GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

Faculty of Physics

Internship Certificate

Max Planck Institute for Solar System Research

Name: Zhengzhan Shang
Student ID number: 23201570
Start of the internship: September, 2022
End of the internship: April, 2023
Average weekly working hour: 7 hours per week

Supervisor signature:

Contents

1 Overview

This is to certify that Zhengzhan Shang has successfully completed an internship at Max Planck Institute for Solar System Research from September 2022 to April 2023. The intern has completed a total of 180 hours during which the weekly working hours were 7 hours. The overview of the tasks done in the internship will be stated in the following.

2 Tasks

I mainly learned the theoretical background and numerical analysis of full spectral methods for flows in spherical coordinates. The programming language I used was *Python*. The paper that I followed was *Comparison of Spectral Methods of Flows on Spheres* (1979) by Cha-Mei Tang. I have closely followed the procedures of this paper. Tang compared the pseudospectral method with the full spectral method, which are used for flows in spherical geometry, using spherical harmonics as the spectral expansion function. Tang showed that with the right truncation procedure and pole condition, the full spectral method is a scheme with energy conservation whereas the pseudospectral method is not.

2.1 Analytical Calculation

I started out with recreating the theories needed for the numerical computation using Tang 1979. The equation of motion for an incompressible flow field, in a rotation coordinate frame is given as,

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times (\nabla \times \mathbf{u}) - 2\Omega \times \mathbf{u} - \nabla P + \nu \nabla^2 \mathbf{u}, \tag{1}$$

and the continuity equation,

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}$ is the incompressible flow velocity field, Ω is the rate of angular rotation, P is the pressure, ν is the kinematic viscosity. Two-dimensional flow is considered, then the flow field can be rewritten into latitude θ and longitude ϕ component as,

$$\mathbf{u} = u_\theta(\theta, \phi)\hat{i}_\theta + u_\phi(\theta, \phi)\hat{i}_\phi, \tag{3}$$

Furthermore, the vorticity $\zeta\hat{i}_r = (\nabla_{\theta,\phi} \times \mathbf{u})$ and streamfunction ψ can be used to describe the flow field. The velocity field is expressible as $\mathbf{u} = -\nabla_{\theta,\phi} \times (\psi\hat{i}_r)$, that is,

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \quad \text{and} \quad u_\phi = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \tag{4}$$

where r is the r-direction in spherical coordinate. Scalar variables like ψ, ζ, P are independent of coordinate frames and can be expanded as spherical harmonics. For example, ψ, can be expanded as,

$$\psi(\theta, \phi) = \sum_{|m|=0}^{N} \sum_{n=|m|}^{N} \psi_{nm} Y_n^m(\theta, \phi), \tag{5}$$

where Y_n^m is the surface harmonic of degree n and order m, and ψ_{nm} is the surface harmonics coefficients. Y_n^m is defined using Legendre polynomials $P_n^m(\cos\theta)$,

$$Y_n^m(\theta,\phi) = b_{nm}P_n^m(\cos\theta)e^{im\phi}, \tag{6}$$

where $b_{nm} = [(2n+1)(n-m)!]^{1/2}[4\pi(n+m)]^{-1/2}$, and

$$P_n^m(\cos\theta) = \frac{\sin^m\theta}{2^m}\sum_{j=0}^{n-m}\frac{(n+m+j)!(-1)^j}{(n-m-j)!(m+j)!(j)!}(\frac{1-\cos\theta}{2})^j. \tag{7}$$

To project the equation (for example, $\psi(\theta,\phi)$) onto its surface harmonics coefficient (ψ_{nm}), a double integral is used,

$$\psi_{nm} = \int_0^{2\pi}\int_0^{\pi}\psi(\theta,\phi)(Y_n^m(\theta,\phi))^*\sin\theta d\theta d\phi, \tag{8}$$

and to reconstruct the equation (for example, $\psi(\theta,\phi)$) from its surface harmonics coefficient (ψ_{nm}), we can use Equation 5.

The vector components of the flow field $\mathbf{u}$ can be written using scalars $U(\theta,\phi)$, $V(\theta,\phi)$ as,

$$u_\theta = \frac{U(\theta,\phi)}{\sin\theta} \quad \text{and} \quad u_\phi = \frac{V(\theta,\phi)}{\sin\theta}. \tag{9}$$

By substituting this into the continuity equation Equation 2,

$$-\frac{\partial U}{\partial\cos\theta} + \frac{1}{\sin^2\theta}\frac{V}{\partial\phi} = 0. \tag{10}$$

Further substituting Equation 9 into the equation of motion Equation 1, the U ($\hat{\theta}$-direction) and V ($\hat{\phi}$-direction) component can be separated,

$$\frac{\partial U}{\partial t} = \sin^2\theta\frac{\partial P}{\partial\cos\theta} + V\zeta + 2\Omega\cos\theta V + \nu\nabla^2 U, \tag{11}$$

and

$$\frac{\partial V}{\partial t} = -\frac{\partial P}{\partial\phi} - U\zeta - 2\Omega\cos\theta U + \nu(\nabla^2 V + 2\cos\theta\zeta), \tag{12}$$

with

$$\zeta = -\frac{\partial V}{\partial\cos\theta} - \frac{1}{\sin^2\theta}\frac{\partial U}{\partial\phi}, \tag{13}$$

and ζ_{nm} can also be obtained using u_{nm},

$$\zeta_{nm} = -in(n+1)m^{-1}u_{nm}. \tag{14}$$

Since U, V, P, and ζ are all scalars, which means they can all be expanded into spherical harmonics using their corresponding coefficients u_{nm}, v_{nm}, p_{nm}, and ζ_{nm}.

Using the Gaussian quadrature in θ and Fourier transform in ϕ, we can have $U'(\theta, \phi)$ and $V'(\theta, \phi)$ as,

$$U'(\theta, \phi) = V\zeta + 2\Omega V \cos\theta \quad \text{and} \quad V'(\theta, \phi) = -U\zeta - 2\Omega U \sin\theta. \tag{15}$$

The corresponding surface harmonics coefficient u'_{nm} and v'_{nm} can be obtained using Equation 8.

To calculate pressure P, we have to keep in mind that the pressure counteracts any divergent flow produced by the convective and rotational terms. By substituting Equation 15 back into Equation 11 and Equation 12, after ignoring the dissipative terms (the terms that have ν) and taking the divergence, we have,

$$\nabla^2 P = \frac{\partial U'}{\partial \cos\theta} + \frac{1}{\sin^2\theta} \frac{\partial V'}{\partial \phi}, \tag{16}$$

then, we can obtain the surface harmonics $(\nabla^2 P)_{nm}$ of $\nabla^2 P$ using Equation 8. Finally, surface harmonics coefficients for pressure, can be written using $(\nabla^2 P)_{nm}$,

$$p_{nm} = \frac{-1}{n(n+1)} (\nabla^2 P)_{nm}. \tag{17}$$

From p_{nm}, we can reconstruct $P(\theta, \phi)$ using Equation 5. After pressure is obtained, it can be put back into Equation 11 and Equation 12. The equivalent form of Equation 11 and Equation 12, in term of surface harmonics with time steps can be written as,

$$\begin{aligned} \frac{u_{nm}^{t+\Delta t} - u_{nm}^{t-\Delta t}}{2\Delta t} =& u'_{nm} + (n+2)(n+m+1)(2n+3)^{-1} p_{n+1,m} \\ &+ (n-m)(1-n)(2n-1)^{-1} p_{n-1,m} - \nu n(n+1) u_{nm}^{t}, \end{aligned} \tag{18}$$

$$v_{nm}^{t+\Delta t} = \frac{1}{im} [B_{np}(n-1) u_{n-1,m}^{t+\Delta t} + B_{nm}(n+1) u_{n+1,m}^{t+\Delta t}], \tag{19}$$

where

$$B_{np}(n) = -n(n-m+1)(2n+1)^{-1} \quad \text{and} \quad B_{nm} = (n+1)(n+m)(2n+1)^{-1}. \tag{20}$$

It is worth to note that v_{nm}^{t} advances in time with u_{nm}^{t}. Also, for simplification, instead of

$$\frac{u_{nm}^{t+\Delta t} - u_{nm}^{t-\Delta t}}{2\Delta t},$$

I used

$$\frac{u_{nm}^{t+\Delta t} - u_{nm}^{t}}{\Delta t}.$$

After the theoretical background is introduced, the objective now would be to calculate the surface harmonics coefficients (u_{nm}, v_{nm}, u'_{nm}, v'_{nm}, ζ_{nm}, and p_{nm}), then using them in Equation 18 and Equation 19 for the numerical computation.

2.2 Numerical Calculation

It is crucial for me to project surface harmonics and reconstruct the original function, so the first thing I had to do was to define a function in *Python* that enables me to project an arbitrary function into its surface harmonic, then another function that allows me to reconstruct the original function using its surface harmonic coefficient. I first defined a $2D$ mesh grid using pre-defined θ and ϕ, along with some constant terms like r, ν, Ω. But, due to pole conditions, θ cannot be zero, instead, I made θ starting very small (10^{-4}). Then by, using Equation 8 and Equation 5, I calculated the sum using a double loop, and integrals using Simpson's rule in *Scipy*. In order to test whether the code is working, I took the difference between the original arbitrary function and reconstructed function using the surface harmonics. The difference is minimised to 10^{-2} order.

The next step would be calculate U, V, ζ, U', V', and P using the equations obtained from previous section. One thing to note is that every m and n used in a loop is a different m and n and have to be computed separately. The partial derivative in Equation 16 is calculated using the *np.gradient* function in *Numpy*. When computing ζ_{nm} and p_{nm}, m was set to start at 1 due to pole conditions.

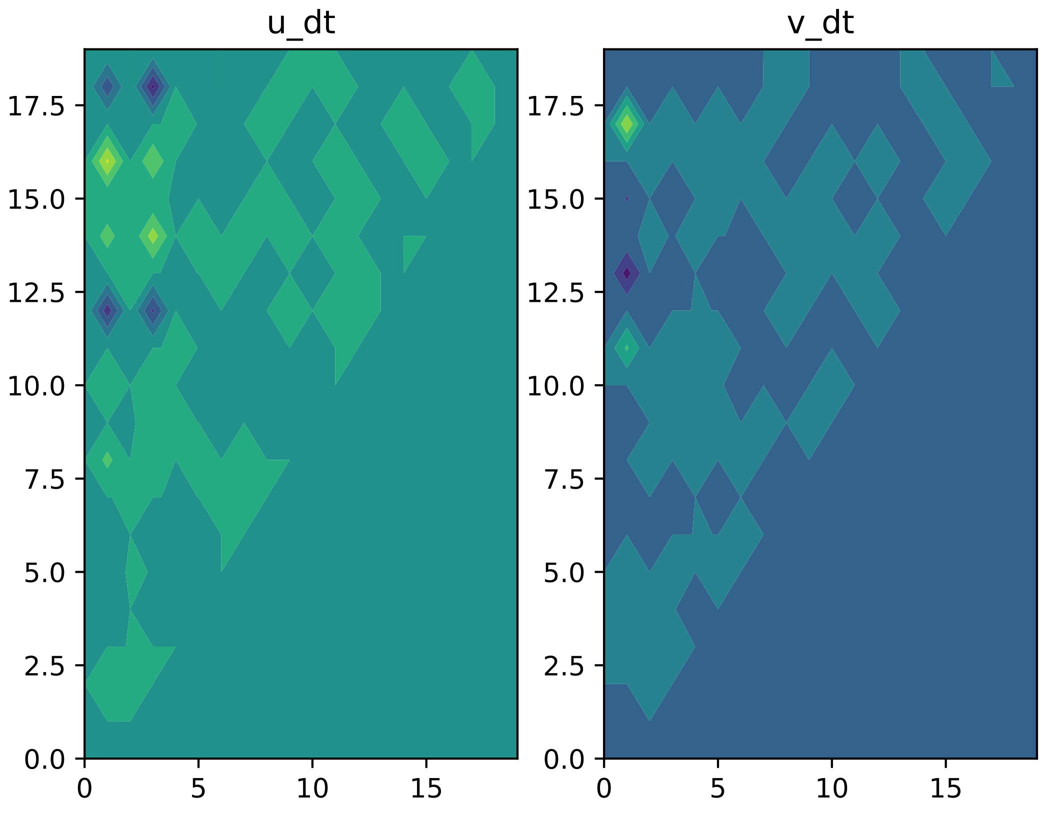

Figure 1: Final plot for calcuated u_{nm} and v_{nm} for a total of 100 time steps, which are denoted as u_{dt} and v_{dt} respectively.

In addition, for Equation 18, every time step requires a new set of parameters like ζ_{nm} and p_{nm}. To simplify the time loop, I have also defined a function for calculating ζ_{nm} and a function for calculating p_{nm}. For the actual code for computing Equation 18, I used a triple loop in t, m, and n. For each time step, a new set of ζ_{nm} and p_{nm} is calculated using the pre-defined functions, and Equation 19 is also calculated for each time step. Pole condition is applied inside this triple loop for time, m, and n.

For initial condition, I used the calculated $u_0 = u^t_{nm}$ as the first time step to compute the $u_1 = u^{t+\Delta t}_{nm}$, then use this new u_1 as the new u^t_{nm}, and so on. The total time steps was kept to be 100. The plot of u_{nm} and v_{nm} are shown in Figure 1.

3 Conclusion

I have learned the theoretical background for the full spectral method. Furthermore, I have successfully recreated the full spectral method used for flows in spherical geometry.