Fixed Income Markets
MFE Summer 2018
Assignment 4

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1. To calibrate the Vasicek model using the given data, we firstly initiate a random set of parameters, say $\kappa = 0.05, \mu = 0.1$, and we use this set of parameters to solve for r_0 exactly for each value of CMT(5). In specific, we solve the following equation:

$$100 = \sum_{i=0.5}^{5} CMT(5)e^{-i \cdot r(i)} + 100 \cdot e^{-5 \cdot r(5)}$$
$$r(T) = -\frac{A(T)}{T} + \frac{B(T)}{T} r_0$$
$$B(T) = \frac{1 - e^{-\kappa T}}{\kappa}$$
$$A(T) = (B(T) - T)\left(\hat{\mu} - \frac{\sigma^2}{2\kappa^2} - \frac{\sigma^2 B(T)^2}{4\kappa}\right)$$

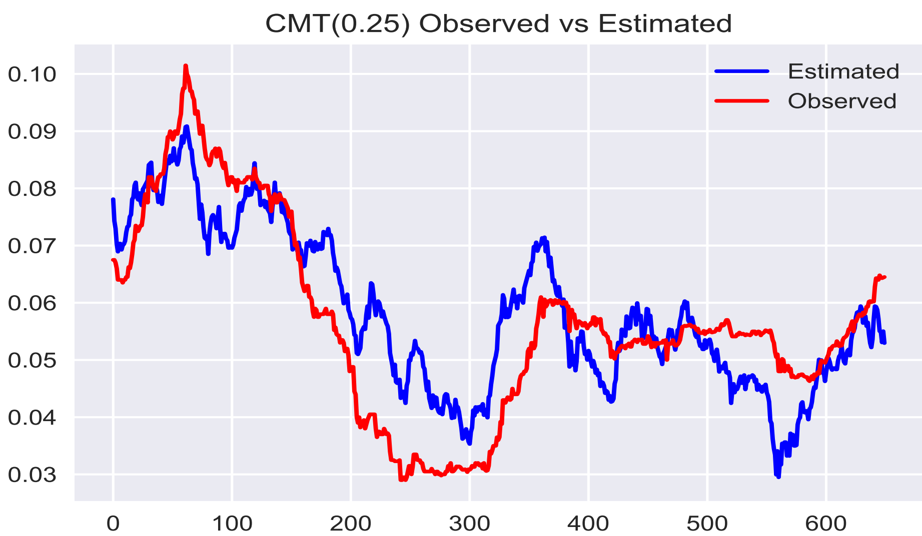
We can see from the above equations that all the parameters are set and variables are given in the data, except for r_0. Therefore we can solve for r_0. Then for each iteration, we use the same r_0 for each point in time to find the estimated value of CMT, and use numpy functionality to minimize the error subject to changing κ and μ.

(a) After the iterative minimization, the estimated parameters are as follows:

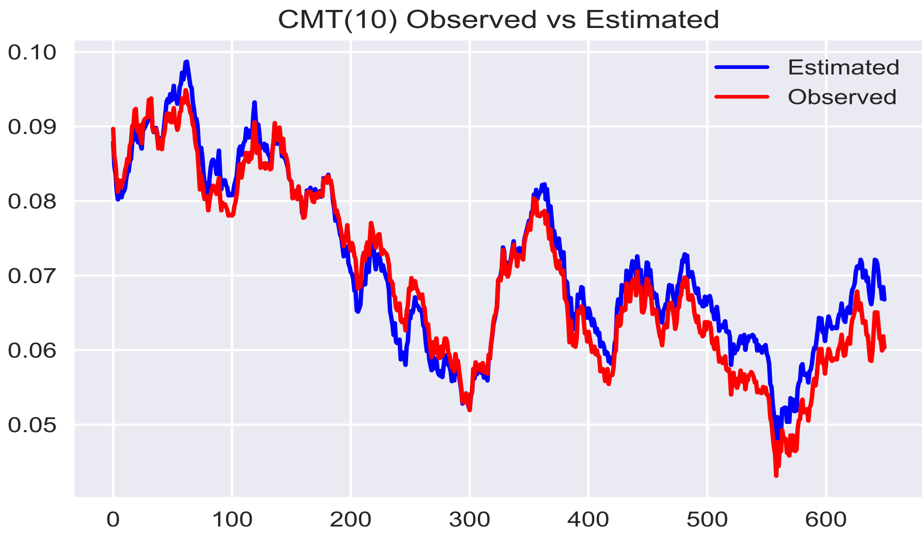
$\kappa = 0.03988$ $\qquad\qquad$ $\mu = 0.1735$ $\qquad\qquad$ $r = 0.07614$

(b) Using the estimated parameters, the following graphs are produced:

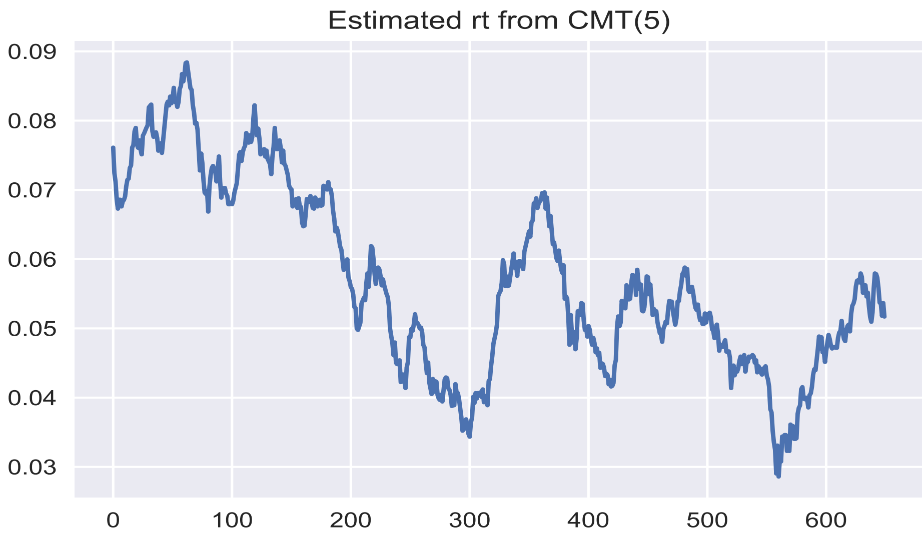
(i)



(ii)



(iii)



(c) The results are summarized as the following:

$$SampleMean = 0.05686 \qquad SampleStd = 0.01344 \qquad Sample\Delta rStd = 0.00152$$

The model implied mean should be close to $\hat{\mu} = 0.1735$, and the model implied std should be close to $\sqrt{\frac{\sigma^2}{2\kappa}} = 0.07825$, and the std for δr should be close to $\sqrt{\sigma^2 \cdot \Delta t} = 0.0030$. We can see that the values are not at all close, reason being that we do not observe λ, that is the market price of risk. As a result, the sample values are measure under the P measure, whereas the model implied values are under Q measure. Therefore, these two sets of parameters do not coincide.

2. (a) The forward rate for a 3-year loan beginning in 2 years is:

$$f = \frac{\log(Z_1/Z_2)}{(5-2)} = 0.05606$$

Where $Z_1 = 0.9097$, $Z_2 = 0.7689$ are the model implied discount function given by the following formula:

$$Z(t, t+T) = e^{A(T)-B(T)r}$$

and A(T), B(T) defined same as above.

(b) The future price for a 3-year zero-coupon bond to be delivered in 2 years can be calculated as follows:

$$\begin{aligned}
E_0[r_2] &= r_0 e^{-2\kappa} + \mu(1 - e^{-2\kappa}) = 0.05264 \\
VAR_0[r_2] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t)}\right) = 8.65 \cdot 10^{-5} \\
Z_{t=2,T=5} &= 100 \cdot E_0\left[e^{A(3)-B(3)r_2}\right] = 84.51 \\
r_{implied} &= \log(100/Z_{t=2,T=5})/3 = 0.05610
\end{aligned}$$

We can notice that the implied r is different from the calculated forward rate. The reason lies in the difference of forward and future, in that the PnL realized by the futures contract is marked to market, whereas the PnL realized by the forward is settled at expiry, i.e. $T = 5$. Taking into the consideration of discounting of the PnL, the implied yields for the two contracts are therefore different.

(c) To value the 4 year option on 10 year coupon bond, we first need to find r^* that make the value of the coupon bond equals to 100 at time $T = 4$. Solving numerically, we have: $r^* = 0.055213$. Then we use the corresponding r^* and model implied term structure to find the strikes of the remaining cash flow. We then use the closed form solution for the zero coupon bond call option for Vasicek Model to calculable the time zero call price for each cash flow. The results are:

$$\begin{array}{llllll}
S_1 = 5.6720 & S_2 = 5.3542 & S_3 = 5.0499 & S_4 = 4.7605 & S_5 = 4.4862 & S_6 = 74.6773 \\
C_1 = 0.0112 & C_2 = 0.0170 & C_3 = 0.0197 & C_4 = 0.0207 & C_5 = 0.0208 & C_6 = 0.3579
\end{array}$$

Summing over all the C_i, we get: $Call = 0.4474$.

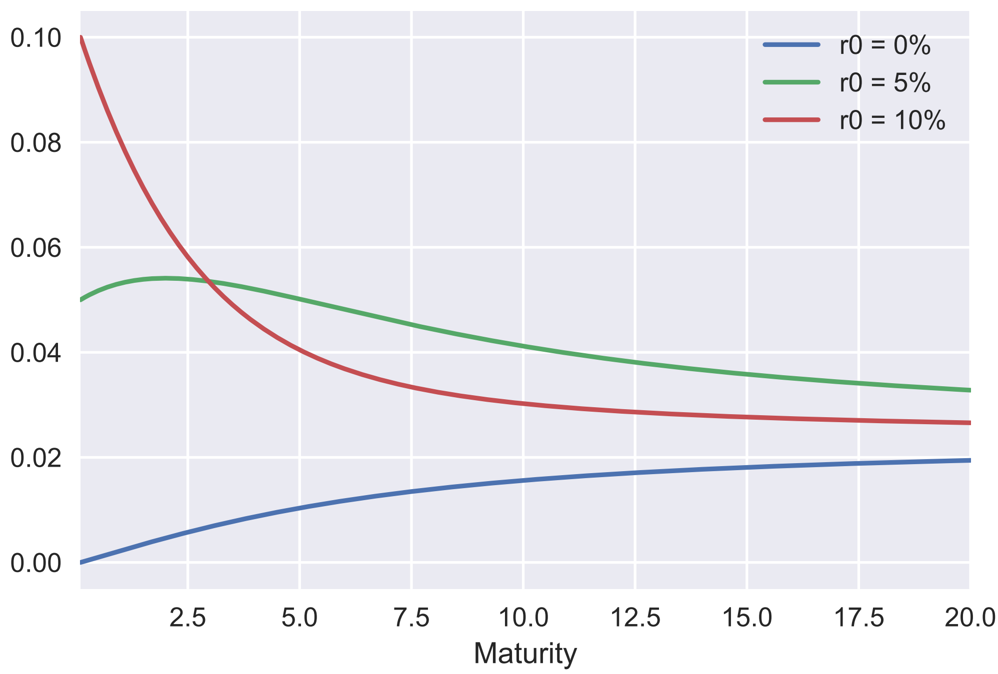
(d) To calculate the hedging ratio, we can begin by estimating the Δ of the call option with respect to the underlying. The delta can be estimated numerically by the following method:

$$\begin{aligned}
Bond_{up} &= P(r_0 = 0.04 + \delta r) = 102.56 \\
Bond_{dn} &= P(r_0 = 0.04 - \delta r) = 102.92 \\
Call_{up} &= C(r_0 = 0.04 + \delta r) = 0.438 \\
Call_{dn} &= C(r_0 = 0.04 - \delta r) = 0.457 \\
\Delta &= \frac{Call_{up} - Call_{dn}}{Bond_{up} - Bond_{dn}} = 0.05
\end{aligned}$$

i.e. we need to short 0.05 unit of the underlying bond to hedge out the interest rate risk of the call option.

3. We can observe from the model that x and y are symmetric, in the sense that they are interchangeable in the model. Therefore, there is no real difference between κ_1 and κ_2, μ_1 and μ_2, σ_1 and σ_2. Therefore, the estimated values for κ_1 and κ_2 are interchangeable, i.e. the estimation is not unique.

4. **(a)** We begin by defining T, the longest maturity, as well as δt, the discrete incrimination of t. Then we build a tree with length $N = \frac{T}{\delta t}$ and three rows. After this, for each point in time t, we initiate a zero coupon bond that matures at time t and use the risk-neutral transition matrix to calculate the risk neutral valuation of the bond one period before, until we have reached $t = 0$. By doing this, we have constructed a discount function for each initial value of r_0. We then use the post rate using codes from assignment 1 to find the spot rate. In implementation, we assume $T = 20$ and $\delta t = 0.1$. The plotted term structures are shown as follows:



(b) Assuming that it is an European option, we construct a similar tree as above with $T = 3$, $\delta t = 0.01$ (we have set δt to be smaller since T is relatively smaller). Then we discount backward, adding coupon payments along the way, back to $t = 0.5$ where the option expires. Subtract the $t = 0.5$ price of the bond by 100, and continue to discount back to $t = 0$ WITHOUT adding the coupon payment, we then get the time zero option price for each initial short rate. The results are:

$$C(r_0 = 0) = 9.93323 \qquad C(r_0 = 5\%) = 0.111378 \qquad C(r_0 = 10\%) = 2.1974$$

5. **(a)** at time 0, the portfolio value is:

$$V(0) = m \cdot e^{-\tau r_0} - e^{-2\tau r_0} + n \cdot e^{-3\tau r_0}$$

In order to let $V(0) = 0$, we need to have: $m - e^{-\tau r_0} + n \cdot e^{-2\tau r_0} = 0$.

(b) Holding this portfolio to time τ, with interest rate r, the value of potfolio is:

$$V(\tau) = m - e^{-\tau r} + n \cdot e^{-2\tau r}$$

(c) Differentiate V with respect to r and equate the derivative to zero at r_0, we have:

$$\frac{\partial V}{\partial r}|r_0 = \tau e^{-\tau r_0} - 2\tau n e^{-2\tau r_0} = 0 \Rightarrow n = \frac{1}{2}e^{\tau r_0} \quad and \quad m = \frac{1}{2}e^{-\tau r_0}$$

Also, we know that this is indeed minimum of V because $V(\tau)$ is a quadratic function of $e^{-\tau r}$, with coefficient of the second order term positive, so its a U shape parabola, with only global minimum.

(d) The dollar duration of the portfolio is:

$$D_{\$} = \frac{\tau}{2}e^{-2\tau r_0} - 2\tau e^{-2\tau r_0} + \frac{3\tau}{2}e^{-2\tau r_0} = \left(\frac{\tau}{2} - 2\tau + \frac{3\tau}{2}\right) e^{-2\tau r_0} = 0$$

(e) The arbitrage exist in the following sense:

$$V(0) = 0, \quad V_{min}(\tau) = \frac{1}{2}e^{-\tau r_0} - e^{-\tau r_0 +} + \frac{1}{2}e^{-\tau r_0} = 0$$

The initial investment is 0 and the minimum payoff at τ is also zero, meaning one could get positive payoff without initial investment, thus arbitrage exists. We could treat the portfolio as longing a barbell consisting of the τ and 3τ ZCB and shorting a bullet with maturity 2τ. The barbell and bullet has same value and duration, but the barbell has higher convexity than the bullet. Thus whenever there's parallel shift in interest rates, the barbell outperforms the bullet, and the portfolio would have positive payoff.