Time Integrated Beam pattern in Equatorial Coordinates

the documentation on TIBEC

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1 Conventions

- The **antenna spherical system**, $\{\theta_a, \phi_a\}$ is the default spherical coordinate system used in the farfield source file for describing the linearly scaled farfield patterns.
- The **antenna Cartesian system**, $\{x_a, y_a, z_a\}$ is derived from the antenna spherical system:
  - The origins of the two systems are the same.
  - $\hat{z}_a$: parallel to the direction from the origin to the pointing centre.
  - $\hat{x}_a$: parallel to the radial direction at $(\theta_e = \pi/2, \phi_e = 0)$.
  - $\hat{y}_a$: given by the right hand convention.
- The **horizontal (or local) Cartesian system**, $\{x_h, y_h, z_h\}$ is defined using Cardinal directions and the zenith relative to the antenna:
  - It shares the same origin as the antenna Cartesian system.
  - $\hat{z}_h$: identical to $\hat{\mathbf{Z}}$, the direction from the antenna to the zenith.
  - $\hat{x}_h$: identical to $\hat{\mathbf{S}}$, the direction of the true South of the antenna.
  - $\hat{y}_h$: identical to $\hat{\mathbf{E}}$, the direction of the true East of the antenna.
- The **equatorial spherical system**, $\{\theta_e, \phi_e\}$ is the same thing as the $\{\mathrm{RA}, \mathrm{DEC}\}$ system but with $\theta_e = \frac{\pi}{2} - \mathrm{DEC}$ and $\phi = \mathrm{RA}$, where all the angle objects are in radians.
- The **equatorial Cartesian system**, $\{x_e, y_e, z_e\}$ is defined based on the equatorial spherical system:
  - The origins of the two systems are the same.
  - $\hat{z}_e$: parallel to the direction from the origin to the north polar.
  - $\hat{x}_e$: parallel to the radial direction at $(\theta_e = \pi/2, \phi_e = 0)$.
  - $\hat{y}_e$: given by the right hand convention.
- The **antenna alignment** is given as the antenna axes expressed in local coordinates. In this work, the x-Axis and z-Axis are given as $\hat{x}_a = [S_x, E_x, Z_x]^T$ and $\hat{z}_a = [S_z, E_z, Z_z]^T$ so that

$$\hat{x}_a = S_x\hat{\mathbf{S}} + E_x\hat{\mathbf{E}} + Z_x\hat{\mathbf{Z}} \tag{1}$$

and

$$\hat{z}_a = S_z\hat{\mathbf{S}} + E_z\hat{\mathbf{E}} + Z_z\hat{\mathbf{Z}}. \tag{2}$$

For example, $\hat{x}_a = [1,0,0]^T$ means the x-Axis points to the true South, and $\hat{z}_a = [0,0,1]^T$ means the z-Axis is aligned with the direction of zenith. And $\hat{y}_a$ is specified by $\hat{z}_a \times \hat{x}_a$.

We define the pointing matrix $\mathbf{P}$ for later use

$$\mathbf{P} = \begin{pmatrix} S_x & S_y & S_z \\ E_x & E_y & E_z \\ Z_x & Z_y & Z_z \end{pmatrix} \tag{3}$$

Note that all basis vectors are unit vectors. $\mathbf{P}$ is thus an orthogonal matrix.

- The relation between the horizontal Cartesian basis vectors $\{\hat{\mathbf{S}}, \hat{\mathbf{E}}, \hat{\mathbf{Z}}\}$ and the equatorial Cartesian basis vectors $\{\hat{x}_e, \hat{y}_e, \hat{z}_e\}$:

$$\begin{aligned}
\hat{\mathbf{S}} &= \cos\tilde{\theta}_e \cos\tilde{\phi}_e\, \hat{x}_e + \cos\tilde{\theta}_e \sin\tilde{\phi}_e\, \hat{y}_e - \sin\tilde{\theta}_e\, \hat{z}_e \\
\hat{\mathbf{E}} &= -\sin\tilde{\phi}_e\, \hat{x}_e + \cos\tilde{\phi}_e\, \hat{y}_e \\
\hat{\mathbf{Z}} &= \sin\tilde{\theta}_e \cos\tilde{\phi}_e\, \hat{x}_e + \sin\tilde{\theta}_e \sin\tilde{\phi}_e\, \hat{y}_e + \cos\tilde{\theta}_e\, \hat{z}_e
\end{aligned}$$

where $\{\tilde{\theta}_e, \tilde{\phi}_e\}$ are the equatorial spherical coordinates of the zenith (local to the antenna), so they are functions of the local sidereal time (LST, in seconds) and the latitude (lat, in radians) of the antenna:

$$\begin{aligned}
\tilde{\theta}_e &= \frac{\pi}{2} - \text{lat} \\
\tilde{\phi}_e &= \frac{\text{LST}}{3600} \cdot 15 \cdot \frac{\pi}{180}
\end{aligned}$$

For later convenience, we define the rotation matrix, $\mathbf{R}$ as

$$\mathbf{R}_{\text{eq-h}}(\theta,\phi) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{4}$$

$\mathbf{R}_{\text{eq-h}}$ is an orthogonal matrix, which implies $\mathbf{R}_{\text{h-eq}} \equiv \mathbf{R}_{\text{eq-h}}^{-1} = \mathbf{R}_{\text{eq-h}}^{T}$.

- One should notice that the origin of the equatorial systems is separated from the origin of other systems by a distance equal to the radius of the Earth. Since what we observe it the power of astrophysical sources, this difference in origin is considered negligible, i.e., we assume the above systems have the same origin.

2 Transformations

- from the antenna spherical system to the antenna Cartesian system
  - Coordinates mapping ($r = 1$):

$$\begin{aligned}
x_a &= r\sin\theta_a\cos\phi_a \\
y_a &= r\sin\theta_a\sin\phi_a \\
z_a &= r\cos\theta_a
\end{aligned}$$

– Field components:

$$\begin{pmatrix} E_{x_a} \\ E_{y_a} \\ E_{z_a} \end{pmatrix} = \mathbf{R}_{\text{s-c}}(\theta_a, \phi_a) \begin{pmatrix} 0 \\ E_{\theta_a} \\ E_{\phi_a} \end{pmatrix} \tag{5}$$

where $\mathbf{R}_{\text{s-c}}$ is the rotation matrix transforming the field components from the spherical system to the Cartesian system:

$$\mathbf{R}_{\text{s-c}}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \tag{6}$$

- from the antenna Cartesian system to the horizontal Cartesian system
  – Coordinates mapping:

$$\begin{pmatrix} x_h \\ y_h \\ z_h \end{pmatrix} = \mathbf{P} \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \tag{7}$$

  – Field components:

$$\begin{pmatrix} E_{x_h} \\ E_{y_h} \\ E_{z_h} \end{pmatrix} = \mathbf{P} \begin{pmatrix} E_{x_a} \\ E_{y_a} \\ E_{z_a} \end{pmatrix} \tag{8}$$

  where $\mathbf{P}$ is the pointing matrix defined above.

- from the horizontal Cartesian system to the equatorial Cartesian system
  – Coordinates mapping:

$$\begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix} = \mathbf{R}_{\text{h-eq}} \begin{pmatrix} x_h \\ y_h \\ z_h \end{pmatrix} \tag{9}$$

  – Field components:

$$\begin{pmatrix} E_{x_e} \\ E_{y_e} \\ E_{z_e} \end{pmatrix} = \mathbf{R}_{\text{h-eq}} \begin{pmatrix} E_{x_h} \\ E_{y_h} \\ E_{z_h} \end{pmatrix} \tag{10}$$

  where $\mathbf{R}_{\text{h-eq}}$ is the rotation matrix defined above.

- from the equatorial Cartesian system to the equatorial spherical system
  – Coordinates mapping:

$$\theta_e = \text{numpy.arctan2}(\frac{\sqrt{x_e^2 + y_e^2}}{z_e})$$

$$\phi_e = \begin{cases} \text{numpy.arctan2}(\frac{y_e}{x_e}) & \text{if numpy.arctan2}(\frac{y_e}{x_e}) \geq 0 \\ \text{numpy.arctan2}(\frac{y_e}{x_e}) + 2\pi & \text{if numpy.arctan2}(\frac{y_e}{x_e}) < 0 \end{cases}$$

  – Field components:

$$\begin{pmatrix} E_{r_e} \\ E_{\theta_e} \\ E_{\phi_e} \end{pmatrix} = [\mathbf{R}_{\text{s-c}}(\theta_e, z_e)]^{-1} \begin{pmatrix} E_{x_e} \\ E_{y_e} \\ E_{z_e} \end{pmatrix} \tag{11}$$