Adaptiva robust online portfolio selection

1. 考虑交易成本的投资组合选择问题

m 个资产 n 连续时期投资

资产 i 在时期 t 的收盘价为 $p_{t,i}$

用 $\mathbf{p}_t = (p_{t,1}, \ldots, p_{t,m})^\top$

相对价格变化 $\mathbf{x}_t = \mathbf{p}_t / \mathbf{p}_{t-1}$

在每期搜索投资组合$\mathbf{b}_t \in \Delta_m$，其中$\Delta_m = \left\{\mathbf{b} \in \mathbb{R} \mid \mathbf{b} \geq 0, \mathbf{b}^\top \mathbf{1} = 1\right\}$

初始投资组合设为$\mathbf{b}_1 = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)^\top$

在投资周期 t 执行策略 $\mathbf{b}_t$，那么在投资周期 t 的期末，可以观察到每个资产的收盘价格。那么相对价格 $\mathbf{x}_t$就已知了，并且期末投资组合各资产的权重为

$$\hat{\mathbf{b}_t} = \frac{\mathbf{b}_t \cdot \mathbf{x}_t}{\mathbf{b}_t^\top \mathbf{x}_t},$$

考虑交易成本

在投资周期 t 的期末，给定将要执行策略 $\mathbf{b}_{t+1}$，计入交易成本后剩余的净资金比例 w_t 满足下列方程：

$$w_t + \gamma \left[\sum_{i=1}^{m} \left(\hat{b}_{t,i} - w_t b_{t+1,i}\right)^+ + \sum_{i=1}^{m} \left(w_t b_{t+1,i} - \hat{b}_{t,i}\right)^+\right] = 1, \tag{1}$$

简记为：

$$w_t + \gamma \left\|\hat{\mathbf{b}}_t - w_t \mathbf{b}_{t+1}\right\|_1 = 1$$

记 S_t 为在 t 期末的累积财富，则更新过程即为 $S_{t+1} = S_t w_t \left(\mathbf{b}_{t+1}^\top \mathbf{x}_{t+1}\right)$

整个投资过程的框架如下：

Framework 1: A framework for the OLPS problem with consideration of transaction costs

Initialization: Set $\mathbf{b}_1 = \mathbf{1}/m$, $\mathbf{b}_0 = \mathbf{0}$ and $S_0 = 1$.

for $t = 1, \ldots, n$ **do**
- Obtain the new portfolio weights $\mathbf{b}_t$ based on an OLPS strategy.
- Compute the net proportion w_{t-1} of the rebalancing (from $\hat{\mathbf{b}}_{t-1}$) by solving (1).
- Observe the realized return $\mathbf{x}_t$.
- Update the cumulative wealth as $S_t = S_{t-1} w_{t-1} (\mathbf{b}_t^\top \mathbf{x}_t)$.
- Obtain the current portfolio weights $\hat{\mathbf{b}}_t = \mathbf{b}_t \cdot \mathbf{x}_t / \mathbf{b}_t^\top \mathbf{x}_t$.

end

OLPS策略通常包含预测步骤，即需要在优化步骤中去预测 $\tilde{\mathbf{x}}_t$ ，比如可以采用历史数据的滑动平均值等方法。

Base model is as follows：

$$\begin{array}{ll} \underset{w \geq 0, \mathbf{b} \geq \mathbf{0}}{\operatorname{maximize}} & w \mathbf{b}^\top \tilde{\mathbf{x}}_t - \lambda \left\|\hat{\mathbf{b}}_{t-1} - w \mathbf{b}\right\|_1 \\ \text{subject to} & w + \gamma \left\|\hat{\mathbf{b}}_{t-1} - w \mathbf{b}\right\|_1 = 1, \quad \mathbf{b}^\top \mathbf{1} = 1, \end{array} \tag{2}$$

非线性优化问题需要对问题进行转化，可以证明，该问题与以下优化问题等价：

Theorem 1. *Consider the following convex program*

$$\begin{array}{ll} \underset{\mathbf{b} \geq \mathbf{0}}{\operatorname{maximize}} & \mathbf{b}^\top \tilde{\mathbf{x}}_t - \lambda \left\|\hat{\mathbf{b}}_{t-1} - \mathbf{b}\right\|_1 \\ \text{subject to} & \mathbf{b}^\top \mathbf{1} + \gamma \left\|\hat{\mathbf{b}}_{t-1} - \mathbf{b}\right\|_1 \leq 1, \end{array} \tag{3}$$

and let $\mathbf{b}_{(2)}^*$ *be an optimal solution. Then, the optimal value of (2) is the same as that of (3), and* $\left(w_{(1)}^*, \mathbf{b}_{(1)}^*\right) := \left(\mathbf{1}^\top \mathbf{b}_{(2)}^*, \frac{\mathbf{b}_{(2)}^*}{\mathbf{1}^\top \mathbf{b}_{(2)}^*}\right)$ *is an optimal solution to (2)*

上述的凸优化问题与下列线性规划问题等价，并可使用现有的优化器进行求解。

Corollary 1. *The problem (2) is equivalent to the following LP.*

$$\begin{array}{ll}\underset{\mathbf{b}\geq\mathbf{0},\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0}}{\operatorname{maximize}} & \mathbf{b}^\top\tilde{\mathbf{x}}_t-\lambda(\mathbf{u}+\mathbf{v})^\top\mathbf{1}\\ \text{subject to} & \mathbf{b}^\top\mathbf{1}+\gamma(\mathbf{u}+\mathbf{v})^\top\mathbf{1}\leq 1\\ & \mathbf{u}\geq\hat{\mathbf{b}}_{t-1}-\mathbf{b},\quad \mathbf{v}\geq\mathbf{b}-\hat{\mathbf{b}}_{t-1}\end{array}\tag{4}$$

直接应用Base model (2) 会出现两个问题：

1. 由于算术收益的线性性质，模型可能会只投资单一资产，比如只投资具有最大预测值 $\tilde{x_{t,i}}$ 的资产。
2. 依赖于对下一期收益率的预测，收益预测具有较大的不确定性，导致模型表现不稳定。

因此，采用 robust optimization 方法，而不使用简单的算术收益。

2. 鲁棒椭球策略 (robust ellipsoidal (RELP) strategy)

考虑以下鲁棒优化模型：

$$\begin{array}{ll}\underset{w\geq 0,\mathbf{b}\geq\mathbf{0}}{\operatorname{maximize}} & \min_{\mathbf{x}\in\mathcal{U}}\left\{w\mathbf{b}^\top\mathbf{x}\right\}-\lambda\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1\\ \text{subject to} & w+\gamma\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1=1,\quad \mathbf{b}^\top\mathbf{1}=1\end{array}\tag{5}$$

在此处 $\mathcal{U}$ 是椭球不确定集，如下：

$$\mathcal{U}\left(\tilde{\mathbf{x}}_t,\kappa\right)=\left\{\mathbf{x}\in\mathbb{R}^m\mid\left(\mathbf{x}-\tilde{\mathbf{x}}_t\right)^\top\mathbf{\Sigma}^{-1}\left(\mathbf{x}-\tilde{\mathbf{x}}_t\right)\leq\kappa^2\right\}\tag{6}$$

- $\tilde{\mathbf{x}}_t\in\mathbb{R}^m$ 椭球中心参数，最近 $m+1$ 个样本的滑动平均估计
- $\mathbf{\Sigma}\succ\mathbf{0}$ 椭球形状参数，最近 $m+1$ 个样本的协方差估计，捕获市场走势，且满足正定性的要求
- $\kappa\geq 0$ 椭球大小参数。

(5) 直接求解困难，涉及椭球内部最大化的问题。考虑以下命题：

Proposition 1. *Given the ellipsoidal uncertainty (6), we have*

$$\min_{\mathbf{x}\in\mathcal{U}(\tilde{\mathbf{x}}_t,\kappa)}\left\{\mathbf{b}^\top\mathbf{x}\right\}=\tilde{\mathbf{x}}^\top\mathbf{b}-\kappa\|\mathbf{U}\mathbf{b}\|_2,$$

where $\mathbf{U}$ is the upper triangular matrix of the Cholesky factorization of $\mathbf{\Sigma}$, i.e., $\mathbf{\Sigma}=\mathbf{U}^\top\mathbf{U}$

由该命题，可以得到与标准差对应的 l_2 norm为：

$$\|\mathbf{U}\mathbf{b}\|_2=\sqrt{(\mathbf{U}\mathbf{b})^\top(\mathbf{U}\mathbf{b})}=\sqrt{\mathbf{b}^\top\mathbf{U}^\top\mathbf{U}\mathbf{b}}=\sqrt{\mathbf{b}^\top\mathbf{\Sigma}\mathbf{b}}.$$

考虑了 l_2 norm 也是考虑了预测的风险，即预测的不确定性，会给出更分散的投资组合。

Lemma 1. *Let $\sigma=\sqrt{\sum_{i=1}^m\sigma_i^2}$, where $\{\sigma_i^2\}$ are the diagonal entries of $\mathbf{\Sigma}=\mathbf{U}^\top\mathbf{U}$. Then, for any $\{\mathbf{b}_1,\mathbf{b}_2\}\subseteq\mathbb{R}^m$, whe have*

$$\left|\|\mathbf{U}\mathbf{b}_1\|_2-\|\mathbf{U}\mathbf{b}_2\|_2\right|\leq\sigma\left\|\mathbf{b}_1-\mathbf{b}_2\right\|_2.$$

Theorem 2. *Assume that $\max_i\{\tilde{x}_{t,i}\}>\kappa\sigma+\lambda$, where $\sigma=\sqrt{\sum_{i=1}^m\sigma_i^2}$ are $\{\sigma_i^2\}$ are the diagonal entries of $\mathbf{\Sigma}$. Consider the following program.*

$$\begin{array}{ll}\underset{\mathbf{b}\geq\mathbf{0}}{\operatorname{maximize}} & \mathbf{b}^\top\tilde{\mathbf{x}}-\lambda\left\|\hat{\mathbf{b}}_{t-1}-\mathbf{b}\right\|_1-\kappa\|\mathbf{U}\mathbf{b}\|_2\\ \text{subject to} & \mathbf{b}^\top\mathbf{1}+\gamma\left\|\hat{\mathbf{b}}_{t-1}-\mathbf{b}\right\|_1\leq 1\end{array}\tag{7}$$

and let $\mathbf{b}_{(3)}^*$ be an optimal solution. Then, $\left(w_{(1)}^*,\mathbf{b}_{(1)}^*\right):=\left(\mathbf{1}^\top\mathbf{b}_{(3)}^*,\frac{\mathbf{b}_{(3)}^*}{\mathbf{1}^\top\mathbf{b}_{(3)}^*}\right)$ is an optimal solution to (5).

Algorithm 2: The RELP strategy with the SB and top-K adaptive schemes on parameter $\phi \in \{\lambda, \kappa\}$

Initialization: Set $\mathbf{b}_1 = \mathbf{1}/m$, $\mathbf{b}_0 = \mathbf{0}$ and $S_0 = 1$. Pick a range of values of ϕ, i.e., $\{\phi_1, \dots, \phi_R\}$. Here, ϕ is a generic parameter representing λ or κ.

Initialization for SB scheme**:** Choose window size W, indifference zone parameter δ and the initial tracking portfolio r'.

Initialization for Top-K scheme**:** Choose K and initial $\overline{\phi}_0$.

for $t = 1, \dots, n$ **do**

- Obtain the new portfolio weights $\mathbf{b}_t^r$ for expert by solving (9), e.g., using an off-the-shelf optimization solver, with ϕ_r for $r \in \{1, \dots, R\}$.
- (For $t = 1$, we use $\mathbf{b}_1^r = \mathbf{1}/m$; for $t \in \{2, \dots, m+1\}$, we use $\mathbf{b}_t^r = \hat{\mathbf{b}}_{t-1}^r$.)
- **if** SB scheme **then**
  - Set $\mathbf{b}_t = \mathbf{b}_t^{r'}$.
- **else if** Top-K scheme **then**
  - Obtain the new portfolio weights $\mathbf{b}_t$ by solving (9), e.g., using an off-the-shelf optimization solver, with $\phi = \overline{\phi}_{t-1}$.
- **end**
- Compute the net proportion w_{t-1} of the rebalance (from $\hat{\mathbf{b}}_{t-1}$), as well as w_{t-1}^r for each expert (from $\hat{\mathbf{b}}_{t-1}^r$) by solving (1) .
- Observe the realized return $\mathbf{x}_t$.
- Update the cumulative wealth as $S_t = S_{t-1} w_{t-1} (\mathbf{b}_t^\top \mathbf{x}_t)$.
- Update the cumulative wealth for each expert as $S_t^r = S_{t-1}^r w_{t-1}^r [(\mathbf{b}_t^r)^\top \mathbf{x}_t]$ for $r \in \{1, \dots, R\}$.
- **if** SB scheme **then**
  - Based on the recent portfolio wealth $\{S_{t-W+1}^{r'}, \dots, S_t^{r'}\}$ of expert r', compute the sample mean $\bar{S}_t^{r'}$ and sample standard deviation $\hat{\sigma}_t^{r'}$.
  - **if** $\mathcal{R} = \left\{ r'' \in \{1, \dots, R\} \setminus \{r'\} \mid \bar{S}_t^{r''} > \bar{S}_t^{r'} + \max\{\delta, 1.96\hat{\sigma}_t^{r'}\} \right\} \neq \emptyset$ **then**
    - Update the tracking expert by setting $r' \in \operatorname{argmax}_{r \in \mathcal{R}} \{\bar{S}_t^r\}$.
  - **end**
- **else if** Top-K scheme **then**
  - Compute $\overline{\phi}_t$ as the average value of ϕ's of the top-K performing portfolios.
- **end**
- Obtain the current portfolio weights $\hat{\mathbf{b}}_t = \mathbf{b}_t \cdot \mathbf{x}_t / \mathbf{b}_t^\top \mathbf{x}_t$ as well as $\hat{\mathbf{b}}_t^r$ for each expert.

end