Game Theory with Computer Science Applications
Lecture 3: Existence of a Nash Equilibrium

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Game Setting and related Concepts

- Pure Strategy for agent i: $x_i \in X_i$ (discrete, finite set).
- Mixed Strategy for agent i: $p_i(x_i) = \Pr(\text{agent i plays action } x_i)$.
- Utility to i: $U_i(x_i, \mathbf{x}_{-i})$ and $U_i(p_i, \mathbf{p}_{-i})$.
- Some concepts: closed set, bounded set, convex set, continuous functions.

The Nash's Theorem

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Any finite strategic game has a mixed strategy Nash Equilibrium.

Brouwer Fixed Point Theorem

Brouwer Fixed Point Theorem

Let $C \subseteq \mathbb{R}^n$ be a **compact (closed and bounded)** and **convex** set. Let $f\colon C \to C$ be a continuous function. Then f has a fixed pointed in C, *i.e.*, $x \in C$, *s.t.*, $x = f(x)$.

Proof.

For the one-dimensional case. When $n = 1$, the convex and compact sets are closed intervals $[a, b]$. Let $f\colon [a, b] \leftarrow [a, b]$. If $f(a) = a$ or $f(b) = b$ we are done. Suppose $f(a) > a$ and $f(b) < b$. Consider $g(x) = f(x) - x$. Then $g(a) > 0$, $g(b) < 0$. g is continuous because f is continuous. The ***intermediate value theorem*** tells us that there is some $a < x^* < b$, such that $g(x^*) = 0$. □

Proof for Nash Theorem using Brouwer FP Theorem

- Let $(p_1, p_2, \cdots, p_n)$ be a set of strategies.
- Define $r_i(x_i) = (U_i(x_i, p_{-i}) - U_i(p_i, p_{-i}))^+$, i.e., $r_i(x_i)$ is the amount by which the expected utility to i can be increased by changing strategy from p_i to x_i.
- Define

$$f_i(p_i(x_i)) = \frac{p_i(x_i) + r_i(x_i)}{\sum_x (p_i(x) + r_i(x))}.$$

- $(p_1, p_2, \cdots, p_n)$ is a **convex and compact set**.
  $f(p) = (f_1(p_1), f_2(p_2), \cdots, f_n(p_n))$ is a **continuous function**.
  Homework.
- We then have a fixed point p:

$$p_i(x_i) = \frac{p_i(x_i) + r_i(x_i)}{\sum_x (p_i(x) + r_i(x))}.$$

Proof for Nash Theorem using Brouwer (continued)

- We will now show that for such a fixed point,

$$r_i(x_i) = 0 \quad \forall i, x_i.$$

i.e., no increase in utility is possible by changing strategy from p_i to x_i. Thus, such a fixed point is a NE.

- First, we claim that **for each** i**,** $\exists x_i$**, s.t.,** $r_i(x_i) = 0$**.** We will prove this by contradiction. Suppose for some i, $r_i(x_i) > 0$, $\forall x_i$. Then,

$$\begin{aligned} & U_i(x_i, p_{-i}) > U_i(p_i, p_{-i}), \quad \forall x_i. \\ \Rightarrow \quad & \sum_{x_i} p_i(x_i) U_i(x_i, p_{-i}) > U_i(p_i, p_{-i}), \\ \Rightarrow \quad & U_i(p_i, p_{-i}) > U_i(p_i, p_{-i}), \end{aligned}$$

which is a contradiction.

Proof for Nash Theorem using Brouwer (continued)

- Fix i, let x_i be s.t., $r_i(x_i) = 0$. Then,

$$\begin{aligned}
& p_i(x_i) = \frac{p_i(x_i)}{\sum_x (p_i(x) + r_i(x))} \\
\Rightarrow \quad & \sum_x p_i(x) + \sum_x r_i(x) = 1 \\
\Rightarrow \quad & \sum_x r_i(x) = 0 \\
\Rightarrow \quad & r_i(x) = 0, \forall x \in A_i.
\end{aligned}$$

This completes the proof.

Kakutani Fixed-point Theorem

Kakutani Fixed-point Theorem

Let C be a convex and compact subset of $\mathbb{R}^n$. Let f be a correspondence mapping each point in C to a subset of a C, i.e., $f: C \to 2^C$. Suppose the following three conditions hold:

- $f(x) \neq \emptyset, \quad \forall x,$
- $f(x)$ is a convex set, $\forall x$,
- f has a closed graph: if $\{x_n, y_n\} \to \{x, y\}$ with $y_n \in f(x_n)$, then $y \in f(x)$.

Then f has a fixed point in C.

Weierstrass's Theorem

Let A be a non-empty, compact subset of $\mathbb{R}^n$ and let $f: A \to \mathbb{R}$ be a continuous function. Then, there exists an optimal solution to the optimization problem $Min\ f(x), \quad x \in A$.

Proof for Nash Theorem using Kakutani

- We will apply Kakutani's fixed point theorem to establish the existence of a solution to

$$p \in BP(p),$$

  where $p = (p_1, p_2, \cdots, p_n)$ and $BP(p) = (BP(p_{-1}), \cdots, BP(p_{-n}))$.
- We will verify the mapping BP satisfies the conditions required in the Kakutani's fixed-point theorem.
- (1) $BP(p)$ **is a non-empty set for each** p**.**
  This is because $\max_{p_i \in \Delta X_i} U_i(p_i, p_{-i})$ is a maximization problem of a continuous function over the set of the probability distribution over X_i, which is a compact set. The result follows from **Weierstrass' extreme value theorem**.

Proof for Nash Theorem using Kakutani (continued)

- (2) **For each p, $BP(p)$ is a convex set.**
  We recall that

$$U_i(p_i, p_{-i}) = \sum_{x} p_1(x_1) \times \cdots \times p_n(x_n) \times U_i(x_i, x_{-i}).$$

So if $p_i^*, \tilde{p}_i \in BP(p_{-i})$, as $U_i(p_i^*, p_{-i}) = U_i(\tilde{p}_i, p_{-i})$, we can verify that

$$U_i(\alpha \times p_i^* + (1-\alpha) \times \tilde{p}_i, p_{-i}) = U_i(p_i^*, p_{-i}), \quad \forall \alpha \in [0,1].$$

Hence, we have $\alpha \times p_i^* + (1-\alpha) \times \tilde{p}_i \in BP(p_{-i})$

Proof for Nash Theorem using Kakutani (continued)

- (3) **We will now show that BP has a closed graph.** Let $(p_i^n, p_{-i}^n) \to (p_i, p_{-i})$ with $p_i^n \in BP(p_{-i}^n)$. Suppose that $p_i \notin BP(p_{-i})$. Then $\exists \tilde{p}_i$ and $\epsilon > 0$ s.t.,

$$U_i(\tilde{p}_i, p_{-i}) \geq U_i(p_i, p_{-i}) + \epsilon.$$

- We next show that $\tilde{p}_i$ is a better response for p_{-i}^n (for some n) than p_i^n, and thus contradicts $p_i^n \in BP(p_{-i}^n)$.
- For sufficiently large n,

$$\begin{aligned}
U_i(\tilde{p}_i, p_{-i}^n) &\geq U_i(\tilde{p}_i, p_{-i}) - \frac{\epsilon}{2} && (1)\\
&\geq U_i(p_i, p_{-i}) + \epsilon - \frac{\epsilon}{2} && (2)\\
&\geq U_i(p_i^n, p_{-i}^n) - \frac{\epsilon}{4} + \frac{\epsilon}{2} && (3)\\
&= U_i(p_i^n, p_{-i}^n) + \frac{\epsilon}{4}. && (4)
\end{aligned}$$

Proof for Nash Theorem using Kakutani (continued)

- (1) comes from that $p^n_{-i} \to p_{-i}$ and U_i is continuous.
  (3) comes from that for sufficiently large n, $(p_i^n, p^n_{-i}) \to (p_i, p_{-i})$ and U_i is continuous.
- The above result contradicts $p_i^n \in BP(p^n_{-i})$. Thus, BP has a closed graph.
- Nash's Theorem follows from the Kakutani fixed point theorem.