(5)$

9.6.1 Neutrino mass mechanism for Majorana masses

The goal is to obtain the Seesaw mechanism, where $m_\nu = m_L - m_D(m_R)^{-1}m_D^T$. There are three steps. We will discuss in sequence the left-handed neutrino Majorana mass, the right-handed Majorana mass, and the Dirac mass.

The left-handed Majorana mass is associated with m_L, and comes from from a $\psi_{\overline{5}}\psi_{\overline{5}}\chi$ term. However, the choice of Higgs representations does not allow for this term, and we cannot construct a $SU(5)$-invariant term due to Eq I.10. Hence, this is equivalent to setting $m_L = 0$.

Next, we consider the right-handed neutrino Majorana mass matrix. There are four possible terms allowed by the D_4 assignments, shown in Eq 9.16. For clarity, we suppress factors of $\frac{1}{\Lambda_F}$; it is taken that all terms will have mass dimension 4.

$$\begin{aligned} -\mathcal{L}_{\text{RH maj}} &= y_1\Psi_1^{(1,2)}\Psi_1^{(1,2)}\chi + y_2\Psi_1^3\Psi_1^{(3)}\chi \\ &\quad + y_3\Psi_1^{(1,2)}\Psi_1^{(3)}\eta^{(1,2)} + y_4\Psi_1^(3)\Psi^{(1,2)}\eta^{(1,2)} \end{aligned} \tag{9.16}$$

In the flavour basis, the mass matrix will take the form:

$$M_R = M_{maj} \sim \begin{pmatrix} 0 & y_1\chi & y_3\eta_2 \\ y_1\chi & 0 & y_4\eta_1 \\ y_3\eta_2 & y_4\eta_1 & y_2\chi \end{pmatrix} \tag{9.17}$$

which will be associated with the sterile right-handed neutrino ψ_1 after SSB.

Now, we will work out the Dirac mass term:

$$\begin{aligned} -\mathcal{L}_{\text{Dirac}} &= y_5\Psi_{\overline{5}}^{(1,2)}\Psi_1^{(1,2)}\chi + y_6\Psi_{\overline{5}}^{(3)}\Psi_1^{(3)}\chi \\ &\quad + y_7\Psi_{\overline{5}}^{(1,2)}\Psi_1^{(3)}\eta^{(1,2)} + y_8\Psi_{\overline{5}}^{(3)}\Psi_1^{(1,2)}\eta^{(1,2)} + y_9\Psi_{\overline{5}}^{(1,2)}\Psi_5^{(1,2)}\chi + y_{10}\Psi_{\overline{5}}^{(3)}\Psi_5^{(3)}\chi \end{aligned} \tag{9.18}$$

This leads to the same mass matrix texture:

$$M_D \sim \begin{pmatrix} 0 & (y_5+y_9)\chi & y_7\eta_2 \\ (y_5+y_9)\chi & 0 & y_8\eta_1 \\ y_7\eta_2 & y_8\eta_1 & (y_6+y_{10})\chi \end{pmatrix} \tag{9.19}$$

The Seesaw relation tells us that $m_\nu \sim m_D(m_R)^{-1}m_D^T$. Within the context of SUSY $SU(5)$, we can use the M_R and M_D matrices and apply the Seesaw relationship.

m_ν is symmetric by construction. Hence, we can diagonalise it using a unitary matrix U_ν to obtain the neutrino mixing matrix.

9.6.2 Charged electrons and the neutrino mixing matrix

Now that we have U_ν, we have one half of the puzzle for computing the neutrino mixing matrix. After computing U_e, the matrix that diagonalises the charged lepton mixing matrix, we will be able to construct $U_{PMNS} = U_e^\dagger U_\nu$, which is the neutrino mixing matrix.

Charged lepton mass terms originate from terms of the form $\Psi_{10}\Psi_{\overline{5}}$, where all generations of Ψ_{10} are singlets under D_4. This will lead to the following Yukawa terms for the charged lepton masses:

$$-\mathcal{L}_{\text{electron}} \sim \overline{e}^{(1,2,3)} \left[y_{11} e^{(1,2)} H_{\overline{5}} \eta^{(1,2)} + y_{12} e^{(3)} H_{\overline{5}} \chi \right] \tag{9.20}$$

which after flavour symmetry breaking gives rise to the mass matrix:

$$M_{\text{clep}} \sim y_{11} \langle H_{\overline{5}} \rangle \begin{pmatrix} \eta^2 & \eta^1 & 0 \\ \eta^2 & \eta^1 & 0 \\ \eta^2 & \eta^1 & 0 \end{pmatrix} + y_{12} \langle H_{\overline{5}} \rangle \begin{pmatrix} 0 & 0 & \chi \\ 0 & 0 & \chi \\ 0 & 0 & \chi \end{pmatrix} \tag{9.21}$$

Since this matrix is non-symmetric, we will need the bi-unitary transformation to diagonalise it into $M_{clep} = U_e^\dagger \hat{M}_{clep} V_e$. We can absorb V_e into the fermion field via a re-definition. This produces the neutrino mixing matrix $U_{PMNS} = U_e^\dagger U_\nu$.

Also, the unitary matrices are non-unique by definition, and the arbitrary V_e field introduces yet another degree of freedom, so we conclude that there are enough degrees of freedom to reproduce the existing neutrino mixing matrices. We should be able to set U_e as the identity matrix to reproduce non-observation of charged lepton flavour violation, as well as to reproduce the U_{PMNS} matrix.

9.6.3 Quark mixing matrix

The flavon scalars are coupled to the quark fields as the different generations of $\Psi_{\overline{5}}$ have been assigned D_4 character. Since quarks have Dirac mass terms only, we only need to consider the following terms in the Lagrangian:

$$-\mathcal{L}_{\text{down quark mass}} = y_{13}\psi_{10}^{(1,2,3)}\psi_{\overline{5}}^{(1,2)}H_{\overline{5}}\eta^{(1,2)} + y_{14}\psi_{10}^{(1,2,3)}\psi_{\overline{5}}^{(3)}H_{\overline{5}}\chi \tag{9.22}$$

This gives the down-quark mass matrix:

$$M_{\text{quark}} \sim y_{13}\langle H_{\overline{5}}\rangle \begin{pmatrix} \eta^2 & \eta^1 & 0 \\ \eta^2 & \eta^1 & 0 \\ \eta^2 & \eta^1 & 0 \end{pmatrix} + y_{14}\langle H_{\overline{5}}\rangle \begin{pmatrix} 0 & 0 & \chi \\ 0 & 0 & \chi \\ 0 & 0 & \chi \end{pmatrix} \tag{9.23}$$

while the up-quark mass matrix comes from the Lagrangian terms:

$$-\mathcal{L}_{\text{up quark mass}} = y_{15}\Psi_{10}\Psi_{10}H_{\overline{50}} \tag{9.24}$$

and does not receive any special mixing pattern as Ψ_{10} does not have any D_4 character.

An interesting question is whether we can recover the CKM matrix. The formula is to diagonalise the up-quark and down-quark mass matrices using the bi-unitary transformation, leading to the unitary matrix $U_{CKM} = U_u^\dagger U_d$, and two arbitrary matrices V_u, V_d absorbed via field re-definition.

Given the available degrees of freedom in choosing $\langle H_{\overline{50}}\rangle$, and given that rotation matrices are not unique, the CKM matrix can be reproduced. That being said, the size of the allowed parameter space remains an open numerical question and is a topic for future work.

9.6.4 Comparisons with similar extensions

This small extension is similar to Ishimori et al. (2009) [89], which applied an S_4 model to SUSY $SU(5)$, assigning singlet and doublet character across different generations. They argue that the S_4 group can uniquely satisfy the Tri-Bimaximal (TBM) mixing pattern exactly – this was before the Daya Bay results in which $\theta_{13} \neq 0$. The justification for the choice of S_4 is based on a group-theoretical argument by [90].

The key difference between this work and Ishimori et al. is their choice of discrete group and character assignments. The three generations of $\psi_{\overline{5}}$ are assigned to a triplet of S_4, while the first and second generations of ψ_{10} are assigned to an S_4 doublet, while the third generation is a singlet. Right-handed neutrinos are assigned as doublets for the first two generations and singlet for the third generation.

This small extension is also similar to Miskaoui et al. (2021) [91], which uses a $D_4 \otimes U(1)_X$ discrete symmetry group for a SUSY $SU(5)$ GUT. The choice of this group was justified in its ability to reproduce existing phenomenology through its specific choice of charge assignments and fermion content.

The key differences lie in the particle content, the choice of charge assignments, as well as the choice of Higgs content. Firstly, they include a new fermion sector, X for down quarks, charged leptons neutrinos, and Y for up quarks. By adding the $U(1)_X$ group, They also use the 24- and $\overline{45}$-dimensional representation of the Higgs to break the $SU(5)$ symmetry – they avoid the double-triplet splitting problem by using the $U(1)_X$ group to forbid the problematic interaction term.

The key advantage of the approach found in this thesis lies in the brevity of its particle content (no new heavy particles), as well as the lack of an arbitrary $U(1)_X$ charge assignment that generates mass hierarchies.

In summary, this extension applies key ideas in the model-building literature in the context of SUSY $SU(5)$ GUT. We have used a discrete symmetry group to generate a neutrino mixing pattern, and have examined some of its consequences. It also appears that there are sufficient degrees of freedom to reproduce phenomenology, though confirmation will have to come from future numerical work.

9.7 Summary

In this chapter, we have introduced the SUSY $SU(5)$ GUT as a progression from minimal $SU(5)$. We have introduced its gauge group, and we have explicitly worked out its irreducible representations and tensor products, showing how the fermions, gauge bosons, and Higgs sector fit into these representations. We have discussed the theoretical promise of $SU(5)$ in gauge coupling unification, the realisation of gauge coupling unification in supersymmetric $SU(5)$, as well as problems related to double-triplet splitting and proton decay.

Subsequently, we have worked out a small extension and built a model that extends SUSY $SU(5)$ GUT. We introduced a D_4 discrete symmetry group in an attempt to explain the neutrino mixing pattern for a specific choice of charge assignments. We have also compared our model to others found in the literature, commenting on some of the differences. Building models in higher theories allows us to further the unification project, while also allowing the degrees of freedom to reproduce existing phenomenology.

Chapter 10

Conclusion

10.1 Summary

This thesis has provided a progressive theoretical treatment of the question of massive neutrinos and their mixing patterns, starting from the Standard Model, before moving on to effective Weinberg operators, Left-Right Symmetric Models, Supersymmetric, $SU(5)$ GUT, and finally Supersymmetric $SU(5)$ GUT.

As original contributions in this thesis, we have proposed an LRSM tree-level diagram in Section 7.5.1 for neutrino masses whose interaction vertices obey the gauge symmetries. This produces another Weinberg operator into the Yukawa Lagrangian that can contribute to neutrino masses in the Left-Right symmetric theory and is justified by our discussion on Effective Field Theory in Appendix H.

In addition, in the discussion on SUSY $SU(5)$, we have proposed a new flavour mixing model in Section 9.6, where a D_4 discrete symmetry group is used to restrict Yukawa couplings in the SUSY $SU(5)$ Yukawa Lagrangian. This generates mass mixing matrices for the charged leptons, neutrinos, and quarks, while also explaining the smallness of the left-handed neutrinos. This model can in principle be used to make predictions for fermion masses, gauge coupling constants, as well as the stability of the proton.

Furthermore, we have worked out a detailed description of the $SU(5)$ irreducible representations and tensor product decompositions in Appendix I, which is not explicitly available in existing literature [92].

We re-emphasise the theoretical importance of studying neutrino masses - this is one of the key windows into Beyond Standard Model physics, as there is no compelling reason why neutrinos should have masses at all within the framework of the Standard Model. At the experimental level, the possibility of effective operators contributing to neutrino masses is currently under investigation at the LHC [93], in which LHC data is combed for decay signals from heavy Majorana neutrino by investigating proton-proton collision data. Examples of such processes are shown in Figure 10.1.

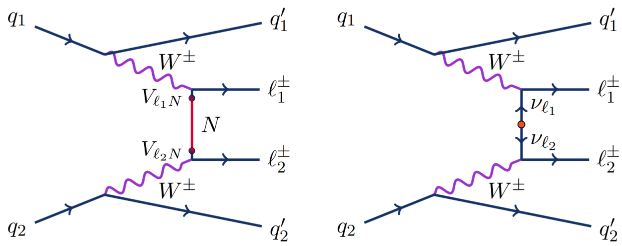


Figure 10.1: Examples of processes mediated by a heavy Majorana neutrino, whose particle signatures are being looked for at the LHC. Extracted from [93]

These experiments, as well as upcoming LHC runs, will be able to further extend the mass range probed for heavy Majorana neutrinos.

10.2 Open questions

The last significant announcement in high-energy physics dates back to the detection of the Higgs boson in 2012. At present, there are multiple research directions hoping to find the BSM physics that was promised in the last century. In this section, we briefly review experiments for neutrino mass and flavour matrices, proton decay experiments, as well as cosmological searches.

As discussed in the 2021 review of Feruglio and Romanino [35], we presently know that $\sin\theta_{13}$ is non-zero with over 5σ confidence. There is a preference for normal mass ordering at a 3σ level.

However, we do not know the mass of the lightest neutrino – the present upper bound is at 0.8 eV [23]. Furthermore, neutrinoless double-beta decay has not yet been detected [21]. In addition, searches for the lightest supersymmetric particle at the LHC have turned up empty-handed [94]. There is much left to learn about the fundamental theory of reality.

10.3 Experimental efforts

10.3.1 Neutrino experiments

In this section, we review experimental efforts that aim to elucidate mysteries surrounding neutrinos. Firstly, the ICARUS experiment at Fermilab aims to detect mass oscillations associated with a hypothetical fourth sterile neutrino to investigate anomalies of ν_e events in the ν_μ beam [95, 96]. ICARUS uses a liquid argon technology with 476 tons of active mass; particles travelling through the argon medium ionise the argon nuclei, and electrons produced drift to wire planes that pick up electric fields, along with scintillator tubes to detect emitted photons [97].

Secondly, the DUNE experiment aims to study flavour physics by measuring neutrino oscillations, as well as using different off-axis angles to produce different neutrino beam spectra. The experiment aims to pin down the CP violation phase δ, as well as constraining parameter space for models of neutrino flavour that typically introduce discrete symmetry groups whose sum rules can be experimentally falsified [98]. The facility is expected to be completed in 2025 [99].

Thirdly, T2K is a long-baseline neutrino experiment in Japan designed to study neutrino oscillations. The Baby MIND detector is a new muon range detector that was recently installed at T2K, to measure cross-sections of charged current neutrino and anti-neutrino interactions to constrain neutrino cross-sections [100, 101]. Understanding neutrino cross-sections has significant implications for astrophysics in understanding core-collapse supernovae, along with studying non-standard neutrino interactions that can lead to flavour violation.

In summary, there are many upcoming experimental searches which will help further constrain parameter and model space, as well as elucidate more properties of the mysterious neutrino, which is to date the best indication of the existence of BSM physics.

10.3.2 Proton decay

Experiments have thus far ruled out proton decay with $\tau_p \geq 10^{34}$ years [86]. The Hyper-Kamiokande experiment [102] aims to further push the current experimental bound, probing up to 10^{35} years after 10 years of runtime.

Proton decay is a prediction of all GUT models – new constraints on the proton decay lifetime can help to definitively rule out the minimal version of $SU(5)$, and lend additional support for SUSY $SU(5)$. This would help to further constrain the size of model space.

10.3.3 Cosmological constraints

High-precision cosmology aims to use power spectra from the Cosmic Microwave Background and Baryon Acoustic Oscillations to constrain the total sum of neutrino masses in the universe. This is because massive neutrinos contribute differently to cosmological observables compared to cold dark matter [103, 104]. Hence, cosmology represents an orthogonal way to set limits on the physical properties of neutrinos and can be used to check experimental results.

10.4 Future Work

In future work, it is still possible to study higher theories. Early realisations of Orbifold GUTs were proposed by Hebecker et al. [105, 106], in which supersymmetric GUTs are realised in 5-spacetime dimensions, and then broken down into MSSM.

These theories embed Minkowski spacetime onto a circle, governed by discrete symmetries with boundary conditions tuned to lead to $SU(5)$ breaking. These objects are also found in string theory and represent new areas for exploration.

Bibliography

[1] CERN. "CERN experiments observe particle consistent with long-sought higgs boson". 2012. URL: https://press.cern/news/press-release/cern/cern-experiments-observe-particle-consistent-long-sought-higgs-boson.

[2] A Astbury. "Found: W and z". In: *Physics Bulletin* 34.10 (1983), p. 434.

[3] Claudio Campagnari and Melissa Franklin. "The discovery of the top quark". In: *Reviews of Modern Physics* 69.1 (1997), p. 137.

[4] Alaina G. Levine. "This Month in physics history". Ed. by David Voss. 2017. URL: https://www.aps.org/publications/apsnews/201711/history.cfm.

[5] Matthew D Schwartz. "Quantum field theory and the standard model". Cambridge university press, 2014.

[6] Lewis H Ryder. "Quantum field theory". Cambridge university press, 1996.

[7] Chong-Sa Lim, Toshiyuki Morii, and Shankar Nath Mukherjee. "The Physics of the Standard Model and Beyond". New Jersey: World Scientific Publishing Company, June 15, 2002. 312 pp. ISBN: 978-981-02-4571-9.

[8] Anthony Zee. "Group theory in a nutshell for physicists". Vol. 17. Princeton University Press, 2016.

[9] Xing Fan et al. "Measurement of the electron magnetic moment". In: *Physical review letters* 130.7 (2023), p. 071801.

[10] Tsung-Dao Lee and Chen-Ning Yang. "Question of parity conservation in weak interactions". In: *Physical Review* 104.1 (1956), p. 254.

[11] Chien-Shiung Wu et al. "Experimental test of parity conservation in beta decay". In: *Physical review* 105.4 (1957), p. 1413.

[12] David Griffiths. “Introduction to elementary particles”. John Wiley & Sons, 2020.

[13] John Ellis, Mary K. Gaillard, and Dimitri V. Nanopoulos. “A Historical Profile of the Higgs Boson”. In: *The Standard Theory of Particle Physics*. WORLD SCIENTIFIC, Aug. 2016, 255–274. DOI: 10.1142/9789814733519_0014. URL: http://dx.doi.org/10.1142/9789814733519_0014.

[14] Sven Krippendorf, Fernando Quevedo, and Oliver Schlotterer. “Cambridge lectures on supersymmetry and extra dimensions”. In: *arXiv preprint arXiv:1011.1491* (2010).

[15] Gianfranco Bertone and Dan Hooper. “History of dark matter”. In: *Reviews of Modern Physics* 90.4 (2018), p. 045002.

[16] Zhi-Zhong Xing and Shun Zhou. “Neutrinos in Particle Physics, Astronomy and Cosmology”. Advanced Topics in Science and Technology in China. Berlin, Heidelberg: Springer, 2011. ISBN: 978-3-642-17559-6 978-3-642-17560-2. DOI: 10.1007/978-3-642-17560-2. URL: https://link.springer.com/10.1007/978-3-642-17560-2 (visited on 08/31/2023).

[17] S Bilenky. “Neutrino oscillations: from an historical perspective to the present status”. In: *Journal of Physics: Conference Series* 718 (May 2016), p. 062005. ISSN: 1742-6588, 1742-6596. DOI: 10.1088/1742-6596/718/6/062005. URL: https://iopscience.iop.org/article/10.1088/1742-6596/718/6/062005 (visited on 10/22/2023).

[18] Archil Kobakhidze et al. “Closing up a light stop window in natural SUSY at LHC”. In: *Physics Letters B* 755 (2016), pp. 76–81.

[19] CMS collaboration et al. “Search for supersymmetry in final states with disappearing tracks in proton-proton collisions at $\sqrt{s} = 13$ TeV”. In: *arXiv preprint arXiv:2309.16823* (2023).

[20] Goran Senjanović. “Proton decay and grand unification”. In: *AIP Conference Proceedings*. Vol. 1200. 1. American Institute of Physics. 2010, pp. 131–141.

[21] Michelle J Dolinski, Alan WP Poon, and Werner Rodejohann. “Neutrinoless double-beta decay: status and prospects”. In: *Annual Review of Nuclear and Particle Science* 69 (2019), pp. 219–251.

[22] KS Hirata et al. "Experimental study of the atmospheric neutrino flux". In: *Physics Letters B* 205.2-3 (1988), pp. 416–420.

[23] "Direct neutrino-mass measurement with sub-electronvolt sensitivity". In: *Nature Physics* 18.2 (2022), pp. 160–166.

[24] Samoil Bilenky. "Introduction to the Physics of Massive and Mixed Neutrinos". 2nd ed. 2018 edition. New York, NY: Springer, Apr. 3, 2018. 293 pp. ISBN: 978-3-319-74801-6.

[25] Armen Tumasyan et al. "Search for long-lived heavy neutral leptons with displaced vertices in proton-proton collisions at $\sqrt{s} = 13$ TeV". In: *Journal of High Energy Physics* 2022.7 (2022), pp. 1–56.

[26] Masataka Fukugita and Tsutomu Yanagida. "Physics of neutrinos: and applications to astrophysics". Vol. 593. Springer, 2003.

[27] Steven Weinberg. "Effective gauge theories". In: *Physics Letters B* 91.1 (Mar. 24, 1980), pp. 51–55. ISSN: 0370-2693. DOI: 10.1016/0370-2693(80)90660-7. URL: https://www.sciencedirect.com/science/article/pii/0370269380906607 (visited on 09/07/2023).

[28] Mikhail Bilenky and Arcadi Santamaria. "One-loop effective lagrangian for an extension of the standard model with a heavy charged scalar singlet". In: *Nuclear Physics B* 420.1 (May 30, 1994), pp. 47–93. ISSN: 0550-3213. DOI: 10.1016/0550-3213(94)90375-1. URL: https://www.sciencedirect.com/science/article/pii/0550321394903751 (visited on 09/07/2023).

[29] Yi Cai et al. "From the Trees to the Forest: A Review of Radiative Neutrino Mass Models". In: *Frontiers in Physics* 5 (2017). ISSN: 2296-424X. URL: https://www.frontiersin.org/articles/10.3389/fphy.2017.00063 (visited on 09/14/2023).

[30] Masataka Fukugita and Atsuto Suzuki. "Physics and astrophysics of neutrinos". Springer Science & Business Media, 2013.

[31] Ernest Ma. "Verifiable Radiative Seesaw Mechanism of Neutrino Mass and Dark Matter". In: *Physical Review D* 73.7 (Apr. 14, 2006), p. 077301. ISSN: 1550-7998, 1550-2368. DOI: 10.1103/PhysRevD.73.077301. arXiv: hep-

ph/0601225. URL: http://arxiv.org/abs/hep-ph/0601225 (visited on 08/27/2023).

[32] Romain Bouchand. "Radiative Neutrino Mass Generation and Renormalization Group Running in the Ma-Model". In: (2012).

[33] G. 't Hooft and M. Veltman. "Scalar one-loop integrals". In: *Nuclear Physics B* 153 (Jan. 1, 1979), pp. 365–401. ISSN: 0550-3213. DOI: 10.1016/0550-3213(79)90605-9. URL: https://www.sciencedirect.com/science/article/pii/0550321379906059 (visited on 09/19/2023).

[34] Particle Data Group et al. "Review of particle physics". In: *Progress of theoretical and experimental physics* 2022.8 (2022), p. 083C01.

[35] Ferruccio Feruglio and Andrea Romanino. "Lepton Flavour Symmetries". Mar. 21, 2021. DOI: 10.48550/arXiv.1912.06028. arXiv: 1912.06028[hep-ph]. URL: http://arxiv.org/abs/1912.06028 (visited on 09/09/2023).

[36] Lorenzo Calibbi and Giovanni Signorelli. "Charged lepton flavour violation: an experimental and theoretical introduction". In: *La Rivista del Nuovo Cimento* 41 (2018), pp. 71–174.

[37] Kenzo Nakamura. "Review of particle physics". In: *Journal of Physics G: Nuclear and Particle Physics* 37.7 A (2010).

[38] S. T. Petcov. "Discrete flavour symmetries, neutrino mixing and leptonic CP violation". In: *The European Physical Journal C* 78.9 (Sept. 4, 2018), p. 709. ISSN: 1434-6052. DOI: 10.1140/epjc/s10052-018-6158-5. URL: https://doi.org/10.1140/epjc/s10052-018-6158-5 (visited on 09/09/2023).

[39] CMS collaboration et al. "Measurement of CKM matrix elements in single top quark t-channel production in proton-proton collisions at $\sqrt{s} = 13$ TeV". In: *arXiv preprint arXiv:2004.12181* (2020).

[40] P. F. Harrison, D. H. Perkins, and W. G. Scott. "Tri-Bimaximal Mixing and the Neutrino Oscillation Data". In: *Physics Letters B* 530.1 (Mar. 2002), pp. 167–173. ISSN: 03702693. DOI: 10.1016/S0370-2693(02)01336-9. arXiv: hep-ph/0202074. URL: http://arxiv.org/abs/hep-ph/0202074 (visited on 10/18/2023).

[41] J. Kubo et al. “The Flavor Symmetry”. In: *Progress of Theoretical Physics* 109.5 (May 2003), 795–807. ISSN: 1347-4081. DOI: 10.1143/ptp.109.795. URL: http://dx.doi.org/10.1143/PTP.109.795.

[42] Jisuke Kubo, Hiroshi Okada, and Fumiaki Sakamaki. “Higgs potential in a minimal S3 invariant extension of the standard model”. In: *Physical Review D* 70.3 (Aug. 2004). ISSN: 1550-2368. DOI: 10.1103/physrevd.70.036007. URL: http://dx.doi.org/10.1103/PhysRevD.70.036007.

[43] Ernest Ma. “Neutrino mass matrix from S4 symmetry”. In: *Physics Letters B* 632.2–3 (Jan. 2006), 352–356. ISSN: 0370-2693. DOI: 10.1016/j.physletb.2005.10.019. URL: http://dx.doi.org/10.1016/j.physletb.2005.10.019.

[44] Tatsuo Kobayashi et al. “Stringy origin of non-Abelian discrete flavor symmetries”. In: *Nuclear Physics B* 768.1–2 (Apr. 2007), 135–156. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2007.01.018. URL: http://dx.doi.org/10.1016/j.nuclphysb.2007.01.018.

[45] Guido Altarelli and Ferruccio Feruglio. “Tri-Bimaximal Neutrino Mixing, A4 and the Modular Symmetry”. In: *Nuclear Physics B* 741.1 (May 2006), pp. 215–235. ISSN: 05503213. DOI: 10.1016/j.nuclphysb.2006.02.015. arXiv: hep-ph/0512103. URL: http://arxiv.org/abs/hep-ph/0512103 (visited on 10/19/2023).

[46] Woulter Denkins. “A4 Family Symmetry”. PhD thesis. Rijks Universiteit Groningen, Aug. 4, 2011. URL: https://fse.studenttheses.ub.rug.nl/9755/1/A4_family_symmetry.pdf (visited on 10/17/2023).

[47] Zhi-zhong Xing. “Quark Mass Hierarchy and Flavor Mixing Puzzles”. en. In: *International Journal of Modern Physics A* 29.30 (Dec. 2014). arXiv:1411.2713 [hep-ph], p. 1430067. ISSN: 0217-751X, 1793-656X. DOI: 10.1142/S0217751X14300671.

[48] Colin D Froggatt and Holger Bech Nielsen. “Hierarchy of quark masses, Cabibbo angles and CP violation”. In: *Nuclear Physics B* 147.3-4 (1979), pp. 277–298.

[49] F. Capozzi et al. “Global analysis of oscillation parameters”. In: *Journal of Physics: Conference Series* 1312.1 (Sept. 1, 2019), p. 012005. ISSN: 1742-6588, 1742-6596. DOI: 10.1088/1742-6596/1312/1/012005. URL: https://iopscience.iop.org/article/10.1088/1742-6596/1312/1/012005 (visited on 11/01/2023).

[50] P. F. de Salas et al. “Status of neutrino oscillations 2018: 3 hint for normal mass ordering and improved CP sensitivity”. In: *Physics Letters B* 782 (July 10, 2018), pp. 633–640. ISSN: 0370-2693. DOI: 10.1016/j.physletb.2018.06.019. URL: https://www.sciencedirect.com/science/article/pii/S0370269318304672 (visited on 11/01/2023).

[51] S. Gariazzo et al. “Neutrino masses and their ordering: global data, priors and models”. In: *Journal of Cosmology and Astroparticle Physics* 2018.3 (Mar. 2018), p. 011. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2018/03/011. URL: https://dx.doi.org/10.1088/1475-7516/2018/03/011 (visited on 11/01/2023).

[52] FJ Hasert et al. “Observation of neutrino-like interactions without muon or electron in the Gargamelle neutrino experiment”. In: *Nuclear Physics B* 73.1 (1974), pp. 1–22.

[53] Jogesh C Pati and Abdus Salam. “Lepton number as the fourth color"”. In: *Physical Review D* 10.1 (1974), p. 275.

[54] G. Senjanovic and R. N. Mohapatra. “Exact left-right symmetry and spontaneous violation of parity”. In: *Physical Review D* 12.5 (Sept. 1, 1975), pp. 1502–1505. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.12.1502. URL: https://link.aps.org/doi/10.1103/PhysRevD.12.1502 (visited on 09/18/2023).

[55] Rabindra N Mohapatra and Jogesh C Pati. “Left-right gauge symmetry and an “isoconjugate” model of CP violation”. In: *Physical Review D* 11.3 (1975), p. 566.

[56] R. N. Mohapatra and J. C. Pati. “"Natural" left-right symmetry”. In: *Physical Review D* 11.9 (May 1, 1975), pp. 2558–2561. ISSN: 0556-2821. DOI: 10.1103/

PhysRevD.11.2558. URL: https://link.aps.org/doi/10.1103/PhysRevD.11.2558 (visited on 09/18/2023).

[57] Melissa Harris. "Left-Right Symmetric Model". In: ().

[58] N. G. Deshpande et al. "Left-right-symmetric electroweak models with triplet Higgs field". In: *Physical Review D* 44.3 (Aug. 1, 1991), pp. 837–858. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.44.837. URL: https://link.aps.org/doi/10.1103/PhysRevD.44.837 (visited on 09/24/2023).

[59] P. S. Bhupal Dev et al. "Vacuum structure of the left-right symmetric model". Nov. 16, 2018. arXiv: 1811.06869[hep-ph]. URL: http://arxiv.org/abs/1811.06869 (visited on 09/21/2023).

[60] Alessio Maiezza, Goran Senjanovi, and Juan Carlos Vasquez. "Higgs Sector of the Left-Right Symmetric Theory". In: *Physical Review D* 95.9 (May 8, 2017), p. 095004. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.95.095004. arXiv: 1612.09146[hep-ex,physics:hep-ph]. URL: http://arxiv.org/abs/1612.09146 (visited on 09/21/2023).

[61] Akira Kokado and Takesi Saito. "New mixing angles in the left-right symmetric model". In: *Physical Review D* 92.12 (Dec. 8, 2015), p. 125008. ISSN: 1550-7998, 1550-2368. DOI: 10.1103/PhysRevD.92.125008. URL: https://link.aps.org/doi/10.1103/PhysRevD.92.125008 (visited on 09/24/2023).

[62] Rabindra N. Mohapatra and Goran Senjanovi. "Neutrino masses and mixings in gauge models with spontaneous parity violation". In: *Physical Review D* 23.1 (Jan. 1, 1981). Publisher: American Physical Society, pp. 165–180. DOI: 10.1103/PhysRevD.23.165. URL: https://link.aps.org/doi/10.1103/PhysRevD.23.165 (visited on 08/20/2023).

[63] B. S. Balakrishna and R. N. Mohapatra. "Radiative fermion masses from new physics at TeV scale". In: *Physics Letters B* 216.3 (Jan. 12, 1989), pp. 349–352. ISSN: 0370-2693. DOI: 10.1016/0370-2693(89)91129-5. URL: https://www.sciencedirect.com/science/article/pii/0370269389911295 (visited on 09/21/2023).

[64] Rabindra N. Mohapatra and Yongchao Zhang. "TeV scale universal seesaw, vacuum stability and heavy Higgs". In: *Journal of High Energy Physics* 2014.6 (June 11, 2014), p. 72. ISSN: 1029-8479. DOI: 10.1007/JHEP06(2014)072. URL: https://doi.org/10.1007/JHEP06(2014)072 (visited on 09/18/2023).

[65] Chandan Hati et al. "Neutrino Masses and Leptogenesis in LeftRight Symmetric Models: A Review From a Model Building Perspective". In: *Frontiers in Physics* 6 (2018). ISSN: 2296-424X. URL: https://www.frontiersin.org/articles/10.3389/fphy.2018.00019 (visited on 10/05/2023).

[66] Cesar Bonilla et al. "Neutrino phenomenology in a left-right D 4 symmetric model". In: *Physical Review D* 102.3 (Aug. 7, 2020), p. 036006. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.102.036006. URL: https://link.aps.org/doi/10.1103/PhysRevD.102.036006 (visited on 10/05/2023).

[67] Matteo Bertolini. "Lectures on Supersymmetry". en. In: (2024).

[68] Julius Wess and Jonathan Bagger. "Supersymmetry and Supergravity". Revised edition. Princeton, N.J: Princeton University Press, Mar. 3, 1992. 260 pp. ISBN: 978-0-691-02530-8.

[69] Gordon L Kane. "Perspectives on supersymmetry". Vol. 18. World Scientific, 2010.

[70] Joseph D. Lykken. "Introduction to Supersymmetry". en. In: arXiv:hep-th/9612114 (Dec. 1996). arXiv:hep-th/9612114. URL: http://arxiv.org/abs/hep-th/9612114.

[71] Stephen P. Martin. "A Supersymmetry Primer". In: vol. 18. July 1998, pp. 1–98. DOI: 10.1142/9789812839657_0001. arXiv: hep-ph/9709356. URL: http://arxiv.org/abs/hep-ph/9709356 (visited on 10/30/2023).

[72] Sidney Coleman and Jeffrey Mandula. "All possible symmetries of the S matrix". In: *Physical Review* 159.5 (1967), p. 1251.

[73] Rudolf Haag, Jan T Łopuszański, and Martin Sohnius. "All possible generators of supersymmetries of the S-matrix". In: *Nuclear Physics B* 88.2 (1975), pp. 257–274.

[74] Stuart Raby. “Supersymmetric Grand Unified Theories”. Vol. 939. Lecture Notes in Physics. Cham: Springer International Publishing, 2017. ISBN: 978-3-319-55253-8. DOI: 10.1007/978-3-319-55255-2. URL: http://link.springer.com/10.1007/978-3-319-55255-2.

[75] Benjamin C. Allanach and F. Quevedo. “Supersymmetry”. In: 2016. URL: https://www.damtp.cam.ac.uk/user/examples/3P7.pdf.

[76] Pran Nath. “Supersymmetry, supergravity, and unification”. Cambridge University Press, 2016.

[77] Rabindra N Mohapatra. “Unification and supersymmetry”. In: *Particle Physics* (1986), p. 336.

[78] Marina von Steinkirch. “Introduction to Group Theory for Physicists”. In: (2011), p. 108.

[79] Howard Georgi and Sheldon L Glashow. “Unity of all elementary-particle forces”. In: *Physical Review Letters* 32.8 (1974), p. 438.

[80] Gernot Eichmann. “Lecture Notes for QCD and Hadron Physics, Appendix A”. In: (2020). URL: http://cftp.ist.utl.pt/~gernot.eichmann/2020-QCDHP/App-SU(N).pdf.

[81] Eugene Golowich. “Scalar-mediated proton decay”. In: *Physical Review D* 24.11 (1981), p. 2899.

[82] Ilja Doršner, Svjetlana Fajfer, and Nejc Košnik. “Heavy and light scalar leptoquarks in proton decay”. In: *Physical Review D* 86.1 (2012), p. 015013.

[83] A Masiero et al. “Naturally massless Higgs doublets in supersymmetric SU (5)”. In: *Physics Letters B* 115.5 (1982), pp. 380–384.

[84] Benjamin Grinstein. “A supersymmetric SU (5) gauge theory with no gauge hierarchy problem”. In: *Nuclear Physics B* 206.3 (1982), pp. 387–396.

[85] K. Kobayashi et al. “Search for nucleon decay via modes favored by supersymmetric grand unification models in Super-Kamiokande-I”. In: *Physical Review D* 72.5 (Sept. 2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.72.052007. URL: http://dx.doi.org/10.1103/PhysRevD.72.052007.

[86] A Takenaka et al. "Search for proton decay via p→ e+ π 0 and p→ μ+ π 0 with an enlarged fiducial volume in Super-Kamiokande I-IV". In: *Physical Review D* 102.11 (2020), p. 112011.

[87] Ugo Amaldi, Wim de Boer, and Hermann Fürstenau. "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP". In: *Physics Letters B* 260.3-4 (1991), pp. 447–455.

[88] Guido Altarelli, Ferruccio Feruglio, and Isabella Masina. "From minimal to realistic supersymmetric SU(5) grand unification". en. In: *Journal of High Energy Physics* 2000.11 (Jan. 2001), p. 040. ISSN: 1126-6708. DOI: 10.1088/1126-6708/2000/11/040.

[89] Hajime Ishimori, Yusuke Shimizu, and Morimitsu Tanimoto. "S 4 flavor symmetry of quarks and leptons in SU (5) GUT". In: *Progress of theoretical physics* 121.4 (2009), pp. 769–787.

[90] C. S. Lam. "Unique horizontal symmetry of leptons". In: *Physical Review D* 78.7 (Oct. 2008). ISSN: 1550-2368. DOI: 10.1103/physrevd.78.073015. URL: http://dx.doi.org/10.1103/PhysRevD.78.073015.

[91] M Miskaoui and MA Loualidi. "Leptogenesis, fermion masses and mixings in a SUSY SU (5) GUT with D4 flavor symmetry". In: *Journal of High Energy Physics* 2021.11 (2021), pp. 1–39.

[92] Naoki Yamatsu. "Finite-Dimensional Lie Algebras and Their Representations for Unified Model Building". 2020. arXiv: 1511.08771 [hep-ph].

[93] CMS collaboration et al. "Probing heavy Majorana neutrinos and the Weinberg operator through vector boson fusion processes in proton-proton collisions at sqrt(s) = 13 TeV". In: *arXiv preprint arXiv:2206.08956* (2022).

[94] Armen Tumasyan et al. "Search for supersymmetry in final states with a single electron or muon using angular correlations and heavy-object identification in proton-proton collisions at $\sqrt{s} = 13$ TeV". In: *Journal of High Energy Physics* 2023.9 (2023), pp. 1–58.

[95] A Aguilar et al. "Evidence for neutrino oscillations from the observation of ν_e appearance in a ν_μ beam". In: *Physical Review D* 64.11 (2001), p. 112007.

[96] Filippo Varanini, ICARUS Collaboration, et al. "ICARUS status and near future". In: *Journal of Physics: Conference Series*. Vol. 718. 6. IOP Publishing. 2016, p. 062064.

[97] Jacob Zettlemoyer. "Current Status and Prospects of the ICARUS Experiment". en. In: (2023). URL: `https://indico.fnal.gov/event/59656/contributions/268980/attachments/168509/225794/JCZ_ICARUS_UsersMeeting.pdf`.

[98] DUNE collaboration et al. "Deep Underground Neutrino Experiment (DUNE): Far detector technical design report. Volume I. Introduction to DUNE". In: *Journal of Instrumentation* 15.8 (2020).

[99] DUNE collaboration et al. "The Deep Underground Neutrino Experiment: Status and Prospects". In: (2023). URL: `https://indico.cern.ch/event/1303630/contributions/5626510/attachments/2767654/4821212/231207_TAU2023_tzanov_final.pdf`.

[100] M. Antonova et al. "Baby MIND: a magnetized segmented neutrino detector for the WAGASCI experiment". In: *Journal of Instrumentation* 12.07 (July 2017), C07028–C07028. ISSN: 1748-0221. DOI: `10.1088/1748-0221/12/07/c07028`. URL: `http://dx.doi.org/10.1088/1748-0221/12/07/C07028`.

[101] Saba Parsa. "Baby MIND detector first physics run". In: *arXiv preprint arXiv:2004.05245* (2020).

[102] Ke Abe et al. "Hyper-Kamiokande design report". In: *arXiv preprint arXiv:1805.04163* (2018).

[103] Guillermo Franco Abellán et al. "Improved cosmological constraints on the neutrino mass and lifetime". In: *Journal of High Energy Physics* 2022.8 (2022), pp. 1–25.

[104] F Couchot et al. "Cosmological constraints on the neutrino mass including systematic uncertainties". In: *Astronomy & Astrophysics* 606 (2017), A104.

[105] Arthur Hebecker and John March-Russell. "A minimal $S1/(Z2 \times Z2')$ orbifold GUT". In: *Nuclear Physics B* 613.1-2 (2001), pp. 3–16.

[106] Arthur Hebecker and John March-Russell. "The structure of GUT breaking by orbifolding". In: *Nuclear Physics B* 625.1-2 (2002), pp. 128–150.

[107] Ashok Das. "Lectures on quantum field theory". World Scientific, 2020.

[108] Richard P Feynman and Murray Gell-Mann. "Theory of the Fermi interaction". In: *Physical Review* 109.1 (1958), p. 193.

[109] Takeshi Araki et al. "Measurement of neutrino oscillation with KamLAND: Evidence of spectral distortion". In: *Physical review letters* 94.8 (2005), p. 081801.

[110] Y Ashie et al. "Measurement of atmospheric neutrino oscillation parameters by Super-Kamiokande I". In: *Physical Review D* 71.11 (2005), p. 112005.

[111] Hajime Ishimori et al. "Non-Abelian Discrete Symmetries in Particle Physics". In: *Progress of Theoretical Physics Supplement* 183 (2010), pp. 1–163. ISSN: 0375-9687. DOI: 10.1143/PTPS.183.1. arXiv: 1003.3552[hep-ph,physics:hep-th]. URL: http://arxiv.org/abs/1003.3552 (visited on 10/10/2023).

[112] Vladimir Tello. "Connections between the high and low energy violation of Lepton and Flavor numbers in the minimal left-right symmetric model". PhD thesis. SISSA, Trieste, 2012.

[113] Timo Weigand. "Quantum Field Theory I + II". In: (2014).

[114] Walter Pfeifer. "The Lie algebras su (N)". Springer, 2003. URL: https://walterpfeifer.ch/liealgebra/LieAlg_wieBuch4.pdf.

Appendix A

Calculations for introducing QFT

A.1 Klein-Gordon Equation

Throughout this thesis, we use the mostly positive Minkowski metric:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.1}$$

We start with the Einstein relation, where $-p^{\mu}p_{\mu} = m^2$. One may refer to Section 2.2 of [6].

$$E^2 = |p|^2 + m^2 \tag{A.2}$$

replacing with QM operators:

$$H^2 = P^2 + m^2 \tag{A.3}$$

inserting a state ket $|\phi(t)\rangle$ and inserting the time-dependent Schrodinger Equation:

$$-\frac{\partial^2}{\partial t^2}|\phi(t)\rangle = P^2|\phi(t)\rangle + |\phi(t)\rangle m^2 \tag{A.4}$$

Taking the position representation by inserting $\langle x|$:

$$-\frac{\partial^2}{\partial t^2}\phi(t,x) = -c^2\nabla^2\phi(t,x) + \phi(t,x)m^2 \tag{A.5}$$

$$\Box\phi(t,x) = m^2\phi(t,x) \tag{A.6}$$

$$(-\Box + m^2)\phi(t,x) = 0 \tag{A.7}$$

which gives us the Klein-Gordon (KG) equation.

A.2 Negative probabilities from the Klein-Gordon equation

For comparison purposes, we first compute the probability conservation current in QM. We start with the time-dependent Schrodinger Equation and its complex conjugate.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{A.8}$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* \tag{A.9}$$

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \tag{A.10}$$

Apply the vector identity $\nabla\cdot(f\vec{A}) = f\nabla\cdot\vec{A} + (\nabla f)\cdot\vec{A}$ and split the Laplacian:

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\hbar}{2mi}\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0 \tag{A.11}$$

We can define the probability density $\rho = |\psi|^2$ and the probability current $\vec{J}$ as $\frac{\hbar}{2mi}\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*)$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \tag{A.12}$$

which expresses the conservation of probability current. This expresses that probability density is positive, which is acceptable.

Let us do a similar procedure using the KG equation and its conjugate.

$$(\Box - m^2)\phi = 0 \qquad (\Box - m^2)\phi^* = 0 \tag{A.13}$$

$$\phi^*(\Box - m^2) = 0 \qquad \phi(\Box - m^2)\phi^* = 0 \tag{A.14}$$

$$\phi^*\Box\phi - \phi\Box\phi^* = 0 \tag{A.15}$$

$$\phi^*\partial_\mu\partial^\mu\phi - \phi\partial_\mu\partial^\mu\phi^* = 0 \tag{A.16}$$

$$\phi^*\partial_\mu\partial^\mu\phi - (\partial_\mu\phi)(\partial^\mu\phi^*) - \phi\partial_\mu\partial^\mu\phi^* + (\partial_\mu\phi)(\partial^\mu\phi^*) = 0 \tag{A.17}$$

Using the reverse product rule,

$$\partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = 0 \tag{A.18}$$

Splitting the 4-derivative into the time and spatial components:

$$\frac{\partial}{\partial t}\left(-\phi^*\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi^*}{\partial t}\right) + \nabla\cdot(\phi^*\nabla\phi - \phi\nabla\phi^*) = 0 \tag{A.19}$$

Define the probability density ρ:

$$\rho = \frac{i}{2m}\left(-\phi^* \frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi^*}{\partial t}\right) \tag{A.20}$$

and the probability current $\vec{J}$:

$$\vec{J} = \frac{1}{2mi}\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right) \tag{A.21}$$

which allows us to write the conservation of 4-current $\partial_\mu J^\mu = 0$.

Now, we refer to Section 2.2 of [6]. We check that the plane wave is a solution of the Klein-Gordon equation.

$$\begin{aligned}
0 &= (-\Box + m^2)\phi(t,x) & \text{(A.22)}\\
&= (-\Box + m^2)e^{-i\omega t + i\vec{k}\cdot\vec{x}} & \text{(A.23)}\\
&= (-\Box + m^2)e^{-ik\cdot x} & \text{(A.24)}\\
&= \left(-\omega^2 + \vec{k}^2 + m^2\right)\phi(t,x) & \text{(A.25)}\\
&= 0 & \text{(A.26)}
\end{aligned}$$

Any arbitrary solution can be written as a combination of plane waves, so we just need to check one plane wave. Note that by definition of the wave vector, $\omega^2 = k^2 + m^2$, which is seen from the invariance of $p_\mu p^\mu$.

Consider $\phi(t,x) \propto e^{\pm ik\cdot x}$ as a wavefunction, and let us compute the probability density. Assuming that $\phi(x) = e^{-ik\cdot x}$,

$$\begin{aligned}
\rho &= \frac{i}{2m}\left(e^{-ik\cdot x}\frac{\partial e^{-ik\cdot x}}{\partial t} - e^{-ik\cdot x}\frac{\partial e^{ik\cdot x}}{\partial t}\right) & \text{(A.27)}\\
&= \frac{i}{2m}(ik^0 + ik^0) & \text{(A.28)}
\end{aligned}$$

Since $k^0 = \sqrt{\vec{k}^2 + m^2} = \pm\omega_k$,

$$= \mp\frac{\omega_k}{m} \tag{A.29}$$

which is not acceptable as probability density is not allowed to be negative.

A.3 Dirac bilinear properties

Once we apply the Hermitian conjugate to the Dirac equation, we obtain the Dirac adjoint. We refer to Chapter 3.4 of [107].

We know that the Dirac adjoint is $\overline{\psi} = \Psi^\dagger \gamma^0$. By combining the adjoint with the original field, we can construct a few Dirac bilinear objects. We assume that the Lorentz transformation of a Dirac spinor is achieved by the operator:

$$\psi' = S\psi \tag{A.30}$$

Requiring that the Dirac equation is covariant under Lorentz transformations, we see that:

$$S^{-1}\gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu \tag{A.31}$$

The Dirac adjoint transforms as:

$$\overline{\psi'} = \overline{\psi} S^{-1} \tag{A.32}$$

The scalar bilinear is formed by $\overline{\Psi}\Psi$:

$$\overline{\psi'}\psi' = \overline{\psi} S^{-1} S \psi = \overline{\psi}\psi \tag{A.33}$$

The vector bilinear is formed by $\overline{\Psi}\gamma^\mu \Psi$, which transforms as a Lorentz vector:

$$\overline{\psi'}\gamma^\mu \psi' = \overline{\psi} S^{-1} \gamma^\mu S \psi \tag{A.34}$$

$$= \Lambda^\mu_\nu \overline{\psi} \gamma^\nu \psi \tag{A.35}$$

The pseudoscalar bilinear looks like $\overline{\psi}\gamma_5\psi$:

$$\overline{\psi'}\gamma_5\psi' = \overline{\psi}' \frac{i}{24} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \psi' \tag{A.36}$$

$$= \overline{\psi} \frac{i}{24} \epsilon_{\mu\nu\alpha\beta} S^{-1} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta S \psi \tag{A.37}$$

$$= \overline{\psi} \frac{i}{24} \epsilon_{\mu\nu\alpha\beta} S^{-1} \gamma^\mu S S^{-1} \gamma^\nu S S^{-1} \gamma^\alpha S S^{-1} \gamma^\beta S \psi \tag{A.38}$$

$$= \overline{\psi} \frac{i}{24} \epsilon_{\mu\nu\alpha\beta} \Lambda^\mu_\sigma \Lambda^\nu_\rho \Lambda^\alpha_\lambda \Lambda^\beta_\delta \gamma^\sigma \gamma^\rho \gamma^\lambda \gamma^\delta \psi \tag{A.39}$$

Using the property of the determinant where $\epsilon_{\mu\nu\alpha\beta} \Lambda^\mu_\sigma \Lambda^\nu_\rho \Lambda^\alpha_\lambda \Lambda^\beta_\delta = \det(\Lambda) \epsilon_{\sigma\rho\lambda\delta}$:

$$= \det(\Lambda) \overline{\psi} \frac{i}{24} \epsilon_{\sigma\rho\lambda\delta} \gamma^\sigma \gamma^\rho \gamma^\lambda \gamma^\delta \psi \tag{A.40}$$

$$= \det(\Lambda)\overline{\psi}\gamma_5\psi \tag{A.41}$$

Under a parity transformation, $\det(\Lambda) = -1$. This property is connected to the parity violation observed in the β-decay. The axialvector bilinear is the vector version of the pseudoscalar bilinear, and this is the actual object that Feynman and Gellman used to construct the weak current in 1958 [108].

A.4 Gauge boson masses

In this section, we follow [7] and proceed with our calculation to show how the gauge bosons obtain masses, and to see the Higgs boson mass term appear explicitly.

We use the Kibble parameterisation with the choice of:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = e^{i\frac{\vec{\tau}\cdot\vec{\xi}}{2v}} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H) \end{pmatrix} \tag{A.42}$$

where v is the vev, and H represents the Higgs degree of freedom. To deal with the pre-factor in the parameterisation of the Higgs, we choose a $SU(2)$ transformation where

$$U(x) = e^{-i\frac{\vec{\tau}\cdot\vec{\xi}(x)}{2v}} \tag{A.43}$$

This choice is the unitary gauge. We transform the rest of the fields using the unitary gauge. To see where the gauge boson masses come from, we look at the Higgs sector of the Lagrangian.

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\phi)^\dagger(D_\mu\phi) - V(\phi^\dagger\phi) \tag{A.44}$$

Expanding the covariant derivative and substituting our choice of the Higgs vev alignment, we obtain

$$= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{v^2}{2}\begin{pmatrix} 0 & 1 \end{pmatrix}\left(g\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu + \frac{g'}{2}Y_\mu\right)^\dagger\left(g\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu + \frac{g'}{2}Y_\mu\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.45}$$

Using the explicit expression for the Pauli matrices,

$$\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu = \frac{1}{2}(\tau^1 W^1_\mu + \tau^2 W^2_\mu + \tau^3 W^3_\mu) = \frac{1}{2}\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix} \tag{A.46}$$

we obtain after matrix multiplication

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{v^2}{8}\left(g^2 W^1_\mu W^{1\mu} + g^2 W^2_\mu W^{2\mu} + (gW^3_\mu - g'Y'_\mu)^2\right) + \dots \tag{A.47}$$

Rewriting $W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right)$

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2 v^2}{4} W^+_\mu W^{-\mu} + \frac{v^2}{8}(gW^3_\mu - g'Y'_\mu)^2 + \dots \tag{A.48}$$

where the mass of the $W^\pm$ boson is $\frac{gv}{2}$. We rewrite the third term into quadratic form in order to diagonalise it in the next step.

$$\frac{v^2}{8}(gW^3_\mu - g'Y'_\mu)^2 = \frac{v^2}{8}\begin{pmatrix} W^3_\mu & Y_\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3\mu} & Y^\mu \end{pmatrix} \tag{A.49}$$

We can use the following basis

$$\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W^{3\mu} \\ Y^\mu \end{pmatrix} \tag{A.50}$$

where we write $\tan\theta_W = \frac{g'}{g}$ and θ_W is the Weinberg mixing angle. This will give us

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2 v^2}{4} W^+_\mu W^{-\mu} + \frac{v^2}{8}(g^2 + g'^2) Z_\mu Z^\mu + 0 A_\mu A^\mu + \dots \tag{A.51}$$

where we have a Z boson that is heavier than $W^\pm$, and a massless photon A^μ. Now looking only at terms from the Higgs potential:

$$V(\phi^\dagger\phi) = -\frac{\mu^2 v^2}{4} + \frac{1}{2}(2\mu^2)H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4 \tag{A.52}$$

where we identify the mass of the Higgs boson as $m_H = \sqrt{2\mu^2}$ that will be determined via experiment.

Appendix B

Neutrino oscillations and neutrino masses

For this section, we will show the key steps in the calculation involving neutrino oscillations referencing Chapter 7 of [17], with the goal of showing how observations of neutrino oscillations imply non-zero neutrino mass.

B.1 Heisenberg Uncertainty Principle

In neutrino oscillation experiments, neutrino energies $E \gg m_\nu$. For ultra-relativistic neutrinos, we have:

$$|p_i - p_k| \simeq \frac{m_k^2 - m_i^2}{2E}, (i \neq k) \tag{B.1}$$

with observed mass-squared differences of:

$$\Delta m_{12}^2 \simeq 8 \cdot 10^{-5}\mathrm{eV}^2, \quad \Delta m_{23}^2 \simeq 2.4 \cdot 10^{-3}\mathrm{eV}^2 \tag{B.2}$$

The Heisenberg uncertainty relation implies that:

$$\Delta p \simeq \frac{1}{d} \tag{B.3}$$

where d is the characteristic dimension of a neutrino wave packet.

Since the momentum of the ultrarelativistic neutrinos is much less than the momentum uncertainty, it is impossible to resolve the production of neutrinos with small mass-squared differences. As a result, coherent superpositions of states of neutrinos with definite masses are produced and detected.

Instead, experiments involving large distances between neutrino sources and detectors have to be used. For neutrino beams of energy $\Delta E \simeq \frac{\Delta m^2_{ki}}{2E}$ and $\Delta t \simeq L$, which is the distance between neutrino source and detector, the time-energy uncertainty principle tells us that:

$$\Delta E \Delta t \gtrsim 1 \tag{B.4}$$

which implies the ratio $\frac{L}{E}$ should be tuned such that

$$\Delta m^2_{ki} \frac{L}{2E} \gtrsim 1 = 2\Delta_{ki} \tag{B.5}$$

which makes small mass-squared differences observable.

B.2 Two-Neutrino Oscillations

We will consider the simplest case of two-neutrino mixing as a pedagogical demonstration, where the detected neutrino of flavour l is a superposition of two mass eigenstates

$$\nu_l = \sum_i U_{li} \nu_i \tag{B.6}$$

where l indexes the flavour $(e, \mu, \tau)$and i the mass eigenstate, with U the orthogonal mixing matrix:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{B.7}$$

where θ is the mixing angle.

We will now derive the equation describing the transition probability between neutrino flavours at the detector. Let us assume that at $t = 0$, a flavour neutrino state $|\nu_l\rangle$ is produced. To satisfy the time-dependent Schrodinger Equation, the flavour state $|\nu_l\rangle_t$ with l indexing flavour, is written as:

$$|\nu_l\rangle_t = e^{-iHt} |\nu_l\rangle \tag{B.8}$$

where H is the free Hamiltonian. We insert a complete basis of flavour neutrinos:

$$|\nu_l\rangle_t = \sum_{l'} |\nu_{l'}\rangle \langle \nu_{l'}| e^{-iHt} |\nu_l\rangle \tag{B.9}$$

We identify the sandwich term as a probability amplitude A for the transition from ν_l to $\nu_{l'}$. We make another basis expansion over the mass eigenstates, labelled by i

$$A(\nu_l \to \nu_{l'}) = \langle \nu_{l'} | e^{-iHt} | \nu_l \rangle \tag{B.10}$$

$$= \sum_i \langle \nu_{l'} | e^{-iHt} | i \rangle \langle i | \nu_l \rangle \tag{B.11}$$

where $H|i\rangle = E_i|i\rangle$

$$= \sum_i \langle \nu_{l'} | i \rangle | e^{-iE_i t} \langle i | \nu_l \rangle \tag{B.12}$$

Identifying the scalars as elements of rotation matrices where:

$$\langle \nu_{l'i} | i \rangle = U_{l'i} \tag{B.13}$$

$$\langle i | \nu_l \rangle = U^*_{li} \tag{B.14}$$

it follows that

$$A(\nu_l \to \nu_{l'}) = \sum_i U_{l'i} e^{-iE_i t} U^*_{li} \tag{B.15}$$

and $A(\nu_l \to \nu_{l'})$ is the probability amplitude of finding the flavour l' at time t from a beam of flavour l at $t = 0$, with the rotation U matrices describing the mixture of mass eigenstates for each flavour.

Hence, the probability of the transition from $l \to l'$ is the square of the amplitude:

$$P(\nu_l \to \nu_{l'}) = \left| \sum_i U_{l'i} e^{-iE_i t} U^*_{li} \right|^2 \tag{B.16}$$

By applying the unitarity condition as well as doing some rewriting, the transition probability can eventually be written as:

$$P(\nu_l \to \nu_{l'}) = \delta_{l'l} - 4 \sum_i |U_{li}|^2 (\delta_{l'l} - |U_{l'i}|^2) \sin^2 \Delta_{pi} \tag{B.17}$$

$$+ 8 \sum_{i>k} \mathrm{Re}(U_{l'i} U^*_{li} U^*_{l'k} U_{lk}) \cos(\Delta_{pi} - \Delta_{pk}) \tag{B.18}$$

$$+ \mathrm{Im}(U_{l'i} U^*_{li} U^*_{l'k} U_{lk}) \sin(\Delta_{pi} - \Delta_{pk})] \sin \Delta_{pi} \sin \Delta_{pk} \tag{B.19}$$

where p is a fixed index.

In the two neutrino case, there are only two mass eigenstates to consider, with one squared mass difference $\Delta m^2 = m_2^2 - m_1^2$.

To implement this, we can fix $p = 1, i = 2$. There are no terms in the sum involving $i > k$, and the expression simplifies accordingly. We rewrite Δ_{pi} into $\Delta = \frac{\Delta m^2 L}{4E}$. Hence, the survival probability of ν_l can be written as:

$$P(\nu_l \to \nu_l') = 4|U_{l2}|^2(1 - |U_{l2}|^2)\sin^2\Delta, \quad (l' \neq l) \tag{B.20}$$

Substituting $U_{l2} = sin\theta$ and writing Δ explicitly:

$$P(\nu_l \to \nu_l') = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E} \tag{B.21}$$

We can hence introduce the characteristic length L^{osc}:

$$L^{(\text{osc})} = 4\pi \frac{E}{\Delta m^2} \tag{B.22}$$

which is one period of the neutrino amplitude oscillation. Hence, the observation of neutrino oscillation within a finite detector size necessarily implies that the neutrino mass is finite. The detections by KamLAND [109] and SuperKamiokande [110] definitively demonstrated the existence of neutrino mass.

Appendix C

Bi-unitary transformation

We reference Appendix A of Bilenky's monograph [24].

Consider M, a general complex $n \times n$ matrix that can be decomposed into $M = UmU^{-1}$ where U is a unitary transformation matrix and m is a diagonal matrix containing the eigenvalues of M. Note that $(MM^\dagger)$ is itself Hermitian:

$$(MM^\dagger)^\dagger = MM^\dagger \tag{C.1}$$

$$MM^\dagger = UmU^{-1}(UmU^{-1})^\dagger \tag{C.2}$$

$$= UmmU^\dagger \tag{C.3}$$

Recall that $U^{-1} = U^\dagger$ for a unitary matrix. Consider another unitary matrix V, where $VV^\dagger = I$.

$$UmmU^\dagger = (UmV^\dagger)(VmU^\dagger) \tag{C.4}$$

$$M = UmV^\dagger \tag{C.5}$$

This is the bi-unitary expansion for an arbitrary complex matrix M.

Appendix D

A Brief Primer on Group Theory

This review chapter is a set of notes from 'Group Theory in a Nutshell' by Anthony Zee [8]. This text provides a strong pedagogical overview of group theory in a physics context. We will be relying on many of these ideas throughout the main text.

Since this is mostly mathematical in nature, this has been relegated to the Appendix.

D.1 Groups

A group G consists of a set of group elements $\{g_\alpha\}$ which we can compose (multiply) together: $g_\alpha \cdot g_\beta = g_\gamma$.

- Associativity: $(g_\alpha \cdot g_\beta) \cdot g_\gamma = g_\alpha \cdot (g_\beta \cdot g_\gamma)$
- Existence of the identity: There exists an identity element I such that $I \cdot g_\alpha = g_\alpha$ and $g_\alpha \cdot I = g_\alpha$
- Existence of the inverse: For each group element g_α there exists a unique group element, the inverse of g_α denoted by g_α^{-1}, such that $g_\alpha^{-1} g_\alpha = I, g_\alpha g_\alpha^{-1} = I$
- Closure: For the composition of two elements $g_\alpha \cdot g_\beta = g_\gamma$, the product is also a group element of G.

For example, the two square roots of $1, \{1, -1\}$ form the group Z_2 under multiplication. The group has an identity element and is closed under multiplication.

D.1.1 Lagrange's theorem

Let a group G with n elements have a subgroup H with m elements. m is a factor of n.

D.1.2 Direct product of groups

Consider two groups F and G whose elements are denoted by f and g, with m and n elements respectively. We can define a new group $H \equiv F \otimes G$, the direct product of F and G consisting of the elements (f, g).

The inverse of (f, g) is (f^{-1}, g^{-1}), and $F \otimes G$ has mn elements.

For example, the group $Z_2 \otimes Z_2$ is known as the Klein 4-group, which is the direct product of two Z_2 groups.

It consists of four elements: $I = (1, 1), A = (-1, 1), B = (1, -1), C = (-1, -1)$. This is different from the Z_4 group, as the square of any element in the Klein 4-group is equal to the identity, but this is not true for Z_4.

Elements of F can be a subgroup of $F \otimes G$, written as (f, I). Elements of G will be written as (I, g). These two objects, (f, I) and (I, g) commute.

The direct product will be repeatedly used to construct larger groups out of smaller ones. For example, the Standard Model gauge group is $SU(3) \otimes SU(2) \otimes U(1)$.

D.2 Rotations and Lie Algebras

D.2.1 Working in the two-dimensional plane

In Cartesian coordinates, a point P is labelled by two real numbers (x, y). Consider a different coordinate axis with the same origin O, representing the same point P.

The two coordinates (x, y) and (x', y') will be related by a linear transformation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{D.1}$$

written more compactly as $\vec{r'} = R(\theta)\vec{r}$, and $R(\theta)$ is the 2D rotation matrix that allows us to change from one set of coordinate labels to another.

R is an orthogonal matrix, where $R^T R = I$. This is a consequence of imposing invariance of vector length under rotation.

D.2.2 Infinitesimal variation

Consider a rotation through an infinitesimal angle θ. To first order,

$$R(\theta) \simeq I + A \tag{D.2}$$

where I is the identity, and A is an infinitesimal matrix that is linear in θ.

Imposing $R^T R = I$, we find

$$R^T R \simeq (I + A)^T (I + A) \simeq (I + A^T + A) = I \tag{D.3}$$

which implies $A^T = -A$, and A is anti-symmetric. There is only one possible 2-by-2 anti-symmetric matrix:

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{D.4}$$

and infinitesimal rotations will have the form

$$R = I + \theta\mathcal{J} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2) \tag{D.5}$$

and the matrix $\mathcal{J}$ can be said to be the generator of the rotation group. This is consistent with the previous trigonometric picture for small angles θ, requiring no trigonometry at all.

What happens when we perform an infinitesimal rotation a large number of times?

$$R(\theta) = \lim_{N\to\infty} (R(\theta/N))^N = \lim_{N\to\infty} \left(1 + \frac{\theta\mathcal{J}}{N}\right)^N = e^{\theta\mathcal{J}} \tag{D.6}$$

It is easy to see that expanding the exponential series will reproduce $R(\theta)$ in trigonometric form.

To summarise, the approach is to:

- Specify what is invariant by rotations (or transformations).
- Solve the condition for infinitesimal transformations, building up finite transformations from infinitesimal transformations

D.2.3 In three dimensions

The rotation group is fixed by writing $R \simeq I + A$ and then requiring that A is anti-symmetric. For $N = 3$, there are three distinct anti-symmetric matrices:

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{D.7}$$

Hence, we can write:

$$R(\theta) = \exp(\theta_i \mathcal{J}_i) \tag{D.8}$$

for $i = x, y, z$.

D.2.4 Lie Algebra

While a Lie group is characterised by multiplication, its Lie Algebra is characterised by commutation. In general, rotations do not commute. The commutator $[A, B] \equiv AB - BA$ measures the lack of commutativity.

By explicit computation, we find that for orthogonal rotations,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{D.9}$$

where coefficients ϵ_{ijk} are real.

Hence, the commutation relation tells us about how the rotations multiply with each other.

D.2.5 Structure constants

Consider a group with elements $g(\theta_1, \theta_2, \cdots)$ with continuous parameters such that $g(0, 0, \cdots) = I$.

Similar to the discussion of the rotation group, we will:

- Expand the group elements around the identity such that $g \simeq I + A$
- Write $A = i\theta_a T_a$ as a linear combination of the generators T_a, which depends on the nature of the group.
- Choose two group elements near the identity: $g_1 \simeq I + A, g_2 \simeq I + B$, and $g_1 g_2 g_1^{-1} \simeq I + B + [A, B]$, where the commutator $[A, B]$ captures what happens near the identity.
- We can write $B = i\theta'_b T_b$ as a linear combination of the generators T_b. These generators are related by the commutation relation $[T_a, T_b] = i f_{abc} T_c$

The commutation relations between the generators define a Lie algebra, and f_{abc} are the structure constants of the group. The Lie algebra determines the Lie group.

A continuous group is characterised by its Lie algebra, which will be useful for understanding the structure of gauge-invariant Lagrangians.

Stated more formally, a Lie algebra is a linear space spanned by linear combinations $\theta_i \mathcal{J}_i$ of the generators.

D.3 Representation Theory

Just as we can multiply group elements to form other group elements, we want to represent group elements by matrices such that can multiply them to obtain other matrix representations of the group, where $D(g_1)D(g_2) = D(g_1 g_2)$.

For example, the rotation group $SO(3)$ is defined by 3×3 orthogonal matrices, and are known as defining or fundamental representations.

Given that a group can have many different representations, we denote this as $D^{(r)}(g)$ for the matrix representing the element g in representation r.

We can also define the character $\chi^{(r)}$ of this representation:

$$\chi^{(r)}(g) \equiv \mathrm{tr}(D^{(g)}(g)) \tag{D.10}$$

Elements of a group, each with characters, can be divided into equivalence classes. Two elements g_1 and g_2 are said to be equivalent ($g_1 \sim g_2$) if there exists another element f such that

$$g_1 = f^{-1} g_2 f \tag{D.11}$$

Using trace cyclicity, $g_1 \sim g_2$ implies that they have the same character. This forms an equivalency class.

D.3.1 Reducible and Irreducible representations

We can construct representations of arbitrary dimensions using direct sums of lower dimensional irreducible representations, by stacking irreducible representations in block diagonal form.

We can also use the direct product of matrices to construct a larger representation out of smaller ones.

Given two representations of dimension d_r and d_s, with representation matrices $D^{(r)}(g)$ and $D^{(s)}(g)$ respectively, we can define the direct product representation defined by the direct product matrices $D(g) = D^{(r)}(g) \otimes D^{(s)}(g)$.

These objects are realised in quantum mechanics as wave functions.

D.4 Constructing character tables

This section lists the key results for constructing character tables:

The dimensions of the irreducible representations are:

$$\sum_r d_r^2 = N(G)$$

where $N(G)$ is the number of group elements, and d_r the dimension of the representation r.

The orthogonality relations are:

$$\sum_c n_c (\chi^{(r)})^* \chi^{(s)}(c) = N(G) \delta^{rs}$$

$$\sum_r \chi^{(r)}(c)^* \chi^{(r)}(c') = \frac{N(G)}{n_c} \delta c c'$$

D.4.1 A_3 example character table

Consider A_3, the group of even permutations of three objects. It contains three objects: the identity I, and the permutations $c = (123)$ and $a = (132)$, which are the clockwise and anti-clockwise permutations.

There are three equivalence classes, and using Eq D.4, we can only have $1^2+1^2+1^2 = 3$, with three 1-dimensional irreducible representations labelled as $1, 1', 1''$. In one dimension, the representation matrix is a number. Since $c^3 = I$, c can only be one of the roots of unity: $1, \omega \equiv e^{i2\pi/3}, \omega^*\omega = \omega^2$.

The character table is hence shown in Table D.1.

n_c		1	$1'$	$1''$
1	I	1	1	1
1	c=(123)	1	ω	ω^*
1	a=(132)	1	ω^*	ω

Table D.1: Character table of A_3

D.5 Tensor representations

To construct the irreducible representations of $SO(N)$, we cannot stack smaller representations together in a block diagonal matrix, as such a representation is by definition reducible. To obtain larger irreducible representations, we introduce tensors.

Consider T^{ij}, which carries two indices, with i, j running from 1 to N in N-dimensional space.

If these objects transform under rotations according to

$$T^{ij} \to T'^{ij} = R^{ik} R^{jl} T^{kl} \tag{D.12}$$

then T transforms like a tensor, similar to the transformation law of a vector.

D.5.1 Invariant symbols

The orthogonality in $SO(N)$ is manifest in $\delta^{ij} R^{ik} R^{jl} = \delta^{kl}$, which is equivalent to $R^T R = I$.

The requirement that $\det R = 1$ has some consequences. The determinant can be written using the anti-symmetric symbol $\varepsilon^{ijk\ldots n}$.

The determinant is defined for any matrix R by

$$\varepsilon^{ijk\ldots n} R^{ip} R^{jq} R^{kr} \ldots R^{ns} = \varepsilon^{pqr\ldots s} \det R \tag{D.13}$$

and for rotations, $\det R = 1$, such that when it acts on rotation matrices, they disappear.

The symbols δ and ε can be thought of as invariant symbols: when acted on by rotation matrices, they turn into themselves.

D.5.2 Dual tensors

For an anti-symmetric tensor A^{ij}, we can define another anti-symmetric tensor $B^{k\ldots n} = \varepsilon^{ijk\ldots n} A^{ij}$ with $N-2$ indices. These tensors are dual to each other. In classical electrodynamics, these are the field strength tensor $F^{\mu\nu}$ and the dual field strength tensor $F^{*\mu\nu}$.

Hence, we can construct a large class of irreducible representations of $SO(N)$, known as the tensor representations.

D.5.3 Irreducible representations of $SO(3)$

The number of independent components contained in $S^{i_1 i_2 \ldots i_j}$ gives the dimension of the irreducible representation furnished by $S^{i_1 i_2 \ldots i_j}$ labelled by the integer j.

We find that the dimension d goes up linearly with j, following $d = 2j + 1$.

Structure constants of a Lie algebra furnish a representation of the adjoint algebra, whose dimension is given by the number of generators.

D.5.4 Lie algebra of $SO(3)$ and Ladder Operators

The Lie algebra of $SO(3)$ is a prototype for all Lie algebras. The Lie algebra is found to be:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{D.14}$$

To represent the Lie algebra, we want to find three matrices J_x, J_y, J_z such that the commutation relations above are satisfied. Since rotations are given by exponentials of linear combinations of J, this gives matrices representing the rotation group.

Since J_x, J_y, J_z do not mutually commute, they cannot be simultaneously diagonalised. We choose to work in a basis where J_z is diagonal.

Similar to transforming between a transversely polarised electromagnetic wave to a circularly polarised electromagnetic wave where $z = x + iy, z^* = x - iy$, we define $J_\pm \equiv J_x \pm iJ_y$

We rewrite the commutation relations as:

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z \tag{D.15}$$

We also write the eigenvector of J_z with eigenvalue m as $|m\rangle$.

Consider the state $J_+|m\rangle$ and act on it with J_z:

$$\begin{aligned} J_z J_+|m\rangle &= (J_+J_z + [J_z, J_+])|m\rangle = (J_+J_z + J_+)|m\rangle = (mJ_+ + J_+)|m\rangle && \text{(D.16)} \\ &= (m+1)J_+|m\rangle && \text{(D.17)} \end{aligned}$$

Hence, $J_+|m\rangle$ is an eigenvector of J_z with eigenvalue $m+1$, and $J_+|m\rangle = c_{m+1}|m+1\rangle$ with a complex number determined by some normalisation constant.

Similarly, $J_-|m\rangle = b_{m-1}|m-1\rangle$ with another unknown normalisation constant. These are the ladder operators: J_+ is a raising operator, and J_- is a lowering operator.

Since the representation is finite-dimensional, the ladder must terminate. Denote the maximum value of m using j, and there exists a state $|j\rangle$ such that $J_+|j\rangle = 0$.

$$\begin{aligned} 0 &= \langle j|J_-J_+|j\rangle && \text{(D.18)} \\ &= \langle j|J_+J_- - 2J_z|j\rangle && \text{(D.19)} \\ &= |c_j|^2 - 2j && \text{(D.20)} \end{aligned}$$

It can be shown that the ladder terminates, and there are a total of $(2j+1)$ states. This enumerates the representations of the group $SO(3)$ using the method of Lie algebra.

For $j = \frac{1}{2}$, we have a 2-dimensional representation with the states $|-\frac{1}{2}\rangle$ and $|\frac{1}{2}\rangle$. This will describe the electron spin.

D.6 Tensors and Representations of Special Unitary Groups $SU(N)$

The discussion of the $SU(N)$ group is similar to the $SO(N)$ group in the sense that going to the $SU(N)$ group is similar to going from real to complex numbers.

For $SO(N)$,

$$O^T O = 1, \qquad \det O = 1$$

The first condition ensures that the length squared is invariant under transformation.

The goal is to generalise the discussion for $SO(N)$ by considering linear transformations on complex vectors ψ, where ψ consists of N complex numbers. We want linear transformations

$$\psi \to \psi' = U\psi$$

where $\sum_{j=1}^{N} \psi^{j*}\psi = \psi^\dagger \psi$ is invariant. This also implies that for two arbitrary complex vectors ζ and ψ, the quadratic forms $\zeta^\dagger\psi$ and $\psi^\dagger\zeta$ are invariant.

$$(\zeta')^\dagger(\psi')^\dagger = \zeta^\dagger U^\dagger U\psi \tag{D.21}$$

$$= \zeta^\dagger\psi \tag{D.22}$$

$$\implies U^\dagger U = I \tag{D.23}$$

D.6.1 The group $U(N)$

The group $U(N)$ consists of the $N \times N$ matrices that are unitary:

$$U^\dagger U = I \tag{D.24}$$

For unitary matrices, taking determinant,

$$U^\dagger U = I \tag{D.25}$$

$$\det(U^\dagger U) = (\det U^\dagger)(\det U) \tag{D.26}$$

$$= (\det U)^*(\det U) \tag{D.27}$$

$$= |\det U|^2 \tag{D.28}$$

$$= 1 \tag{D.29}$$

which implies the freedom to choose $\det U = e^{i\alpha}$. By imposing $\det U = 1$, we remove this degree of freedom.

These two conditions define the group $SU(N)$.

D.7 Symmetries in Lagrangians and Hamiltonians

If a symmetry is present, the action S has to be invariant under a symmetry transformation. Rephrased, the action transforms like a singlet, and S belongs to the trivial representation of the symmetry group.

Noether's theorem states that each conservation law is associated with a continuous symmetry of the Lagrangian.

D.8 $SU(2)$

The group $SU(2)$ came with the discovery of electron spin. We will first discuss interesting properties of $SU(2)$.

D.8.1 $SU(2)$ is locally isomorphic to $SO(3)$

Any 2-by-2 Hermitian traceless matrix can be written as a linear combination of the three Pauli matrices, shown below:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{D.30}$$

where $X = \vec{x} \cdot \vec{\sigma}$ three real coefficients. Explicitly,

$$X = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{D.31}$$

which is traceless and Hermitian. The determinant of X, which is invariant, is given by $-\vec{x}^2$, which is the length squared of the vector.

Consider $X' \equiv U^\dagger X U$, where U is a unitary matrix with determinant 1, which is an element of $SU(2)$.

It is clear that X' is also Hermitian and traceless:

$$(X')^\dagger = (U^\dagger X U)^\dagger \tag{D.32}$$
$$= U^\dagger X^\dagger U \tag{D.33}$$
$$= U^\dagger X U \tag{D.34}$$
$$= X' \tag{D.35}$$
$$\mathrm{Tr}(X') = \mathrm{Tr}(U^\dagger X U) \tag{D.36}$$
$$= \mathrm{Tr}(X U U^\dagger) \tag{D.37}$$
$$= \mathrm{Tr}(X) \tag{D.38}$$
$$= 0 \tag{D.39}$$

Computing the determinant of the new vector X',

$$\det X' = -(\vec{x}')^2 \tag{D.40}$$
$$= \det U^\dagger X U \tag{D.41}$$
$$= (\det U^\dagger)(\det X)(\det U) \tag{D.42}$$
$$= \det X \tag{D.43}$$
$$= -(\vec{x})^2 \tag{D.44}$$

so these two vectors have the same length as well: the 3-vector $\vec{x}$ is transformed into the 3-vector $\vec{x}'$, where the length is kept invariant. This defines a rotation R. Hence, we can associate an element of $SO(3)$ with any element U of $SU(2)$.

where a rotation by 2π has only reached $-I$. To reach I, φ would have to go all the way to 4π.

By the time we get around $SU(2)$ once, the corresponding rotation has gone around $SO(3)$.

D.9 $SU(2)$ and electron spin

By proposing that electrons have spin $\frac{1}{2}$, it means that the electron wave function has two components and transforms like $\psi \to e^{i\vec{\phi}\cdot\vec{\sigma}/2}\psi$

For a rotation around, for instance, the z-axis, ψ flips sign to $-\psi$. This works out because observable quantities in quantum mechanics are bilinear in ψ and $\psi^\dagger$, with the probability density being $\psi^\dagger\psi$. Hence, under a 2π rotation, ψ flips its sign.

Appendix E

A_4 Product Rules

We will expand on the calculations found in Altarelli and Feruglio [45].

In one dimension, we can represent the singlets as follows:

$$1 \quad S = 1 \quad T = 1 \tag{E.1}$$

$$1' \quad S = 1 \quad T = e^{i4\pi/3} \equiv \omega^2 \tag{E.2}$$

$$1'' \quad S = 1 \quad T = e^{i2\pi/3} \equiv \omega \tag{E.3}$$

We infer that $1' \otimes 1'' = 1, 1' \otimes 1' = 1''$.

Writing the tensor product of two triplets as $(a_1, a_2, a_3) \otimes (b_1, b_2, b_3) = a \otimes b = (ab)$, we use $\odot$ to represent element-wise multiplication to simplify presentation. This operation is also known as the Hadamard product.

Noting that under the transformation $(Ta) \otimes (Tb)$, we find:

$$(Ta) \otimes (Tb) = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{pmatrix} \odot (ab) \tag{E.4}$$

This implies that a linear combination of $\{a_1b_1, a_2b_3, a_3b_2\}$ transforms into itself, $\{a_1b_2, a_2b_1, a_3b_3\}$ transforms with an ω^2 in front, similar to $1'$, and $\{a_1b_3, a_2b_2, a_3b_1\}$ with ω in front, similar to $1''$.

Under the transformation $(Sa) \otimes (Sb)$:

$$
\begin{aligned}
a_1 b_1 &\to \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \odot (ab) \\
a_1 b_2 &\to \frac{1}{9}\begin{pmatrix} -2 & 1 & -2 \\ 4 & -2 & 4 \\ 4 & -2 & 4 \end{pmatrix} \odot (ab) \\
a_1 b_3 &\to \frac{1}{9}\begin{pmatrix} -2 & -2 & 1 \\ 4 & 4 & -2 \\ 4 & 4 & -2 \end{pmatrix} \odot (ab) \\
a_2 b_1 &\to \frac{1}{9}\begin{pmatrix} -2 & 4 & 4 \\ 1 & -2 & -2 \\ -2 & 4 & 4 \end{pmatrix} \odot (ab) \\
a_2 b_2 &\to \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 1 \end{pmatrix} \odot (ab) \\
a_2 b_3 &\to \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ -2 & -2 & 1 \\ 4 & 4 & -2 \end{pmatrix} \odot (ab) \\
a_3 b_1 &\to \frac{1}{9}\begin{pmatrix} -2 & 4 & 4 \\ -2 & 4 & 4 \\ 1 & -2 & -2 \end{pmatrix} \odot (ab) \\
a_3 b_2 &\to \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ 4 & -2 & 4 \\ -2 & 1 & -2 \end{pmatrix} \odot (ab) \\
a_3 b_3 &\to \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \odot (ab)
\end{aligned}
\tag{E.5}
$$

Using the hint from the invariance under $T(ab)T$, we can see that the following combinations are indeed invariant under the transformation $S(ab)S$.

$$
\begin{aligned}
1 \equiv (ab) &= (a_1b_1 + a_2b_3 + a_3b_2) \\
&\to \frac{1}{9}\left(\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & -2 \\ -2 & -2 & 1 \\ 4 & 4 & -2 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 4 \\ 4 & -2 & 4 \\ -2 & 1 & -2 \end{pmatrix}\right) \odot (ab) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \odot (ab) \\
&= (a_1b_1 + a_2b_3 + a_3b_2) \\
1' \equiv (ab)' &= (a_1b_2 + a_2b_1 + a_3b_3) \\
&\to \frac{1}{9}\left(\begin{pmatrix} -2 & 1 & -2 \\ 4 & -2 & 4 \\ 4 & -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 4 \\ 1 & -2 & -2 \\ -2 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\right) \odot (ab) \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \odot (ab) \\
&= (a_1b_2 + a_2b_1 + a_3b_3) \\
1'' \equiv (ab)'' &= (a_1b_3 + a_2b_2 + a_3b_1) \\
&\to \frac{1}{9}\left(\begin{pmatrix} -2 & -2 & 1 \\ 4 & 4 & -2 \\ 4 & 4 & -2 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 4 \\ -2 & 4 & 4 \\ 1 & -2 & -2 \end{pmatrix}\right) \odot (ab) \\
&= (a_1b_3 + a_2b_2 + a_3b_1)
\end{aligned}
$$

To complete the Lie Algebra where $3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A$, we need to construct two more triplets from the tensor product using 6 independent combinations. We will only verify that these combinations are valid.

$$
2a_1b_1 - a_2b_3 - a_3b_2 \to \frac{1}{9}\begin{pmatrix} -6 & -6 & -6 \\ -6 & 12 & 3 \\ -6 & 3 & 12 \end{pmatrix} \odot (ab)
$$

$$2a_3b_3 - a_1b_2 - a_2b_1 \to \frac{1}{9}\begin{pmatrix} 12 & 3 & -6 \\ 3 & 12 & -6 \\ -6 & -6 & -6 \end{pmatrix} \odot (ab)$$
$$2a_2b_2 - a_1b_3 - a_3b_1 \to \frac{1}{9}\begin{pmatrix} 12 & -6 & 3 \\ -6 & -6 & -6 \\ 3 & -6 & 6 \end{pmatrix} \odot (ab)$$

which forms the symmetric triplet 3_S. We now compute the combinations for the antisymmetric triplets:

$$\begin{aligned} a_2b_3 - a_3b_2 &\to \frac{1}{9}\begin{pmatrix} 0 & 6 & -6 \\ -6 & 0 & -6 \\ 6 & 3 & 0 \end{pmatrix} \odot (ab) \\ a_1b_2 - a_2b_1 &\to \frac{1}{9}\begin{pmatrix} 0 & -3 & -6 \\ 3 & 0 & 6 \\ 6 & -6 & 0 \end{pmatrix} \odot (ab) \\ a_1b_3 - a_3b_1 &\to \frac{1}{9}\begin{pmatrix} 0 & -6 & -3 \\ 6 & 0 & -6 \\ 3 & 6 & 0 \end{pmatrix} \odot (ab) \end{aligned} \tag{E.6}$$

whose anti-symmetric terms form the anti-symmetric triplet 3_A.

Appendix F

D_4 product rules

This section references [66, 111] and shows the computations more explicitly.

The D_4 group has order 8 and represents the symmetry of a square. This is generated by the $\pi/2$ rotation A and reflection B, where $A^4 = B^2 = I$, and $BAB = A^{-1}$.

The character table is shown below:

For the one-dimensional irreducible representations, the multiplication rules are:

$$1^a \times 1 = 1^a, 1' \times 1'' = 1''', 1' \times 1''' = 1'', 1'' \times 1''' = 1' \tag{F.1}$$

We choose the two-dimensional representation of the generators A and B to be:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{F.2}$$

We will now obtain the product decomposition rules for two doublets $2 \otimes 2$. With some foresight, we should be able to decompose $2 \otimes 2 = 1 + 1' + 1'' + 1'''$ into the four singlets.

Consider $x = (x_1, x_2), y = (y_1, y_2)$. We want to find combinations that are invariant under A or B transformations. Let $(xy) = x \otimes y$ and $\odot$ denote the Hadamard

Class	h	1	$1'$	$1''$	$1'''$	2
C_1	1	1	1	1	1	2
C_2	4	1	-1	-1	1	0
C_1'	2	1	1	1	1	-2
C_2'	2	1	1	-1	-1	0
C_2''	2	1	-1	1	-1	0

Table F.1: Characters of D_4 representation element-wise product.

$$
\begin{aligned}
(Ax) \otimes (Ay) &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \odot (xy) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
&= \begin{pmatrix} -x_1y_1 & x_1y_2 \\ x_2y_1 & -x_2y_2 \end{pmatrix} \\
(Bx) \otimes (By) &= \begin{pmatrix} x_2y_2 & x_2y_1 \\ x_1y_2 & x_1y_1 \end{pmatrix}
\end{aligned}
$$

We observe some invariant combinations and make the following assignments with the help of the character table.

$$
x_1y_2 + x_2y_1 \sim 1, \quad x_1y_2 - x_2y_1 \sim 1', \quad x_1y_1 + x_2y_2 \sim 1'', \quad x_1y_1 + x_2y_2 \sim 1''' \quad \text{(F.3)}
$$

Appendix G

Scalar potential breaking in Left-Right Symmetric Models

G.1 Introduction

In this section, we will explicitly show important results relating to the breaking of the scalar potential of the Left-Right Symmetric Model.

The most general scalar potential invariant under $SU(2)_L \otimes SU(2)_R$ contains 17 terms, and respects both $\mathcal{C}$ and $\mathcal{P}$ parities [112].

Given the large number of parameters, there are many extra degrees of freedom. Hence, we will work backwards from a desired symmetry breaking to infer the required constraints.

We will show that requiring a minimised potential implies the seesaw relationship between the v_L and v_R, in which a large right-handed energy scale naturally implies a small electroweak scale without fine-tuning.

G.2 The scalar potential

$$\begin{aligned}
V = &-\mu_1^2 \mathrm{Tr}(\Phi^\dagger\Phi) - \mu_2^2\left(\mathrm{Tr}(\tilde{\Phi}\Phi^\dagger) + \mathrm{Tr}(\tilde{\Phi}^\dagger\Phi)\right) - \mu_3^2\left(\mathrm{Tr}(\Delta_L\Delta_L^\dagger) + \mathrm{Tr}(\Delta_R\Delta_R^\dagger)\right) + \lambda_1 \mathrm{Tr}(\Phi^\dagger\Phi)^2 \\
&+ \lambda_2\left(\mathrm{Tr}(\tilde{\Phi}\Phi^\dagger)^2 + \mathrm{Tr}(\tilde{\Phi}^\dagger\Phi)^2\right) + \lambda_3 \mathrm{Tr}(\tilde{\Phi}\Phi^\dagger)\mathrm{Tr}(\tilde{\Phi}^\dagger\Phi) + \lambda_4 \mathrm{Tr}(\Phi^\dagger\Phi)\left(\mathrm{Tr}(\tilde{\Phi}\Phi^\dagger) + \mathrm{Tr}(\tilde{\Phi}^\dagger\Phi)\right) \\
&+ \rho_1\left(\mathrm{Tr}(\Delta_L\Delta_L^\dagger)^2 + \mathrm{Tr}(\Delta_R\Delta_R^\dagger)^2\right) + \rho_2\left(\mathrm{Tr}(\Delta_L\Delta_L)\mathrm{Tr}(\Delta_L^\dagger\Delta_L^\dagger) + \mathrm{Tr}(\Delta_R\Delta_R)\mathrm{Tr}(\Delta_R^\dagger\Delta_R^\dagger)\right) \\
&+ \rho_3 \mathrm{Tr}(\Delta_L\Delta_L^\dagger)\mathrm{Tr}(\Delta_R\Delta_R^\dagger) + \rho_4\left(\mathrm{Tr}(\Delta_L\Delta_L)\mathrm{Tr}(\Delta_R^\dagger\Delta_R^\dagger) + \mathrm{Tr}(\Delta_L^\dagger\Delta_L^\dagger)\mathrm{Tr}(\Delta_R\Delta_R)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_1 \mathrm{Tr}(\Phi^\dagger\Phi)\left(\mathrm{Tr}(\Delta_L\Delta_L^\dagger) + \mathrm{Tr}(\Delta_R\Delta_R^\dagger)\right) + \alpha_2\left(\mathrm{Tr}(\Delta_L\Delta_L^\dagger)\mathrm{Tr}(\tilde{\Phi}\Phi^\dagger) + \mathrm{Tr}(\Delta_R\Delta_R^\dagger)\mathrm{Tr}(\tilde{\Phi}^\dagger\Phi)\right)\\
&+ \alpha_2^*\left(\mathrm{Tr}(\tilde{\Phi}\Phi^\dagger)\mathrm{Tr}(\Delta_L\Delta_L^\dagger) + \mathrm{Tr}(\tilde{\Phi}^\dagger\Phi)\mathrm{Tr}(\Delta_R\Delta_R^\dagger)\right) + \alpha_3\left(\mathrm{Tr}(\Phi\Phi^\dagger\Delta_L\Delta_L^\dagger) + \mathrm{Tr}(\Phi^\dagger\Phi\Delta_R\Delta_R^\dagger)\right)\\
&+ \beta_1\left(\mathrm{Tr}(\Phi\Delta_R\Phi^\dagger\Delta_L^\dagger) + \mathrm{Tr}(\Phi^\dagger\Delta_L\Phi\Delta_R^\dagger)\right) + \beta_2\left(\mathrm{Tr}(\tilde{\Phi}^\dagger\Delta_R\Phi^\dagger\Delta_L^\dagger) + \mathrm{Tr}(\tilde{\Phi}^\dagger\Delta_L\Phi\Delta_R^\dagger)\right)\\
&+ \beta_3\left(\mathrm{Tr}(\Phi\Delta_R\tilde{\Phi}^\dagger\Delta_L^\dagger) + \mathrm{Tr}(\Phi^\dagger\Delta_L\tilde{\Phi}\Delta_R^\dagger)\right) \qquad \text{(G.1)}
\end{aligned}
$$

We have used traces to keep the scalar potential independent of the choice of basis for the Higgs scalars. Also, we have defined the Higgs conjugate bi-doublet $\tilde{\Phi}$ as:

$$
\begin{aligned}
\tilde{\Phi} &= \sigma_2\Phi^*\sigma_2\\
&= \begin{pmatrix} -\phi_2^0 & \phi_1^+ \\ \phi_2^- & -\phi_1^0 \end{pmatrix} \qquad \text{(G.2)}
\end{aligned}
$$

Schematically, there are four sectors of the scalar potential. The μ sector contains the quadratic potentials, while the λ sector contains the quartic potentials of the Φ fields; we eventually recover the familiar Higgs doublet from the Φ field. These two sectors will produce the Mexican hat potential required for SSB with non-zero vevs.

The ρ sector contains the quartic potentials for the scalar triplets. The α sector mixes the allowed quadratic terms, while the β sector mixes all of the scalar terms.

We assume the following vev alignments after SSB and see what happens to the scalar potential.

$$
\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 e^{i\theta_2}, \end{pmatrix}, \langle\Delta_L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ v_L e^{i\theta_L} & 0 \end{pmatrix}, \langle\Delta_R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix} \qquad \text{(G.3)}
$$

Working out term by term, and assuming that all terms of the scalar potential are real, we find:

$$
\begin{aligned}
V = &- \mu_1^2\frac{1}{2}(\kappa_1^2 + \kappa_2^2)\\
&+ \mu_2^2(2\kappa_1\kappa_2\cos\theta_2)\\
&- \mu_3^2\frac{1}{2}(v_L^2 + v_R^2)\\
&+ \lambda_1\frac{1}{4}(\kappa_1^2+\kappa_2^2)^2\\
&+ \lambda_2\left(2(\kappa_1\kappa_2)^2\cos 2\theta_2\right)\\
&+ \lambda_3(\kappa_1\kappa_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_4(\kappa_1^2+\kappa_2^2)(2\kappa_1\kappa_2\cos\theta_2) \\
&+ \rho_1\frac{1}{4}(v_L^4+v_R^4) \\
&+ \rho_3\frac{1}{4}v_L^2v_R^2 \\
&+ \alpha_1\frac{1}{4}\left((\kappa_1^2+\kappa_2^2)(v_L^2+v_R^2)\right) \\
&- \alpha_2\kappa_1\kappa_2\cos\theta_2\left(v_R^2+v_L^2\right) \\
&+ \alpha_3\frac{1}{4}\kappa_2^2(v_L^2+v_R^2) \\
&+ \beta_1\frac{1}{2}\kappa_1\kappa_2v_Lv_R\cos(\theta_2-\theta_L) \\
&- \beta_2\frac{1}{2}\kappa_1^2v_Lv_R\cos\theta_L \\
&- \beta_3\frac{1}{2}\kappa_2^2v_Lv_R\cos\theta_L
\end{aligned}
$$

G.2.1 Seesaw relation between vevs

Let us compute the derivatives $\frac{\partial V}{\partial x}, x \in \{\kappa_1,\kappa_2,v_L,v_R,\theta_2,\theta_L\}$ explicitly, and then set them to 0.

This leads to the relation:

$$(v_L^2-v_R^2)[\beta_2\kappa_1^2\cos\theta_L+\cos(\theta_2-\theta_L)\beta_1\kappa_2\kappa_1+\cos(2\theta_2-\theta_L)\beta_3\kappa_2^2-(2\rho_1-\rho_3)v_Lv_R]=0 \tag{G.4}$$

which for $v_L^2-v_R^2\neq 0$ leads to the seesaw relationship:

$$\beta_2\kappa_1^2\cos\theta_L+\cos(\theta_2-\theta_L)\beta_1\kappa_2\kappa_1+\cos(2\theta_2-\theta_L)\beta_3\kappa_2^2=(2\rho_1-\rho_3)v_Lv_R \tag{G.5}$$

Since κ_1,κ_2 are on the electroweak scale, $\kappa_1,\kappa_2\sim v$, so schematically, we obtain:

$$\beta'v^2=\rho'v_Lv_R \tag{G.6}$$

which is a seesaw relationship of the vevs. For large v_R (and heavy W_R), this leads naturally to light neutrino masses.

G.2.2 Discussion of naturalness of seesaw mechanism

This line of argument follows Section 3A of [58].

The parameters β_i and ρ_i are assumed to be of order 1. If they are too large, they would violate unitarity and lead to a non-perturbative theory.

The neutrino seesaw mechanism leads to the conclusion that

$$m_\nu \simeq \sqrt{2}\left(h_M v_L - \frac{h_D^2 \kappa_+^2}{2 h_M v_R}\right)$$

where h_M, h_D represent Yukawa coupling constants, and $\kappa_+^2 = \kappa_1^2 + \kappa_2^2$

We introduce γ, where

$$\gamma = \frac{\beta_2 \kappa_1^2 + \beta_1 \kappa_1 \kappa_2 + \beta_3 \kappa_2^2}{(2\rho_1 - \rho_3)\kappa_+^2} \tag{G.7}$$

such that we can rewrite the neutrino mass as:

$$m_\nu \simeq \sqrt{2}\left(h_M \gamma - \frac{1}{2}\frac{h^2 D}{h_M}\frac{\kappa_+^2}{v_R}\right) \tag{G.8}$$

If we choose $v_R \sim 3$ TeV for an observable W_R boson, we find that this requires $\gamma \lesssim 10^{-9}$ which represents extreme fine-tuning.

If we choose $\gamma \sim 1$, we find that $v_R \gtrsim 10^8$ GeV, and $m_{W_R}, m_{Z'}$ will be in the range $10^7 \sim 10^8$ GeV. These heavy degrees of freedom can be integrated out of the Lagrangian, leaving us with effective Majorana neutrino terms. Since there is no *a priori* justification for why fundamental particles need to be at the electroweak scale, we can accept having heavy W_R, Z' gauge bosons.

Appendix H

Effective Field Theories

We will follow closely Timo Weigand's lecture notes [113]

The old renormalisation approach is to use a high-scale cutoff Λ to regulate divergent integrals without any physical meaning. By using renormalisation, the cutoff dependence is removed in physical amplitudes, allowing us to take $\Lambda \to \infty$.

The Wilsonian approach re-interprets renormalisation and is the basis of the modern approach to renormalisation. According to Wilson, QFT is an effective field theory accurate below an intrinsic cutoff Λ_0, and is inaccurate above Λ_0. This is inspired by condensed matter physics, in which distances smaller than the lattice spacing a are simply not needed in our physical description.

The modern interpretation of QFT views it as an effective theory with a cutoff energy.

To illustrate explicitly how the high energy degrees of freedom are captured in the effective coupling constants at lower energies, we consider the scalar ϕ^4 Lagrangian in d Euclidean dimensions:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{H.1}$$

and impose a momentum cutoff Λ_0. In the path integral quantisation, this is the same as restricting the measure to Fourier modes $\tilde{\phi}(k)$ with $0 \leq |k| \leq \Lambda_0$:

$$Z = \int \prod_{|k|\leq\Lambda_0} d\tilde{\phi}(k) e^{-\int d^d x_E (\mathcal{L} + J\cdot\phi)} \tag{H.2}$$

which allows us to compute amplitudes up to an energy scale Λ_0. Consider processes at energy scales:

$$\Lambda = b\Lambda_0, \qquad 0 < b < 1 \tag{H.3}$$

Computations will give rise to large log-corrections if $\Lambda \ll \Lambda_0$. To avoid this, we can integrate out the degree of freedom between Λ and Λ_0, such that the path integral is defined only in terms of the remaining degrees of freedom below Λ.

To do so, split each Fourier mode as:

$$\tilde{\phi}(k) = \phi(k) + \hat{\phi}(k) \tag{H.4}$$

where:

$$\phi(k) = \begin{cases} \tilde{\phi}(k), & 0 \leq |k| \leq b\Lambda_0 \\ 0, & b\Lambda_0 \leq |k| \leq \Lambda_0 \end{cases} \tag{H.5}$$

$$\hat{\phi}(k) = \begin{cases} 0, & 0 \leq |k| \leq b\Lambda_0 \\ \tilde{\phi}(k), & b\Lambda_0 \leq |k| \leq \Lambda_0 \end{cases} \tag{H.6}$$

which are two functions with two different domains, which when put together, are identical to the original function. We take $J = 0$ for brevity, and rewrite the partition function:

$$\begin{aligned} Z &= \int \mathcal{D}\phi\mathcal{D}\hat{\phi}e^{-\int d^dx\left[\frac{1}{2}(\partial\phi+\partial\hat{\phi})^2+\frac{1}{2}m^2(\phi+\hat{\phi})^2+\frac{\lambda}{4!}(\phi+\hat{\phi})^4\right]} \\ &= \int \mathcal{D}\phi e^{-\int d^dx\left(\frac{1}{2}\partial\phi^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4\right)} && \text{(H.7)} \\ &\times \int \mathcal{D}\hat{\phi}e^{-\int d^dx\left[\frac{1}{2}\partial\hat{\phi}^2+\frac{1}{2}m^2\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{\lambda}{6}\phi\hat{\phi}^3\frac{\lambda}{4}\phi^2\hat{\phi}^2+\frac{\lambda}{6}\hat{\phi}\phi^3\right]} && \text{(H.8)} \end{aligned}$$

where we can drop terms like $\phi\hat{\phi}$ as the different momentum modes are orthogonal and will vanish in future steps.

The partition function hence takes the form:

$$Z = \int [\mathcal{D}\phi]_{|k|\leq b\Lambda_0} e^{-S[\phi]} \int [\mathcal{D}\hat{\phi}]_{b\Lambda_0\leq|k|\leq\Lambda_0} e^{-\hat{S}[\phi,\hat{\phi}]} \tag{H.9}$$

Crucially, this can be written as an effective Lagrangian without any dependence on the higher energy field $\hat{\phi}$:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda_0} e^{-\int d^dx\mathcal{L}_{\text{eff}}} \tag{H.10}$$

such that all terms inside $\hat{S}[\phi, \hat{\phi}]$, except the kinetic term, are interpreted as a perturbation, which is justified for $\lambda \ll 1$ and $m \ll b\Lambda_0$.

Perturbatively, we can write:

$$\int \mathcal{D}\hat{\phi} e^{-\hat{S}[\phi,\hat{\phi}]} = \int \mathcal{D}\hat{\phi} e^{-\int d^dx \frac{1}{2}(\partial\hat{\phi})^2} \left[1 - \int d^dx \left(\frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\lambda}{6}\phi\hat{\phi}^3 + \frac{\lambda}{4}\phi^2\hat{\phi}^2 + \frac{\lambda}{6}\hat{\phi}\phi^3\right)\right] \tag{H.11}$$

This is computed using Wick's theorem in perturbation theory, where we contract only pairs of $\hat{\phi}$. In the resulting propagator, only momenta above the cutoff $b\Lambda_0 \le |k| \le \Lambda$ appear.

Applying Wick's theorem to every term in the perturbation, we will eventually obtain higher order terms. For example, the term:

$$\left\langle -\int d^dx \frac{\lambda}{4}\phi^2\hat{\phi}^2 \right\rangle = \dots \tag{H.12}$$

$$= \int d^d \left(-\frac{1}{2}\phi^2(x)\mu\right) \tag{H.13}$$

where μ is a term containing factors from the spherical integral, and a mass dimension of Λ^{d-2}.

As another example, higher order interactions

$$\frac{1}{4}\left\langle \frac{\lambda}{6}\phi^3\hat{\phi}\frac{\lambda}{6}\phi^3\hat{\phi} \right\rangle \sim \phi^6 \tag{H.14}$$

where higher order interaction terms in ϕ generate higher order interactions in the effective action.

Instead of carrying out the computation for all the terms in the perturbative series, we can think of higher order effects as arbitrary corrections of the form:

$$\int d^dx \mathcal{L}_{\text{eff}}^{b\Lambda_0} = \int d^dx [\frac{1}{2}(1+\Delta Z)(\partial\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 \tag{H.15}$$

$$+ \Delta C(\partial\phi)^4 + \Delta D\phi^6 + \dots] \tag{H.16}$$

where the coefficients $\Delta Z, \Delta m, \Delta\lambda, \Delta C, \Delta D$ can be computed explicitly using perturbation theory, receiving contributions only from the momentum regime $b\Lambda_0 \le |k| \le \Lambda_0$. In so doing, this 'averages out' the higher energy field, and only lower energy momenta appear in the loops as all the high energy effects are encoded in the effective Lagrangian.

The effect of integrating out the degrees of freedom $b\Lambda_0 \le |k| \le \Lambda_0$ turns out to have the same effect as renormalising the couplings in the Lagrangian.

Due to the difficulty of the full computation, we will not specify the full form of every coefficient $\Delta Z, \Delta m, \ldots$. Rather, all possible terms are generated, and we can identify any operator we wish to construct (e.g. the dimension-5 Weinberg operator) with a term in this perturbative series.

Also, higher dimensional terms are suppressed by factors of Λ to keep the series dimensionally consistent, which can be seen from dimensional analysis.

What happens to these high-dimensional terms at low energies? We can argue in the following way. Consider a dimensionful coupling $\int d^d x C_{d_i} O_{d_i}$ where O_{d_i} is a local operator of mass dimension d_i and C_{d_i} of mass dimension $d - d_i$.

Here, d is the regularising parameter $d = 4 - \epsilon$, and d_i is a dimension that we can freely choose. For example, a term like $m^2\phi^2$ has an operator dimension of $d_i = 2$, and a coefficient dimension of $d - d_i = 2$. Likewise, a term like $\overline{\psi}\psi HH$ has an operator dimension of $d_i = 5$, and a coefficient dimension of $d - d_i = -1$.

Consider a dimensionless coupling in terms of Λ, the energy scale.

$$g_i(\Lambda) = C_i(\Lambda)\Lambda^{-(d-d_i)} \tag{H.17}$$

with an RG flow:

$$g_i(\Lambda) = \left[g_i(\Lambda_0) + \lambda^{d_i/2}(\Lambda_0) \left(\left(\frac{\Lambda_0}{\Lambda} \right)^{d_i - d} - 1 \right) \right] \left(\frac{\Lambda_0}{\Lambda} \right)^{d-d_i} \tag{H.18}$$

where λ is an effective coupling defined by an observable computed from the effective action with cutoff Λ. This is similar to that found in discussions of renormalisation in ϕ^4-theory.

The scaling due to $\left(\frac{\Lambda_0}{\Lambda}\right)^{d-d_i} = \left(\frac{\Lambda_0}{\Lambda}\right)^{[C_i]}$ dominates. If C_i is non-renormalisable (i.e. the term is a high dimension operator), then the initial value of $g_i(\Lambda_0)$ is irrelevant at low-energy scales $\Lambda \ll \Lambda_0$.

Hence, non-renormalisable couplings become irrelevant at low energies, and effectively vanish. At higher energy scales, these terms become relevant.

This is the basis for introducing higher dimensional operators into the Lagrangian and then calling them effective operators: introducing these operators is akin to describing a higher energy theory with fields at higher energy scales.

The argument in this Appendix is based on ϕ^4-theory. While possible, performing the same argument to include fermion and gauge fields would be extremely tedious. Nevertheless, we can assume that the same conceptual flow is true.

Hence, for any other field theory (Standard Model, LRSM, GUT) we are allowed to add higher dimensional operators. This is because the Wilsonian approach allows us to describe heavier fields at higher energy scales using effective operators at the lower energy scale.

Appendix I

$SU(5)$ representations

In this appendix, we will build the $SU(5)$ irreducible representations with the help of the following references [80, 78, 8, 114]. The goal is to justify the Yukawa terms in the $SU(5)$ Higgs potential that are involved in lepton and quark masses (and hence produce neutrino masses).

I.1 Recursion formula for multiplet dimension

We consider four quantum numbers p, q, r, s each ranging from 0 to 3. This is similar to writing a tensor labelled by four different levels of indices.

To count the number of irreducible representations, the typical procedure is to count the number of distinct configurations of a symmetrised tensor, and to impose the traceless condition by contracting the indices [8].

However, as the dimension increases, doing so is prohibitively difficult. We instead employ an alternative approach. Pfeifer's notes on $SU(N)$ Lie algebras [114] (Section 6.2, page 107) provide a recursive formula for the dimensions of a $SU(N)$-multiplet of rank $N-1$:

$$\begin{aligned} d_N(p_1, p_2, \ldots, p_{N-1}) = \frac{1}{(N-1)!}&(p_{N-1}+1)(p_{N-1}+p_{N-2}+2)(p_{N-1}+p_{N-2}+p_{N-3}+3) \\ &\cdots(p_{N-1}+\ldots+p_1+N-1)d_{N-1}(p_1, p_2, \ldots, p_{N-2}) \end{aligned} \tag{I.1}$$

Written explicitly, and replacing p_i with Roman letters:

$$d_2(p) = p+1 \tag{I.2}$$

$$d_3(p,q) = \frac{1}{2}(q+1)(p+q+2) \cdot d_2(p) \tag{I.3}$$

$$d_4(p,q,r) = \frac{1}{6}(r+1)(r+q+2)(r+q+p+3) \cdot d_3(p,q) \tag{I.4}$$

$$d_5(p,q,r,s) = \frac{1}{24}(s+1)(s+r+2)(s+r+q+3)(s+r+q+p+4) \cdot d_4(p,q,r) \tag{I.5}$$

I.2 Computing the dimensions of irreducible representations

For brevity, we only show multiplets with dimensions of 100 or less.

(p,q,r,s)	Dimension
(0, 0, 0, 0)	1
(0, 0, 0, 1)	5
(1, 0, 0, 0)	5
(0, 0, 1, 0)	10
(0, 1, 0, 0)	10
(0, 0, 0, 2)	15
(2, 0, 0, 0)	15
(1, 0, 0, 1)	24
(0, 0, 0, 3)	35
(3, 0, 0, 0)	35
(0, 0, 1, 1)	40
(1, 1, 0, 0)	40
(0, 1, 0, 1)	45
(1, 0, 1, 0)	45
(0, 0, 2, 0)	50
(0, 2, 0, 0)	50
(1, 0, 0, 2)	70
(2, 0, 0, 1)	70
(0, 1, 1, 0)	75

Table I.1: Quantum numbers and dimensions of irreducible representations in $SU(5)$ up to dimension-75

We observe that most of them have a complex representation e.g. 5 and $\overline{5}$ being represented by $(1,0,0,0)$ and $(0,0,0,1)$. In our convention, we denote the multiplet with the largest first element (by pair-wise comparison) as the conjugate, with a bar on top.

I.3 Young tableaux and decomposition rules

The following rules are extracted from Pfeifer's Lie Algebra notes [114].

The standard arrangement of the *Young tableaux* consists of rows of boxes. There are a few rules:

- The length of the row is decreasing.
- Integers from 1 up to N are put in the boxes with repetitions allowed.
- These integers do not increase to the right, and they increase downward in a column.

For $SU(3)$ multiplets, we draw two lines of boxes. For $SU(5)$, we would draw four lines of boxes. The change in the number of boxes in each line represents a quantum number.

It is taken for granted that the number of standard arrangements of the Young tableau is equal to the dimension of the multiplet.

To compute direct products of multiplets, we decompose them using the method of Young for the reason that this method is more efficient for $SU(N)$ once $N \geq 3$.

For completeness, we include the Young procedure without proof (found in Section 4.6 [114]). To compute the tensor decomposition of two tables, we first mark the boxes in the second table with the number of the corresponding row. We graphically add the boxes of the second diagram to the first one in all possible ways that obey the following rules:

- No row may be longer than the row above.
- In an $SU(N)$-multiplet, no column contains more than N boxes.
- In a row, the numbers do not decrease to the right.
- In a column, the numbers increase downwards.

- One goes through all the boxes from the right to the left, and then from the top to the bottom. At each point of the path, the number of boxes encountered with the number i must be less or equal to the number of boxes with $i-1$ (kept as a running total).
- Columns with N boxes are deleted.

I.4 Tensor products

To figure out the allowed Yukawa couplings in the $SU(5)$ superpotential, we will have to consider the decomposition of various tensor products into the irreducible representations of $SU(5)$.

The SM fermions and the right-handed neutrino are 16 particles that are fully represented by the multiplets $\psi_{10}, \psi_5, \psi_1$. For the purposes of looking at neutrino masses and mixing matrices, we are more interested in the particle content of $\psi_{\overline{5}}$, which represents the conjugate down quarks and the lepton doublet.

The following tensor products are trivial:

$$10 \otimes 1 = 10 \tag{I.6}$$

$$5 \otimes 1 = 5 \tag{I.7}$$

$$1 \otimes 1 = 1 \tag{I.8}$$

Using the method of Young we compute a few of these products with their quantum numbers shown explicitly.

$$5 \otimes \overline{5} = 24 \oplus 1 \qquad (0,0,0,1) \otimes (1,0,0,0) = (1,0,0,1) \oplus (0,0,0,0) \tag{I.9}$$

$$\overline{5} \otimes \overline{5} = \overline{10} \oplus \overline{15} \qquad (1,0,0,0) \otimes (1,0,0,0) = (0,1,0,0) \oplus (2,0,0,0) \tag{I.10}$$

$$5 \otimes 5 = 10 \oplus 15 \qquad (0,0,0,1) \otimes (0,0,0,1) = (0,0,1,0) \oplus (0,0,0,2) \tag{I.11}$$

$$10 \otimes \overline{5} = 5 \oplus \overline{45} \qquad (0,0,1,0) \otimes (1,0,0,0) = (0,0,0,1) \oplus (1,0,1,0) \tag{I.12}$$

$$\overline{10} \otimes 5 = 45 \oplus \overline{5} \qquad (0,1,0,0) \otimes (0,0,0,1) = (0,1,0,1) \oplus (1,0,0,0) \tag{I.13}$$

$$5 \otimes 10 = 40 \oplus \overline{10} \qquad (0,0,0,1) \otimes (0,0,1,0) = (0,0,1,1) \oplus (0,1,0,0) \tag{I.14}$$

$$\overline{10} \otimes \overline{5} = \overline{40} \oplus 10 \qquad (0,1,0,0)(1,0,0,0)\otimes = (1,1,0,0) \oplus (0,0,1,0) \tag{I.15}$$

$$10 \otimes 10 = 50 \oplus 45 \oplus \overline{5} \qquad (0,0,1,0) \otimes (0,0,1,0) = (0,0,2,0) \oplus (0,1,0,1) \oplus (1,0,0,0) \tag{I.16}$$

$$\overline{10} \otimes 10 = 1 \oplus 75 \oplus 24 \qquad (0,1,0,0) \otimes (0,0,1,0) = (0,0,0,0) \oplus (0,1,1,0) \oplus (1,0,0,1) \tag{I.17}$$

$$\overline{10} \otimes \overline{10} = 5 \oplus \overline{50} \oplus \overline{45} \qquad (0,1,0,0) \otimes (0,1,0,0) = (0,0,0,1) \oplus (0,2,0,0) \oplus (1,0,1,0) \tag{I.18}$$

These will be useful for the fermion-fermion products in $SU(5)$ when we are building the Yukawa couplings, as these will turn into mass terms after symmetry breaking. It is useful to note that $5 \otimes \overline{5}$ and $10 \otimes \overline{10}$ generates an $SU(5)$ singlet.

To understand fermion-fermion-scalar couplings as they appear in the Yukawa sector, we can construct the following non-trivial representations where H_5 and $H_{\overline{5}}$ correspond to 5- and $\overline{5}$-representations:

$$\psi_{10}\psi_{\overline{5}}H_{\overline{5}} \qquad (10 \otimes \overline{5}) \otimes \overline{5} \tag{I.19}$$

$$(\psi_{10}\psi_{10})H_5 \qquad (10 \otimes 10) \otimes 5 \tag{I.20}$$

For the gauge sector, the gauge bosons are in the 24-dimensional representation of $SU(5)$. It suffices to see that:

$$24 \otimes 5 = 5 \oplus \dots \tag{I.21}$$

$$24 \otimes \overline{5} = \overline{5} \oplus \dots \tag{I.22}$$

$$24 \otimes 10 = 10 \oplus \dots \tag{I.23}$$

$$24 \otimes \overline{10} = \overline{10} \oplus \dots \tag{I.24}$$

which allows the following couplings:

$$\psi_5 A_{24} \psi_{\overline{5}}, \psi_{\overline{10}} A_{24} \psi_{10} \tag{I.25}$$

In [88], Altarelli et al. introduce Higgs multiplets in 50- and 75-dimensional representations to address the double-triplet splitting problem - we want to have a natural explanation for having two light SM Higgs, and three heavy GUT-scale bosons after symmetry breaking.

Since we are only interested in Yukawa couplings that could lead to neutrino mass terms, we are looking for singlets that arise from tensor decompositions i.e. the Higgs multiplet combines with another object, and this combination can be represented as a singlet of $SU(5)$. Let us list the relevant tensor decompositions that we are interested in that lead to a $SU(5)$ singlet:

$$50 \otimes \overline{50} = 1 \oplus \dots \tag{I.26}$$

where there are no compatible tensor products of H_{75} with any combination of two lepton fields. To see this, consider the Young tableaux representations of 50 and $\overline{50}$, with the goal of obtaining the singlet representation of $SU(5)$.

This means that we are allowed the following Yukawa coupling:

$$H_{\overline{50}}(\psi_{10}\psi_{10}) \sim \overline{50} \otimes 50 \tag{I.27}$$

On the other hand, H_{75} is used to break the $SU(5)$ symmetry. Let us check the couplings of the 75-dimensional representation. It is sufficient to see that:

$$75 \otimes 50 = \overline{5} \oplus \dots \tag{I.28}$$

$$75 \otimes \overline{50} = 5 \oplus \dots \tag{I.29}$$

These couplings allow for cubic scalar interactions involved in soft supersymmetry breaking. We write these as:

$$H_{75}H_{50}H_5, H_{75}H_{\overline{50}}H_{\overline{5}} \tag{I.30}$$

Hence, we have enumerated explicitly the Yukawa terms in the $SU(5)$ theory that are useful for studying neutrino masses and their mixing matrices. We have also provided the necessary tools to reproduce these calculations.

Appendix J

Miscellanous Properties

J.1 Charge conjugate properties

We list a few properties of the charge conjugate matrix $\mathcal{C}$ as these will be used in the following calculations.

$$\mathcal{C}^{-1} = \mathcal{C}^{\dagger} = C^{T} = -\mathcal{C} \tag{J.1}$$

For a four-component Dirac spinor ψ, its charge-conjugate ψ^c is written as [26]

$$\begin{aligned} \psi^c &= \mathcal{C}\overline{\psi}^T \\ &= \mathcal{C}\gamma^0\psi^* \end{aligned} \tag{J.2}$$

We make the choice $\mathcal{C} = i\gamma^2\gamma^0$ [26] and recall that $(\gamma^0)^2 = I$

$$\psi^c = i\gamma^2\psi^* \tag{J.3}$$

Lastly, the charge conjugation matrix satisfies [17]:

$$\mathcal{C}(\gamma^\mu)^T\mathcal{C}^{-1} = -\gamma^\mu \tag{J.4}$$

$$\mathcal{C}\gamma_5^T\mathcal{C}^{-1} = \gamma_5 \tag{J.5}$$