

# Adaptiva robust online portfolio selection

## 1. Introduction

考虑交易成本的投资组合选择问题

$m$  个资产  $n$  连续时期投资

资产  $i$  在时期  $t$  的收盘价为  $p_{t,i}$

用  $\mathbf{p}_t = (p_{t,1}, \dots, p_{t,m})^\top$

相对价格变化  $\mathbf{x}_t = \mathbf{p}_t / \mathbf{p}_{t-1}$

在每期搜索投资组合  $\mathbf{b}_t \in \Delta_m$ ，其中  $\Delta_m = \{\mathbf{b} \in \mathbb{R} \mid \mathbf{b} \geq 0, \mathbf{b}^\top \mathbf{1} = 1\}$

初始投资组合设为  $\mathbf{b}_1 = (\frac{1}{m}, \dots, \frac{1}{m})^\top$

在投资周期  $t$  执行策略  $\mathbf{b}_t$ ，那么在投资周期  $t$  的期末，可以观察到每个资产的收盘价格。那么相对价格  $\mathbf{x}_t$  就已知了，并且期末投资组合各资产的权重为

$$\hat{\mathbf{b}}_t = \frac{\mathbf{b}_t \cdot \mathbf{x}_t}{\mathbf{b}_t^\top \mathbf{x}_t},$$

考虑交易成本

在投资周期  $t$  的期末，给定将要执行策略  $\mathbf{b}_{t+1}$ ，计入交易成本后剩余的净资金比例  $w_t$  满足下列方程：

$$w_t + \gamma \left[ \sum_{i=1}^m \left( \hat{b}_{t,i} - w_t b_{t+1,i} \right)^+ + \sum_{i=1}^m \left( w_t b_{t+1,i} - \hat{b}_{t,i} \right)^+ \right] = 1, \tag{1}$$

简记为：

$$w_t + \gamma \left\| \hat{\mathbf{b}}_t - w_t \mathbf{b}_{t+1} \right\|_1 = 1$$

记  $S_t$  为在  $t$  期末的累积财富，则更新过程即为  $S_{t+1} = S_t w_t (\mathbf{b}_{t+1}^\top \mathbf{x}_{t+1})$

整个投资过程的框架如下：

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### Framework 1: A framework for the OLPS problem with consideration of transaction costs

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**Initialization:** Set  $\mathbf{b}_1 = \mathbf{1}/m$ ,  $\mathbf{b}_0 = \mathbf{0}$  and  $S_0 = 1$ .

- 1 **for**  $t = 1, \dots, n$  **do**
  - 2     Obtain the new portfolio weights  $\mathbf{b}_t$  based on an OLPS strategy.
  - 3     Compute the net proportion  $w_{t-1}$  of the rebalancing (from  $\hat{\mathbf{b}}_{t-1}$ ) by solving (1).
  - 4     Observe the realized return  $\mathbf{x}_t$ .
  - 5     Update the cumulative wealth as  $S_t = S_{t-1} w_{t-1} (\mathbf{b}_t^\top \mathbf{x}_t)$ .
  - 6     Obtain the current portfolio weights  $\hat{\mathbf{b}}_t = \mathbf{b}_t \cdot \mathbf{x}_t / \mathbf{b}_t^\top \mathbf{x}_t$ .
  - 7 **end**
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OLPS策略通常包含预测步骤，即需要在优化步骤中去预测  $\tilde{\mathbf{x}}_t$ ，比如可以采用历史数据的滑动平均值等方法。

Base model is as follows：

$$\begin{aligned} & \underset{w \geq 0, \mathbf{b} \geq 0}{\text{maximize}} && w \mathbf{b}^\top \tilde{\mathbf{x}}_t - \lambda \left\| \hat{\mathbf{b}}_{t-1} - w \mathbf{b} \right\|_1 \\ & \text{subject to} && w + \gamma \left\| \hat{\mathbf{b}}_{t-1} - w \mathbf{b} \right\|_1 = 1, \quad \mathbf{b}^\top \mathbf{1} = 1, \end{aligned} \tag{2}$$

非线性优化问题需要对问题进行转化，可以证明，该问题与以下优化问题等价：

**Theorem 1.** Consider the following convex program

$$\begin{aligned} & \underset{\mathbf{b} \geq 0}{\text{maximize}} && \mathbf{b}^\top \tilde{\mathbf{x}}_t - \lambda \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 \\ & \text{subject to} && \mathbf{b}^\top \mathbf{1} + \gamma \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 \leq 1, \end{aligned} \tag{3}$$

and let  $\mathbf{b}_{(2)}^*$  be an optimal solution. Then, the optimal value of (2) is the same as that of (3), and  $(w_{(1)}^*, \mathbf{b}_{(1)}^*) := \left( \mathbf{1}^\top \mathbf{b}_{(2)}^*, \frac{\mathbf{b}_{(2)}^*}{\mathbf{1}^\top \mathbf{b}_{(2)}^*} \right)$  is an optimal solution to (2)

上述的凸优化问题与下列线性规划问题等价，并可使用现有的优化器进行求解。

**Corollary 1.** *The problem (2) is equivalent to the following LP.*

$$\begin{aligned} & \underset{\mathbf{b} \geq 0, \mathbf{u} \geq 0, \mathbf{v} \geq 0}{\text{maximize}} && \mathbf{b}^\top \tilde{\mathbf{x}}_t - \lambda(\mathbf{u} + \mathbf{v})^\top \mathbf{1} \\ & \text{subject to} && \mathbf{b}^\top \mathbf{1} + \gamma(\mathbf{u} + \mathbf{v})^\top \mathbf{1} \leq 1 \\ & && \mathbf{u} \geq \hat{\mathbf{b}}_{t-1} - \mathbf{b}, \quad \mathbf{v} \geq \mathbf{b} - \hat{\mathbf{b}}_{t-1} \end{aligned} \tag{4}$$

直接应用Base model (2) 会出现两个问题：

1. 由于算术收益的线性性质，模型可能会只投资单一资产，比如只投资具有最大预测值  $\tilde{x}_{t,i}$  的资产。
2. 依赖于对下一期收益率的预测，收益预测具有较大的不确定性，导致模型表现不稳定。

因此，采用 robust optimization 方法，而不使用简单的算术收益。

鲁棒优化模型如下：

$$\begin{aligned} & \underset{w \geq 0, \mathbf{b} \geq 0}{\text{maximize}} && \min_{\mathbf{x} \in \mathcal{U}} \{w \mathbf{b}^\top \mathbf{x}\} - \lambda \left\| \hat{\mathbf{b}}_{t-1} - w \mathbf{b} \right\|_1 \\ & \text{subject to} && w + \gamma \left\| \hat{\mathbf{b}}_{t-1} - w \mathbf{b} \right\|_1 = 1, \quad \mathbf{b}^\top \mathbf{1} = 1 \end{aligned} \tag{5}$$

在此处  $\mathcal{U}$  是椭球不确定集，如下：

$$\mathcal{U}(\tilde{\mathbf{x}}_t, \kappa) = \left\{ \mathbf{x} \in \mathbb{R}^m \mid (\mathbf{x} - \tilde{\mathbf{x}}_t)^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \tilde{\mathbf{x}}_t) \leq \kappa^2 \right\} \tag{6}$$

- $\tilde{\mathbf{x}}_t \in \mathbb{R}^m$  椭球中心参数，最近  $m + 1$  个样本的滑动平均估计
- $\boldsymbol{\Sigma} \succ \mathbf{0}$  椭球形状参数，最近  $m + 1$  个样本的协方差估计，捕获市场走势，且满足正定性的要求
- $\kappa \geq 0$  椭球大小参数。

(5) 直接求解困难，涉及椭球内部最大化的问题。考虑以下命题：

**Proposition 1.** *Given the ellipsoidal uncertainty (6), we have*

$$\min_{\mathbf{x} \in \mathcal{U}(\tilde{\mathbf{x}}_t, \kappa)} \{ \mathbf{b}^\top \mathbf{x} \} = \tilde{\mathbf{x}}^\top \mathbf{b} - \kappa \| \mathbf{U} \mathbf{b} \|_2,$$

where  $\mathbf{U}$  is the upper triangular matrix of the Cholesky factorization of  $\boldsymbol{\Sigma}$ , i.e.,  $\boldsymbol{\Sigma} = \mathbf{U}^\top \mathbf{U}$

由该命题，可以得到与标准差对应的  $l_2$  norm为：

$$\| \mathbf{U} \mathbf{b} \|_2 = \sqrt{(\mathbf{U} \mathbf{b})^\top (\mathbf{U} \mathbf{b})} = \sqrt{\mathbf{b}^\top \mathbf{U}^\top \mathbf{U} \mathbf{b}} = \sqrt{\mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b}}.$$

考虑了  $l_2$  norm 也是考虑了预测的风险，即预测的不确定性，会给出更分散的投资组合。

**Lemma 1.** *Let  $\sigma = \sqrt{\sum_{i=1}^m \sigma_i^2}$ , where  $\{\sigma_i^2\}$  are the diagonal entries of  $\boldsymbol{\Sigma} = \mathbf{U}^\top \mathbf{U}$ . Then, for any  $\{\mathbf{b}_1, \mathbf{b}_2\} \subseteq \mathbb{R}^m$ , we have*

$$| \| \mathbf{U} \mathbf{b}_1 \|_2 - \| \mathbf{U} \mathbf{b}_2 \|_2 | \leq \sigma \| \mathbf{b}_1 - \mathbf{b}_2 \|_2.$$

**Theorem 2.** *Assume that  $\max_i \{ \tilde{x}_{t,i} \} > \kappa \sigma + \lambda$ , where  $\sigma = \sqrt{\sum_{i=1}^m \sigma_i^2}$  are  $\{\sigma_i^2\}$  are the diagonal entries of  $\boldsymbol{\Sigma}$ . Consider the following program.*

$$\begin{aligned} & \underset{\mathbf{b} \geq 0}{\text{maximize}} && \mathbf{b}^\top \tilde{\mathbf{x}} - \lambda \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 - \kappa \| \mathbf{U} \mathbf{b} \|_2 \\ & \text{subject to} && \mathbf{b}^\top \mathbf{1} + \gamma \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 \leq 1 \end{aligned} \tag{7}$$

and let  $\mathbf{b}_{(3)}^*$  be an optimal solution. Then,  $\left( w_{(1)}^*, \mathbf{b}_{(1)}^* \right) := \left( \mathbf{1}^\top \mathbf{b}_{(3)}^*, \frac{\mathbf{b}_{(3)}^*}{\mathbf{1}^\top \mathbf{b}_{(3)}^*} \right)$  is an optimal solution to (5).

**Algorithm 2:** The RELP strategy with the SB and top- $K$  adaptive schemes on parameter  $\phi \in \{\lambda, \kappa\}$

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1 Initialization: Set  $\mathbf{b}_1 = \mathbf{1}/m$ ,  $\mathbf{b}_0 = \mathbf{0}$  and  $S_0 = 1$ . Pick a range of
   values of  $\phi$ , i.e.,  $\{\phi_1, \dots, \phi_R\}$ . Here,  $\phi$  is a
   generic parameter representing  $\lambda$  or  $\kappa$ .
2 Initialization for SB scheme: Choose window size  $W$ ,
   indifference zone parameter  $\delta$  and the initial tracking portfolio
    $r'$ .
3 Initialization for Top- $K$  scheme: Choose  $K$  and initial  $\bar{\phi}_0$ .
4 for  $t = 1, \dots, n$  do
5   Obtain the new portfolio weights  $\mathbf{b}'_t$  for expert by solving
     (9), e.g., using an off-the-shelf optimization solver, with  $\phi_r$ 
     for  $r \in \{1, \dots, R\}$ .
6   (For  $t = 1$ , we use  $\mathbf{b}'_1 = \mathbf{1}/m$ ; for  $t \in \{2, \dots, m+1\}$ , we use
      $\mathbf{b}'_t = \hat{\mathbf{b}}'_{t-1}$ .)
7   if SB scheme then
8     Set  $\mathbf{b}_t = \mathbf{b}'_t$ .
9   else if Top- $K$  scheme then
10    Obtain the new portfolio weights  $\mathbf{b}_t$  by solving (9), e.g.,
      using an off-the-shelf optimization solver, with
       $\phi = \bar{\phi}_{t-1}$ .
11  end
12  Compute the net proportion  $w_{t-1}$  of the rebalance (from
      $\hat{\mathbf{b}}_{t-1}$ ), as well as  $w'_{t-1}$  for each expert (from  $\hat{\mathbf{b}}'_{t-1}$ ) by solving
     (1).
13  Observe the realized return  $\mathbf{x}_t$ .
14  Update the cumulative wealth as  $S_t = S_{t-1} w_{t-1} (\mathbf{b}_t^\top \mathbf{x}_t)$ .
15  Update the cumulative wealth for each expert as
      $S'_t = S'_{t-1} w'_{t-1} [(\mathbf{b}'_t)^\top \mathbf{x}_t]$  for  $r \in \{1, \dots, R\}$ .
16  if SB scheme then
17    Based on the recent portfolio wealth  $\{S'_{t-W+1}, \dots, S'_t\}$  of
      expert  $r'$ , compute the sample mean  $\bar{S}'_t$  and sample
      standard deviation  $\hat{\sigma}'_t$ .
18    if  $\mathcal{R} = \{r'' \in \{1, \dots, R\} \setminus \{r'\} \mid \bar{S}'_{t''} >$ 
       $\bar{S}'_t + \max\{\delta, 1.96\hat{\sigma}'_t\}\} \neq \emptyset$  then
19      Update the tracking expert by setting
       $r' \in \operatorname{argmax}_{r \in \mathcal{R}} \{\bar{S}'_r\}$ .
20    end
21  else if Top- $K$  scheme then
22    Compute  $\bar{\phi}_t$  as the average value of  $\phi$ 's of the top- $K$ 
      performing portfolios.
23  end
24  Obtain the current portfolio weights  $\hat{\mathbf{b}}_t = \mathbf{b}_t \cdot \mathbf{x}_t / \mathbf{b}'_t^\top \mathbf{x}_t$  as well
     as  $\hat{\mathbf{b}}'_t$  for each expert.
25 end
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