DiscreteModels

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Beta-Binomial model

Suppose the outcome of each observation is either 1 or 0 according to a unknown parameter θ. This dataset could be modeled as a binomial sampling model.

$$y_1, ..., y_n|\theta \sim Binomial(n, \theta)$$

To find the conjugate prior for θ, we rewrite distribution of sampling model proportional to θ.

$$P(y_1, ..., y_n|theta) \propto \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

Clearly, its conjugate prior belongs to "beta family". (Same as before, we only consider the terms related to θ)

$$\begin{aligned} P(\theta) &= dbeta(\theta, a, b) \\ &\propto \theta^{a-1}(1-\theta)^{b-1} \end{aligned}$$

Then, we could compute its posterior distribution easily.

$$\begin{aligned} P(\theta|y_1, ..., y_n) &\propto P(y_1, ..., y_n|\theta)P(\theta) \\ &\propto \theta^{\sum y_i}(1-\theta)^{n-\sum y_i} \times \theta^{a-1}(1-\theta)^{b-1} \\ &\propto \theta^{(a+\sum y_i)-1}(1-\theta)^{(b+n-\sum y_i)} - 1 \\ \theta|y_1, ..., y_n &\sim Beta(a+\sum y_i, b+n-\sum y_i) \end{aligned}$$

After compute the posterior distribution for θ, we could easily write out the sampler for θ

```r
beta_binomial_sampler = function(S, y, a, b) {
  y.sum = sum(y)
  n = length(y)
  an = a + y.sum
  bn = b + n - y.sum
  return(rbeta(S, an, bn))
}
```

Gamma-Poisson model

Poisson distribution is a natural choice when modelling counts. The only parameter for Poisson is its mean (θ). As before, we also write down its sampling model proportional to θ.

$$\begin{aligned} y_1, ..., y_n|\theta &\sim Poisson(\theta) \\ P(y_1, ..., y_n|\theta) &\propto \theta^{\sum y_i}e^{-n\theta} \end{aligned}$$

Clearly, the conjugate prior for Poisson distribution is Gamma distribution.

$$\theta \sim Gamma(a, b)$$
$$P(\theta) \propto \theta^{a-1} e^{-b\theta}$$

Then, we could compute its posterior distribution easily.

$$\begin{aligned}
P(\theta|y_1, ..., y_n) &\propto \theta^{\sum y_i} e^{-n\theta} \times \theta^{a-1} e^{-b\theta} \\
&\propto \theta^{(a+\sum y_i)} e^{-(b+n)\theta} \\
\theta|y_1, ..., y_n &\sim Gamma(a + \sum y_i, b + n)
\end{aligned}$$

After compute the posterior distribution for θ, we could easily write out the sampler for θ

```r
Gamma_Poisson_sampler = function(S, y, a, b) {
  y.sum = sum(y)
  n = length(y)
  an = a + y.sum
  bn = b + n
  return(rgamma(S, an, bn))
}
```