Adaptiva robust online portfolio selection

1. Introduction

考虑交易成本的投资组合选择问题

m 个资产 n 连续时期投资

资产 i 在时期 t 的收盘价为 $p_{t,i}$

用 $\mathbf{p}_t=(p_{t,1},\ldots,p_{t,m})^\top$

相对价格变化 $\mathbf{x}_t=\mathbf{p}_t/\mathbf{p}_{t-1}$

在每期搜索投资组合$\mathbf{b}_t\in\Delta_m$，其中$\Delta_m=\left\{\mathbf{b}\in\mathbb{R}\mid\mathbf{b}\geq 0,\mathbf{b}^\top\mathbf{1}=1\right\}$

初始投资组合设为$\mathbf{b}_1=\left(\frac{1}{m},\ldots,\frac{1}{m}\right)^\top$

在投资周期 t 执行策略 $\mathbf{b}_t$，那么在投资周期 t 的期末，可以观察到每个资产的收盘价格。那么相对价格 $\mathbf{x}_t$就已知了，并且期末投资组合各资产的权重为

$$\hat{\mathbf{b}_t}=\frac{\mathbf{b}_t\cdot\mathbf{x}_t}{\mathbf{b}_t^\top\mathbf{x}_t},$$

考虑交易成本

在投资周期 t 的期末，给定将要执行策略 $\mathbf{b}_{t+1}$，计入交易成本后剩余的净资金比例 w_t 满足下列方程：

$$w_t+\gamma\left[\sum_{i=1}^m\left(\hat{b}_{t,i}-w_tb_{t+1,i}\right)^++\sum_{i=1}^m\left(w_tb_{t+1,i}-\hat{b}_{t,i}\right)^+\right]=1, \tag{1}$$

简记为：

$$w_t+\gamma\left\|\hat{\mathbf{b}}_t-w_t\mathbf{b}_{t+1}\right\|_1=1$$

记 S_t 为在 t 期末的累积财富，则更新过程即为 $S_{t+1}=S_tw_t\left(\mathbf{b}_{t+1}^\top\mathbf{x}_{t+1}\right)$

整个投资过程的框架如下：

Framework 1: A framework for the OLPS problem with consideration of transaction costs

Initialization: Set $\mathbf{b}_1=\mathbf{1}/m$, $\mathbf{b}_0=\mathbf{0}$ and $S_0=1$.

for $t=1,\ldots,n$ **do**

  Obtain the new portfolio weights $\mathbf{b}_t$ based on an OLPS strategy.

  Compute the net proportion w_{t-1} of the rebalancing (from $\hat{\mathbf{b}}_{t-1}$) by solving (1).

  Observe the realized return $\mathbf{x}_t$.

  Update the cumulative wealth as $S_t=S_{t-1}w_{t-1}(\mathbf{b}_t^\top\mathbf{x}_t)$.

  Obtain the current portfolio weights $\hat{\mathbf{b}}_t=\mathbf{b}_t\cdot\mathbf{x}_t/\mathbf{b}_t^\top\mathbf{x}_t$.

end

OLPS策略通常包含预测步骤，即需要在优化步骤中去预测 $\tilde{\mathbf{x}}_t$ ，比如可以采用历史数据的滑动平均值等方法。

Base model is as follows：

$$\begin{aligned}\underset{w\geq 0,\mathbf{b}\geq\mathbf{0}}{\text{maximize}}\quad & w\mathbf{b}^\top\tilde{\mathbf{x}}_t-\lambda\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1\\ \text{subject to}\quad & w+\gamma\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1=1,\quad\mathbf{b}^\top\mathbf{1}=1,\end{aligned} \tag{2}$$

非线性优化问题需要对问题进行转化，可以证明，该问题与以下优化问题等价：

Theorem 1. *Consider the following convex program*

$$\begin{array}{ll}\underset{\mathbf{b}\geq\mathbf{0}}{\text{maximize}} & \mathbf{b}^\top\tilde{\mathbf{x}}_t-\lambda\left\|\hat{\mathbf{b}}_{t-1}-\mathbf{b}\right\|_1\\ \text{subject to} & \mathbf{b}^\top\mathbf{1}+\gamma\left\|\hat{\mathbf{b}}_{t-1}-\mathbf{b}\right\|_1\leq 1,\end{array}\tag{3}$$

and let $\mathbf{b}^*_{(2)}$ *be an optimal solution. Then, the optimal value of* (2) *is the same as that of* (3), *and* $\left(w^*_{(1)},\mathbf{b}^*_{(1)}\right):=\left(\mathbf{1}^\top\mathbf{b}^*_{(2)},\frac{\mathbf{b}^*_{(2)}}{\mathbf{1}^\top\mathbf{b}^*_{(2)}}\right)$ *is an optimal solution to* (2)

上述的凸优化问题与下列线性规划问题等价，并可使用现有的优化器进行求解。

Corollary 1. *The problem* (2) *is equivalent to the following LP.*

$$\begin{array}{ll}\underset{\mathbf{b}\geq\mathbf{0},\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0}}{\text{maximize}} & \mathbf{b}^\top\tilde{\mathbf{x}}_t-\lambda(\mathbf{u}+\mathbf{v})^\top\mathbf{1}\\ \text{subject to} & \mathbf{b}^\top\mathbf{1}+\gamma(\mathbf{u}+\mathbf{v})^\top\mathbf{1}\leq 1\\ & \mathbf{u}\geq\hat{\mathbf{b}}_{t-1}-\mathbf{b},\quad \mathbf{v}\geq\mathbf{b}-\hat{\mathbf{b}}_{t-1}\end{array}\tag{4}$$

直接应用Base model (2) 会出现两个问题：

1. 由于算术收益的线性性质，模型可能会只投资单一资产，比如只投资具有最大预测值 $\tilde{x_{t,i}}$ 的资产。
2. 依赖于对下一期收益率的预测，收益预测具有较大的不确定性，导致模型表现不稳定。

因此，采用 robust optimization 方法，而不使用简单的算术收益。

鲁棒优化模型如下：

$$\begin{array}{ll}\underset{w\geq 0,\mathbf{b}\geq\mathbf{0}}{\text{maximize}} & \min_{\mathbf{x}\in\mathcal{U}}\left\{w\mathbf{b}^\top\mathbf{x}\right\}-\lambda\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1\\ \text{subject to} & w+\gamma\left\|\hat{\mathbf{b}}_{t-1}-w\mathbf{b}\right\|_1=1,\quad \mathbf{b}^\top\mathbf{1}=1\end{array}\tag{5}$$

在此处 $\mathcal{U}$ 是椭球不确定集，如下：

$$\mathcal{U}\left(\tilde{\mathbf{x}}_t,\kappa\right)=\left\{\mathbf{x}\in\mathbb{R}^m\mid\left(\mathbf{x}-\tilde{\mathbf{x}}_t\right)^\top\mathbf{\Sigma}^{-1}\left(\mathbf{x}-\tilde{\mathbf{x}}_t\right)\leq\kappa^2\right\}\tag{6}$$

- $\tilde{\mathbf{x}}_t\in\mathbb{R}^m$ 椭球中心参数，最近 $m+1$ 个样本的滑动平均估计
- $\mathbf{\Sigma}\succ\mathbf{0}$ 椭球形状参数，最近 $m+1$ 个样本的协方差估计，捕获市场走势，且满足正定性的要求
- $\kappa\geq 0$ 椭球大小参数。

(5) 直接求解困难，涉及椭球内部最大化的问题。考虑以下命题：

Proposition 1. *Given the ellipsoidal uncertainty* (6), *we have*

$$\min_{\mathbf{x}\in\mathcal{U}(\tilde{\mathbf{x}}_t,\kappa)}\left\{\mathbf{b}^\top\mathbf{x}\right\}=\tilde{\mathbf{x}}^\top\mathbf{b}-\kappa\|\mathbf{U}\mathbf{b}\|_2,$$

where $\mathbf{U}$ *is the upper triangular matrix of the Cholesky factorization of* $\mathbf{\Sigma}$, *i.e.,* $\mathbf{\Sigma}=\mathbf{U}^\top\mathbf{U}$

由该命题，可以得到与标准差对应的 l_2 norm为：

$$\|\mathbf{U}\mathbf{b}\|_2=\sqrt{(\mathbf{U}\mathbf{b})^\top(\mathbf{U}\mathbf{b})}=\sqrt{\mathbf{b}^\top\mathbf{U}^\top\mathbf{U}\mathbf{b}}=\sqrt{\mathbf{b}^\top\mathbf{\Sigma}\mathbf{b}}.$$

考虑了 l_2 norm 也是考虑了预测的风险，即预测的不确定性，会给出更分散的投资组合。

Lemma 1. *Let* $\sigma=\sqrt{\sum_{i=1}^m\sigma_i^2}$, *where* $\{\sigma_i^2\}$ *are the diagonal entries of* $\mathbf{\Sigma}=\mathbf{U}^\top\mathbf{U}$. *Then, for any* $\{\mathbf{b}_1,\mathbf{b}_2\}\subseteq\mathbb{R}^m$, *whe have*

$$\left| \|\mathbf{U}\mathbf{b}_1\|_2 - \|\mathbf{U}\mathbf{b}_2\|_2 \right| \leq \sigma \left\| \mathbf{b}_1 - \mathbf{b}_2 \right\|_2 .$$

Theorem 2. *Assume that* $\max_i\{\tilde{x}_{t,i}\} > \kappa\sigma + \lambda$, *where* $\sigma = \sqrt{\sum_{i=1}^{m} \sigma_i^2}$ *are* $\{\sigma_i^2\}$ *are the diagonal entries of* $\mathbf{\Sigma}$. *Consider the following program.*

$$\begin{array}{ll} \underset{\mathbf{b}\geq\mathbf{0}}{\text{maximize}} & \mathbf{b}^\top \tilde{\mathbf{x}} - \lambda \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 - \kappa \|\mathbf{U}\mathbf{b}\|_2 \\ \text{subject to} & \mathbf{b}^\top \mathbf{1} + \gamma \left\| \hat{\mathbf{b}}_{t-1} - \mathbf{b} \right\|_1 \leq 1 \end{array} \tag{7}$$

and let $\mathbf{b}^*_{(3)}$ *be an optimal solution. Then,* $\left(w^*_{(1)}, \mathbf{b}^*_{(1)}\right) := \left(\mathbf{1}^\top \mathbf{b}^*_{(3)}, \frac{\mathbf{b}^*_{(3)}}{\mathbf{1}^\top \mathbf{b}^*_{(3)}}\right)$ *is an optimal solution to* (5).

Algorithm 2: The RELP strategy with the SB and top-K adaptive schemes on parameter $\phi \in \{\lambda, \kappa\}$

Initialization: Set $\mathbf{b}_1 = \mathbf{1}/m$, $\mathbf{b}_0 = \mathbf{0}$ and $S_0 = 1$. Pick a range of values of ϕ, i.e., $\{\phi_1, \ldots, \phi_R\}$. Here, ϕ is a generic parameter representing λ or κ.

Initialization for SB scheme: Choose window size W, indifference zone parameter δ and the initial tracking portfolio r'.
Initialization for Top-K scheme: Choose K and initial $\overline{\phi}_0$.
for $t = 1, \ldots, n$ **do**
  Obtain the new portfolio weights $\mathbf{b}_t^r$ for expert by solving (9), e.g., using an off-the-shelf optimization solver, with ϕ_r for $r \in \{1, \ldots, R\}$.
  (For $t = 1$, we use $\mathbf{b}_1^r = \mathbf{1}/m$; for $t \in \{2, \ldots, m+1\}$, we use $\mathbf{b}_t^r = \hat{\mathbf{b}}_{t-1}^r$.)
  if SB scheme **then**
    Set $\mathbf{b}_t = \mathbf{b}_t^{r'}$.
  else if Top-K scheme **then**
    Obtain the new portfolio weights $\mathbf{b}_t$ by solving (9), e.g., using an off-the-shelf optimization solver, with $\phi = \overline{\phi}_{t-1}$.
  end
  Compute the net proportion w_{t-1} of the rebalance (from $\hat{\mathbf{b}}_{t-1}$), as well as w_{t-1}^r for each expert (from $\hat{\mathbf{b}}_{t-1}^r$) by solving (1) .
  Observe the realized return $\mathbf{x}_t$.
  Update the cumulative wealth as $S_t = S_{t-1} w_{t-1} (\mathbf{b}_t^\top \mathbf{x}_t)$.
  Update the cumulative wealth for each expert as $S_t^r = S_{t-1}^r w_{t-1}^r [(\mathbf{b}_t^r)^\top \mathbf{x}_t]$ for $r \in \{1, \ldots, R\}$.
  if SB scheme **then**
    Based on the recent portfolio wealth $\{S_{t-W+1}^{r'}, \ldots, S_t^{r'}\}$ of expert r', compute the sample mean $\bar{S}_t^{r'}$ and sample standard deviation $\hat{\sigma}_t^{r'}$.
    if $\mathcal{R} = \left\{r'' \in \{1, \ldots, R\} \setminus \{r'\} \mid \bar{S}_t^{r''} > \bar{S}_t^{r'} + \max\{\delta, 1.96\hat{\sigma}_t^{r'}\}\right\} \neq \emptyset$ **then**
      Update the tracking expert by setting $r' \in \mathrm{argmax}_{r \in \mathcal{R}} \{\bar{S}_t^r\}$.
    end
  else if Top-K scheme **then**
    Compute $\overline{\phi}_t$ as the average value of ϕ's of the top-K performing portfolios.
  end
  Obtain the current portfolio weights $\hat{\mathbf{b}}_t = \mathbf{b}_t \cdot \mathbf{x}_t / \mathbf{b}_t^\top \mathbf{x}_t$ as well as $\hat{\mathbf{b}}_t^r$ for each expert.
end