Source location by waveform inversion

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ABSTRACT

Source location by waveform inversion, in this paper, we derive the stagger grid finite difference modelling in matrix form and derive the exact adjoint operator of modelling operator. Locating the source created by hydralic fracturing by fitting the waveform of receiver.

INTRODUCTION

jijun's paper, some least square reverse time migration paper, and source location paper in passive seismic data.

METHOD

We start with the cost function of multi-component seismic data registration

$$J = ||\mathbf{d} - \mathbf{Gm}||_\mathbf{2}^\mathbf{2} + \mu||\mathbf{m}||_\mathbf{1}, \tag{1}$$

this function can be solved by IRLS or FISTA.

We start with the full 3D elastic wave equation.

$$\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z \\
\tau_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}) \\
\tau_{yy} &= (\lambda + 2\mu)\frac{\partial u_y}{\partial y} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}) \\
\tau_{zz} &= (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}) \\
\tau_{xy} &= \mu(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \\
\tau_{xz} &= \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \\
\tau_{yz} &= \mu(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y})
\end{aligned} \tag{2}$$

u_x, u_y, u_z is particle displacement, τ is stress component, f is body force components. using particle velocity to replace the time derivative of particle displacement. we get

$$
\begin{aligned}
\rho\frac{\partial v_x}{\partial t} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\
\rho\frac{\partial v_y}{\partial t} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y \\
\rho\frac{\partial v_z}{\partial t} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z \\
\frac{\partial \tau_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) \\
\frac{\partial \tau_{yy}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_y}{\partial y} + \lambda(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}) \\
\frac{\partial \tau_{zz}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_z}{\partial z} + \lambda(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}) \\
\frac{\partial \tau_{xy}}{\partial t} &= \mu(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}) \\
\frac{\partial \tau_{xz}}{\partial t} &= \mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}) \\
\frac{\partial \tau_{yz}}{\partial t} &= \mu(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y})
\end{aligned}
\tag{3}
$$

To discretize those partial differential equations, we use stagger grid not only in space but also in time domain, i is the x direction index, j is the y direction index, k is the z direction index, n is the t direction index. $\tau_{xx}, \tau_{yy}, \tau_{zz}$ is assigned to integer index in space, but half index in time domain, $\tau_{xy}, \tau_{xz}, \tau_{yz}$ is assigned to half index in space and half index in time domain, v_x, v_y, v_z is half index in space but integer index in time. so discretize as

$$
\begin{aligned}
&\tau_{xx}(i,j,k,n+\frac{1}{2}),\\
&\tau_{yy}(i,j,k,n+\frac{1}{2}),\\
&\tau_{zz}(i,j,k,n+\frac{1}{2}),\\
&\tau_{xy}(i,j+\frac{1}{2},k+\frac{1}{2},n+\frac{1}{2})\\
&\tau_{xz}(i+\frac{1}{2},j+\frac{1}{2},k,n+\frac{1}{2})\\
&\tau_{yz}(i+\frac{1}{2},j,k+\frac{1}{2},n+\frac{1}{2})\\
&v_x(i,j+\frac{1}{2},k,n),\\
&v_y(i,j,k+\frac{1}{2},n),\\
&v_z(i+\frac{1}{2},j,k,n),
\end{aligned}
\tag{4}
$$

reduce to 2D elastic case

$$
\begin{aligned}
\rho\frac{\partial v_x}{\partial t} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x\\
\rho\frac{\partial v_z}{\partial t} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} + f_z\\
\frac{\partial \tau_{xx}}{\partial t} &= (\lambda+2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z}\\
\frac{\partial \tau_{zz}}{\partial t} &= (\lambda+2\mu)\frac{\partial v_z}{\partial z} + \lambda\frac{\partial v_x}{\partial x}\\
\frac{\partial \tau_{xz}}{\partial t} &= \mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x})
\end{aligned}
\tag{5}
$$

using finite difference to approximate partial derivative, as we use PML as absorbing boundary conditions, each v and τ are split into x and $z$components.

for v_x we get

$$\rho\frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x$$

$$split \quad into \quad v_x^x, v_x^z$$

$$\begin{aligned}\rho\frac{\partial v_{xx}}{\partial t} &= \frac{\partial \tau_{xx}}{\partial x} + f_{xx} - dv_{xx}\cdot v_{xx}\\ \rho\frac{\partial v_{xz}}{\partial t} &= \frac{\partial \tau_{xz}}{\partial z} + f_{xz} - dv_{xz}\cdot v_{xz}\end{aligned}$$

$$discretize \quad those \quad equations$$

$$\begin{aligned}\rho(i,j+\frac{1}{2})\frac{v_{xx}(i,j+\frac{1}{2},n+1)-v_{xx}(i,j+\frac{1}{2},n)}{dt} &= a_1\frac{\tau_{xx}(i,j+1,n+\frac{1}{2})-\tau_{xx}(i,j,n+\frac{1}{2})}{dx}\\ &- a_2\frac{\tau_{xx}(i,j+2,n+\frac{1}{2})-\tau_{xx}(i,j-1,n+\frac{1}{2})}{dx}\\ &- \frac{1}{2}dv_{xx}(i,j+\frac{1}{2})\left(v_{xx}(i,j+\frac{1}{2},n+1)+v_{xx}(i,j+\frac{1}{2},n)\right)\\ &+ f_{xx}(i,j+\frac{1}{2},n+\frac{1}{2})\end{aligned} \tag{6}$$

the center of this equation $(i,j+\frac{1}{2},n+\frac{1}{2})$, re-organized as

$$\begin{aligned}\left(\frac{\rho(i,j+\frac{1}{2})}{dt}+\frac{dv_{xx}(i,j+\frac{1}{2})}{2}\right)v_{xx}(i,j+\frac{1}{2},n+1) &= (\frac{\rho(i,j+\frac{1}{2})}{dt}-\frac{dv_{xx}(i,j+\frac{1}{2})}{2})v_{xx}(i,j+\frac{1}{2},n)\\ &+ a_1\frac{\tau_{xx}(i,j+1,n+\frac{1}{2})-\tau_{xx}(i,j,n+\frac{1}{2})}{dx}\\ &- a_2\frac{\tau_{xx}(i,j+2,n+\frac{1}{2})-\tau_{xx}(i,j-1,n+\frac{1}{2})}{dx}\\ &+ f_{xx}(i,j+\frac{1}{2},n+\frac{1}{2})\end{aligned} \tag{7}$$

v_{xx} extrapolation

the center of this equation $(i,j+\frac{1}{2},n+\frac{1}{2})$, set $a_3 = \left(\frac{\rho(i,j+\frac{1}{2})}{dt}+\frac{dv_{xx}(i,j+\frac{1}{2})}{2}\right)$, where $\rho(i,j+\frac{1}{2}) = \frac{\rho(i,j)+\rho(i,j+1)}{2}$we get the final finite difference equation for one-step extrapolation of v_{xx}

is

$$
\begin{aligned}
v_{xx}(i,j+\frac{1}{2},n+1) = {} & \frac{(\frac{\rho(i,j+\frac{1}{2})}{dt} - \frac{dv_{xx}(i,j+\frac{1}{2})}{2})}{a_3} v_{xx}(i,j+\frac{1}{2},n) \\
& + \frac{a_1}{a_3 dx}\left(\tau_{xx}(i,j+1,n+\frac{1}{2}) - \tau_{xx}(i,j,n+\frac{1}{2})\right) \\
& - \frac{a_2}{a_3 dx}\left(\tau_{xx}(i,j+2,n+\frac{1}{2}) - \tau_{xx}(i,j-1,n+\frac{1}{2})\right) \\
& + \frac{1}{a_3} f_{xx}(i,j+\frac{1}{2},n+\frac{1}{2})
\end{aligned}
\tag{8}
$$

after v_{xx} and v_{xz} have been computed, the v_x can be calculated via $v_x = v_{xx} + v_{xz}$

v_{xz} extrapolation

the center of this equation $(i, j+\frac{1}{2}, n+\frac{1}{2})$, set $a_3 = \left(\frac{\rho(i,j+\frac{1}{2})}{dt} + \frac{dv_{xz}(i,j+\frac{1}{2})}{2}\right)$, we can get the scheme for extrapolating v_{xz}, re-organized as

$$
\begin{aligned}
v_{xx}(i,j+\frac{1}{2},n+1) = {} & \frac{(\frac{\rho(i,j+\frac{1}{2})}{dt} - \frac{dv_{xz}(i,j+\frac{1}{2})}{2})}{a_3} v_{xz}(i,j+\frac{1}{2},n) \\
& + \frac{a_1}{a_3 dz}\left(\tau_{xz}(i+\frac{1}{2},j+\frac{1}{2},n+\frac{1}{2}) - \tau_{xz}(i-\frac{1}{2},j+\frac{1}{2},n+\frac{1}{2})\right) \\
& - \frac{a_2}{a_3 dz}\left(\tau_{xz}(i+\frac{3}{2},j+\frac{1}{2},n+\frac{1}{2}) - \tau_{xz}(i-\frac{3}{2},j+\frac{1}{2},n+\frac{1}{2})\right) \\
& + \frac{1}{a_3} f_{xz}(i,j+\frac{1}{2},n+\frac{1}{2})
\end{aligned}
\tag{9}
$$

those two equations are for the updating of $v_x = v_x^x + v_x^z$.

$$
\begin{gathered}
\rho\frac{\partial v_z}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} + f_z \\
split \quad into \quad v_z^x, v_z^z \\
\rho\frac{\partial v_z^x}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + f_z^x - \delta(v_z^x)\cdot v_z^x \\
\rho\frac{\partial v_z^z}{\partial t} = \frac{\partial \tau_{zz}}{\partial z} + f_z^z - \delta(v_z^z)\cdot v_z^z
\end{gathered}
\tag{10}
$$

v_{zx} extrapolation

the center of this equation $(i+\frac{1}{2}, j, n+\frac{1}{2})$, set $a_3 = \left(\frac{\rho(i+\frac{1}{2},j)}{dt} + \frac{dv_{zx}(i+\frac{1}{2},j)}{2}\right)$, where $\rho(i+\frac{1}{2},j) = \frac{rho(i,j)+\rho(i+1,j)}{2}$ we get the final finite difference equation for one-step extrapolation of v_z^x is

$$\begin{aligned} v_{zx}(i+\frac{1}{2}, j, n+1) = &\left(\frac{\rho(i+\frac{1}{2},j)}{a_3 dt} - \frac{dv_{zx}(i+\frac{1}{2},j)}{2a_3}\right) v_{zx}(i+\frac{1}{2}, j, n) \\ &+ \frac{a_1}{a_3 dx}\left(\tau_{xz}(i+\frac{1}{2}, j+\frac{1}{2}, n+\frac{1}{2}) - \tau_{xz}(i+\frac{1}{2}, j-\frac{1}{2}, n+\frac{1}{2})\right) \\ &- \frac{a_2}{a_3 dx}\left(\tau_{xz}(i+\frac{1}{2}, j+\frac{3}{2}, n+\frac{1}{2}) - \tau_{xz}(i+\frac{1}{2}, j-\frac{3}{2}, n+\frac{1}{2})\right) \\ &+ \frac{1}{a_3} f_{zx}(i+\frac{1}{2}, j, n+\frac{1}{2}) \end{aligned} \tag{11}$$

v_{zz} extrapolation

the center of this equation $(i+\frac{1}{2}, j, n+\frac{1}{2})$, set $a_3 = \left(\frac{\rho(i+\frac{1}{2},j)}{dt} + \frac{dv_{zz}(i+\frac{1}{2},j)}{2}\right)$, we get the final finite difference equation for one-step extrapolation of v_{zx} is

$$\begin{aligned} v_{zz}(i+\frac{1}{2}, j, n+1) = &\left(\frac{\rho(i+\frac{1}{2},j)}{a_3 dt} - \frac{dv_{zz}(i+\frac{1}{2},j)}{2a_3}\right) v_{zz}(i+\frac{1}{2}, j, n) \\ &+ \frac{a_1}{a_3 dz}\left(\tau_{zz}(i+1, j, n+\frac{1}{2}) - \tau_{zz}(i, j, n+\frac{1}{2})\right) \\ &- \frac{a_2}{a_3 dz}\left(\tau_{zz}(i+2, j, n+\frac{1}{2}) - \tau_{zz}(i-1, j, n+\frac{1}{2})\right) \\ &+ \frac{1}{a_3} f_{zz}(i+\frac{1}{2}, j, n+\frac{1}{2}) \end{aligned} \tag{12}$$

after v_{zx} and v_{zz} have been computed, the v_z can be calculated via $v_z = v_{zx} + v_{zz}$

τ_{xxx} extrapolation

for τ_{xx} we have

$$
\begin{aligned}
\frac{\partial \tau_{xx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda \frac{\partial v_z}{\partial z} \\
&split \quad into \quad \tau_{xx}^x, \tau_{zz}^z \\
\frac{\partial \tau_{xxx}}{\partial t} &= (\lambda + 2\mu)\frac{\partial v_x}{\partial x} - d\tau_{xxx} \cdot \tau_{xxx} \\
\frac{\partial \tau_{xxz}}{\partial t} &= \lambda \frac{\partial v_z}{\partial z} - d\tau_{xxz} \cdot \tau_{xxz}
\end{aligned} \tag{13}
$$

the center of this equation $(i, j, n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{d\tau_{xxx}(i,j)}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{xxx} is

$$
\begin{aligned}
\tau_{xxx}(i, j, n + \frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{xxx}(i,j)}{2a_3}\right)\tau_{xxx}(i, j, n + \frac{1}{2}) \\
&+ \frac{a_1(\lambda(i,j) + 2\mu(i,j))}{a_3 dx}\left(v_x(i, j + \frac{1}{2}, n+1) - v_x(i, j - \frac{1}{2}, n+1)\right) \\
&- \frac{a_2(\lambda(i,j) + 2\mu(i,j))}{a_3 dx}\left(v_x(i, j + \frac{3}{2}, n+1) - v_x(i, j - \frac{3}{2}, n+1)\right)
\end{aligned} \tag{14}
$$

τ_{xxz} extrapolation

the center of this equation $(i, j, n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{d\tau_{xxz}(i,j)}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{xxz} is

$$
\begin{aligned}
\tau_{xxz}(i, j, n + \frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{xxz}(i,j)}{2a_3}\right)\tau_{xxz}(i, j, n + \frac{1}{2}) \\
&+ \frac{a_1\lambda(i,j)}{a_3 dz}\left(v_z(i + \frac{1}{2}, j, n+1) - v_z(i - \frac{1}{2}, j, n+1)\right) \\
&- \frac{a_2\lambda(i,j)}{a_3 dz}\left(v_z(i + \frac{3}{2}, j, n+1) - v_z(i - \frac{3}{2}, j, n+1)\right)
\end{aligned} \tag{15}
$$

after τ_{xxx} and τ_{xxz} have been computed, the τ_{xx} can be calculated via $\tau_{xx} = \tau_{xxx} + \tau_{xxz}$

τ_{zzx} extrapolation

for τ_{zz} we have

$$
\begin{aligned}
\frac{\partial \tau_{zz}}{\partial t} &= (\lambda+2\mu)\frac{\partial v_z}{\partial z} + \lambda\frac{\partial v_x}{\partial x} \\
&\quad split \quad into \quad \tau_{zzx}, \tau_{zzz} \\
\frac{\partial \tau_{zzx}}{\partial t} &= \lambda\frac{\partial v_x}{\partial x} - d\tau_{zzx}\cdot\tau_{zzx} \\
\frac{\partial \tau_{zzz}}{\partial t} &= (\lambda+2\mu)\frac{\partial v_z}{\partial z} - d\tau_{zzz}\cdot\tau_{zzz}
\end{aligned} \tag{16}
$$

the center of this equation $(i,j,n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{\delta(\tau_{zzx}(i,j)}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{zzx} is

$$
\begin{aligned}
\tau_{zzx}(i,j,n+\frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{zzx}(i,j)}{2a_3}\right)\tau_{zzx}(i,j,n+\frac{1}{2}) \\
&+ \frac{a_1\lambda(i,j)}{a_3 dx}\left(v_x(i,j+\frac{1}{2},n+1) - v_x(i,j-\frac{1}{2},n+1)\right) \\
&- \frac{a_2\lambda(i,j)}{a_3 dx}\left(v_x(i,j+\frac{3}{2},n+1) - v_x(i,j-\frac{3}{2},n+1)\right)
\end{aligned} \tag{17}
$$

τ_{zzz} extrapolation

the center of this equation $(i,j,n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{d\tau_{zzz}(i,j)}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{zzz} is

$$
\begin{aligned}
\tau_{zzz}(i,j,n+\frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{zzz}(i,j)}{2a_3}\right)\tau_{zzz}(i,j,n+\frac{1}{2}) \\
&+ \frac{a_1(\lambda(i,j)+2\mu(i,j))}{a_3 dz}\left(v_z(i+\frac{1}{2},j,n+1) - v_z(i-\frac{1}{2},j,n+1)\right) \\
&- \frac{a_2(\lambda(i,j)+2\mu(i,j))}{a_3 dz}\left(v_z(i+\frac{3}{2},j,n+1) - v_z(i-\frac{3}{2},j,n+1)\right)
\end{aligned} \tag{18}
$$

after τ_{zzx} and τ_{zzz} have been computed, the τ_{zz} can be calculated via $\tau_{zz} = \tau_{zzx} + \tau_{zzz}$

τ_{xzx} extrapolation

for $\tau_x z$ we have

$$\begin{aligned}\frac{\partial \tau_{xz}}{\partial t} &= \mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}) \\ & split \quad into \quad \tau_{xzx}, \tau_{xzz} \\ \frac{\partial \tau_{xzx}}{\partial t} &= \mu\frac{\partial v_z}{\partial x} - d\tau_{xzx}\cdot\tau_{xzx} \\ \frac{\partial \tau_{xzz}}{\partial t} &= \mu\frac{\partial v_x}{\partial z} - d\tau_{xzz}\cdot\tau_{xzz}\end{aligned} \tag{19}$$

the center of this equation $(i+\frac{1}{2}, j+\frac{1}{2}, n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{d\tau_{xzx}(i+\frac{1}{2},j+\frac{1}{2})}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{xzx} is

$$\begin{aligned}\tau_{xzx}(i+\frac{1}{2}, j+\frac{1}{2}, n+\frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{xzx}(i+\frac{1}{2}, j+\frac{1}{2})}{2a_3}\right)\tau^x_{xz}(i+\frac{1}{2}, j+\frac{1}{2}, n+\frac{1}{2}) \\ &+ \frac{a_1\mu(i+\frac{1}{2}, j+\frac{1}{2})}{a_3 dx}\left(v_z(i+\frac{1}{2}, j+1, n+1) - v_z(i+\frac{1}{2}, j, n+1)\right) \\ &- \frac{a_2\mu(i+\frac{1}{2}, j+\frac{1}{2})}{a_3 dx}\left(v_z(i+\frac{1}{2}, j+2, n+1) - v_z(i+\frac{1}{2}, j-1, n+1)\right)\end{aligned} \tag{20}$$

τ_{xzz} extrapolation

the center of this equation $(i+\frac{1}{2}, j+\frac{1}{2}, n+1)$, set $a_3 = \left(\frac{1}{dt} + \frac{d\tau_{xzz}(i+\frac{1}{2},j+\frac{1}{2})}{2}\right)$, we get the final finite difference equation for one-step extrapolation of τ_{xzz} is

$$\begin{aligned}\tau_{xzz}(i+\frac{1}{2}, j+\frac{1}{2}, n+\frac{3}{2}) = &\left(\frac{1}{a_3 dt} - \frac{d\tau_{xzx}(i+\frac{1}{2}, j+\frac{1}{2})}{2a_3}\right)\tau_{xzz}(i+\frac{1}{2}, j+\frac{1}{2}, n+\frac{1}{2}) \\ &+ \frac{a_1\mu(i+\frac{1}{2}, j+\frac{1}{2})}{a_3 dz}\left(v_x(i+1, j+\frac{1}{2}, n+1) - v_x(i, j+\frac{1}{2}, n+1)\right) \\ &- \frac{a_2\mu(i+\frac{1}{2}, j+\frac{1}{2})}{a_3 dz}\left(v_x(i+2, j+\frac{1}{2}, n+1) - v_x(i-1, j+\frac{1}{2}, n+1)\right)\end{aligned} \tag{21}$$

after τ_{xzx} and τ_{xzz} have been computed, the τ_{xz} can be calculated via $\tau_{xz} = \tau_{xzx} + \tau_{xzz}$

the big matrix for one step finite-difference simplified as

$$\mathbf{p^{n+1}} = \mathbf{G}_\tau \mathbf{G_v} \mathbf{p^n} \tag{22}$$

$$\mathbf{G_v} = \begin{pmatrix}
\mathbf{M^{v_{xx}}_{v_{xx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{\mathbf{v_{xx}}}_{\tau_{\mathbf{xx}}} & \mathbf{M}^{\mathbf{v_{xx}}}_{\tau_{\mathbf{xx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M^{v_{xz}}_{v_{xz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{\mathbf{v_{xz}}}_{\tau_{\mathbf{xz}}} & \mathbf{M}^{\mathbf{v_{xz}}}_{\tau_{\mathbf{xz}}} \\
\mathbf{0} & \mathbf{0} & \mathbf{M^{v_{zx}}_{v_{zx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{\mathbf{v_{zx}}}_{\tau_{\mathbf{xz}}} & \mathbf{M}^{\mathbf{v_{zx}}}_{\tau_{\mathbf{xz}}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M^{v_{zz}}_{v_{zz}}} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{\mathbf{v_{zz}}}_{\tau_{\mathbf{zz}}} & \mathbf{M}^{\mathbf{v_{zz}}}_{\tau_{\mathbf{zz}}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{pmatrix} \tag{23}$$

The adjoint operator of G_v is

$$
\mathbf{G_v^T} = \begin{pmatrix}
\mathbf{M}_{\mathbf{v_{xx}}}^{\mathbf{v_{xx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\mathbf{v_{xz}}}^{\mathbf{v_{xz}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_{zx}}}^{\mathbf{v_{zx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_{zz}}}^{\mathbf{v_{zz}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{M}_{\tau_{\mathbf{xx}}}^{\mathbf{v_{xx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{M}_{\tau_{\mathbf{xx}}}^{\mathbf{v_{xx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zz}}}^{\mathbf{v_{zz}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zz}}}^{\mathbf{v_{zz}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{xz}}\,\mathbf{T}} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{zx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{xz}}\,\mathbf{T}} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{zx}}\,\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{pmatrix} \tag{24}
$$

$$
\mathbf{G}_\tau = \begin{pmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xxx}}} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xxx}}} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xxx}}}^{\tau_{\mathbf{xxx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xxz}}} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xxz}}} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xxz}}}^{\tau_{\mathbf{xxz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{zzx}}} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{zzx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zzx}}}^{\tau_{\mathbf{zzx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{zzz}}} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{zzx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zzz}}}^{\tau_{\mathbf{zzz}}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xzx}}} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xzx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xzx}}}^{\tau_{\mathbf{xzx}}} & \mathbf{0} \\
\mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xzz}}} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xzz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xzz}}}^{\tau_{\mathbf{xzz}}}
\end{pmatrix} \tag{25}
$$

so the adjoint of one-step finite difference operator is

$$\mathbf{G^T} = \mathbf{G_v^T} \cdot \mathbf{G_\tau^T} \tag{26}$$

The adjoint operator of G_τ is

$$\mathbf{G_\tau^T} = \begin{pmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xxx}}\mathbf{T}} & \mathbf{0} & \mathbf{M}_{\mathbf{v^x}}^{\tau_{\mathbf{zzx}}\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xzz}}\mathbf{T}} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xxx}}\mathbf{T}} & \mathbf{0} & \mathbf{M}_{\mathbf{v^x}}^{\tau_{\mathbf{zzx}}\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_x}}^{\tau_{\mathbf{xzz}}\mathbf{T}} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xxz}}\mathbf{T}} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{zzz}}\mathbf{T}} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xzx}}\mathbf{T}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xxz}}\mathbf{T}} & \mathbf{0} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{zzz}}\mathbf{T}} & \mathbf{M}_{\mathbf{v_z}}^{\tau_{\mathbf{xzx}}\mathbf{T}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xxx}}}^{\tau_{\mathbf{xxx}}\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xxz}}}^{\tau_{\mathbf{xxz}}\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zzx}}}^{\tau_{\mathbf{zzx}}\mathbf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zzz}}}^{\tau_{\mathbf{zzz}}\mathbf{T}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xzx}}}^{\tau_{\mathbf{xzx}}\mathbf{T}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xzz}}}^{\tau_{\mathbf{xzz}}\mathbf{T}}
\end{pmatrix} \tag{27}$$

using the transpose of each suborder matrix.

The complete equations for modelling

$$\mathbf{d} = \mathbf{RSGm} \tag{28}$$

take 3 step finite difference as an example

$$\begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_3} \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{m_1} \\ \mathbf{m_2} \\ \mathbf{m_3} \end{pmatrix} \tag{29}$$

the adjoint of this equation is

$$\mathbf{m'} = \mathbf{G^T S^T R^T d'} \tag{30}$$

expanded as

$$\begin{pmatrix} \mathbf{m_1} \\ \mathbf{m_2} \\ \mathbf{m_3} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{G^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{G^T} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{S^T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S^T} \end{pmatrix} \begin{pmatrix} \mathbf{R^T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R^T} \end{pmatrix} \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_3} \end{pmatrix} \tag{31}$$

$\mathbf{R^T}$ is the adjoint of sampling operator, it can be done with AddSource. $\mathbf{S}$ is the summation operator, this is used for collapse the splitted variables

$$\begin{pmatrix} \mathbf{v_x} \\ \mathbf{v_z} \\ \tau_{\mathbf{xx}} \\ \tau_{\mathbf{zz}} \\ \tau_{\mathbf{xz}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v_{xx}} \\ \mathbf{v_{xx}} \\ \mathbf{v_{xx}} \\ \mathbf{v_{xx}} \\ \tau_{\mathbf{xxx}} \\ \tau_{\mathbf{xxz}} \\ \tau_{\mathbf{zzx}} \\ \tau_{\mathbf{zzz}} \\ \tau_{\mathbf{xzx}} \\ \tau_{\mathbf{xzz}} \end{pmatrix} \tag{32}$$

the adjoint of summation operator is

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \tag{33}$$

limit amount duration of source wave field, take four step iteration as an example take 3 step finite difference as an example

$$\begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{d_3} \\ \mathbf{d_4} \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{G} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{G} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{m_1} \\ \mathbf{m_2} \end{pmatrix} \tag{34}$$

the adjoint of those modelling process is

$$\begin{pmatrix} \mathbf{m'_1} \\ \mathbf{m'_2} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{G^T} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{G^T} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{G^T} \end{pmatrix} \begin{pmatrix} \mathbf{S^T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S^T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S^T} \end{pmatrix} \begin{pmatrix} \mathbf{R^T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R^T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R^T} \end{pmatrix} \tag{35}$$

this is one way to reduce the size of model parameter, there is another way to achieve the same goal, assuming the source only come as body force, so we can ignore the three pressure component.

we use group sparsity constraints to regularize the cost function

$$||\mathbf{d} - \mathbf{Gm}||_\mathbf{2}^\mathbf{2} + \lambda_{\mathbf{l_{1,1,2}}} ||\mathbf{m}||_{\mathbf{1,1,2}} \tag{36}$$

in this problem, our model parameters $\mathbf{m}$ is a 3D array, the first dimension is z, second dimension is x, third dimension is t. the second part in this equation expanded as

$$||\mathbf{m}||_{\mathbf{1,1,2}} = \sum_{\mathbf{iz=1}}^{\mathbf{nz}} \sum_{\mathbf{ix=1}}^{\mathbf{nx}} \left(\sum_{\mathbf{it=1}}^{\mathbf{nt}} \mathbf{m}_{\mathbf{iz,ix,it}}^{\mathbf{2}} \right)^{\frac{\mathbf{1}}{\mathbf{2}}} \tag{37}$$

Data: The initial guess is 0

Result: 3D Wave field formed by Sources

initialization;

$\mathbf{y_1} = \mathbf{m_0} = \mathbf{0}$;

$\mathbf{m_1} = \mathcal{F}_{\lambda/\mathbf{2}\alpha}\left(\frac{\mathbf{1}}{\alpha}\mathbf{G^T d}\right)$;

$\mathbf{J[1]} = ||\mathbf{d}||_\mathbf{2}^\mathbf{2} + ||\mathbf{m_1}||_{\mathbf{1},\mathbf{1},\mathbf{2}}$;

$\mathbf{y_2} = \mathbf{m_1}$;

$t_{old} = \frac{1+\sqrt{5}}{2} \quad t_{new} = \frac{1+\sqrt{1+4t_{old}^2}}{2}$;

for *k = 2 : niter* **do**

$\mathbf{m_k} = \mathbf{y_k} + \frac{\mathbf{1}}{\alpha}\mathbf{G^T}(\mathbf{d} - \mathbf{G y_k})$;

$\mathbf{m_k} = \mathcal{F}_{\lambda/\mathbf{2}\alpha}(\mathbf{m_k})$;

$\mathbf{J[iter]} = ||\mathbf{d} - \mathbf{G y_k}||_\mathbf{2}^\mathbf{2} + ||\mathbf{m_k}||_{\mathbf{1},\mathbf{1},\mathbf{2}}$;

$\mathbf{y_{k+1}} = \mathbf{m_k} + \left(\frac{\mathbf{t_{old}-1}}{\mathbf{t_{new}}}\right)(\mathbf{m_k} - \mathbf{m_{k-1}})$;

$t_{old} = t_{new}, \quad t_{new} = \frac{1+\sqrt{1+4t_{old}^2}}{2}$

end

the soft thresh holding is computed as follow

$$\tau = \frac{\lambda}{2\alpha} \tag{38}$$

$$\frac{\left(\sqrt{\sum_{it=1}^{nt} m_{iz,ix,it}^2} - \tau\right)_+}{\sqrt{\sum_{it=1}^{nt} m_{iz,ix,it}^2}} m_{iz,ix,it} \tag{39}$$

in this equation $(\quad)_+$ represent get the maximum value between 0 and the variable in bracket.

the elastic wave equation for model order reduction:

$$
\mathbf{G_v Q} = \begin{pmatrix}
\mathbf{M}_{\mathbf{v_{xx}}}^{\mathbf{v_{xx}}}\mathbf{Q}_{\mathbf{v_{xx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xx}}}^{\mathbf{v_{xx}}}\mathbf{Q}_{\tau_{\mathbf{xxx}}} & \mathbf{M}_{\tau_{\mathbf{xx}}}^{\mathbf{v_{xx}}}\mathbf{Q}_{\tau_{\mathbf{xxz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_{\mathbf{v_{xz}}}^{\mathbf{v_{xz}}}\mathbf{Q}_{\mathbf{v_{xz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{xz}}}\mathbf{Q}_{\tau_{\mathbf{xzx}}} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{xz}}}\mathbf{Q}_{\tau_{\mathbf{xzz}}} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_{zx}}}^{\mathbf{v_{zx}}}\mathbf{Q}_{\mathbf{v_{zx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{zx}}}\mathbf{Q}_{\tau_{\mathbf{xzx}}} & \mathbf{M}_{\tau_{\mathbf{xz}}}^{\mathbf{v_{zx}}}\mathbf{Q}_{\tau_{\mathbf{xzz}}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\mathbf{v_{zz}}}^{\mathbf{v_{zz}}}\mathbf{Q}_{\mathbf{v_{zz}}} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\tau_{\mathbf{zz}}}^{\mathbf{v_{zz}}}\mathbf{Q}_{\tau_{\mathbf{zzx}}} & \mathbf{M}_{\tau_{\mathbf{zz}}}^{\mathbf{v_{zz}}}\mathbf{Q}_{\tau_{\mathbf{zzz}}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{xxx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{xxz}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{zzx}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{zzz}}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{xzx}}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\tau_{\mathbf{xzz}}}
\end{pmatrix} \tag{40}
$$

stability condition for finite difference method

$$\Delta t \leqslant \frac{6h}{7\sqrt{3}v_p} \tag{41}$$

frequency dispersion condition

$$f < v_{min}/5h \tag{42}$$

EXAMPLES

synthetic data example

Real data example

CONCLUSIONS

ACKNOWLEDGMENTS

We thank the sponsors

APPENDIX A