# The Problem Set

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**Problem 1-1**

Show that

$$\{P_0\}S\{Q_0\} \text{ and } \{P_1\}S\{Q_1\}$$

implies

$$\{P_0 \wedge P_1\}S\{Q_0 \wedge Q_1\} \text{ and } \{P_0 \vee P_1\}S\{Q_0 \vee Q_1\}.$$

**Problem 2-1**

Prove

$$\begin{array}{l}
|[ \\
\textbf{var}\ x, y : \mathit{int}; \\
\{x = A \wedge y = B\} \\
x := x - y;\ y := x + y;\ x := y - x \\
\{x = B \wedge y = A\} \\
]|.
\end{array}$$

**Problem 2-2**

Determine the weakest $P$ such that

$$\begin{array}{l}
|[ \\
\textbf{var}\ x : \mathit{int}; \\
\{P\} \\
x := x + 1; \\
\textbf{if}\ x > 0\ \rightarrow\ x := x - 1 \\
\ [\,]\ x < 0 \rightarrow x := x + 2 \\
\ [\,]\ x = 1 \rightarrow \mathit{skip} \\
\textbf{fi} \\
\{x \geq 1\} \\
]|.
\end{array}$$

**Problem 2-3**

Prove the correctness of the following program.

$$
\begin{array}{l}
|[ \\
\mathbf{var}\ x, y, z : \mathit{int}; \\
\{\mathit{true}\} \\
\mathbf{do}\ x < y \rightarrow x := x + 1 \\
\ \ [ ]\ y < z \rightarrow y := y + 1 \\
\ \ [ ]\ z < x \rightarrow z := z + 1 \\
\mathbf{od} \\
\{x = y = z\} \\
]|
\end{array}
$$

**Problem 2-4**

The following problem may be used to compute (non-deterministically) natural numbers $x$ and $y$ such that $x * y = N$. Prove:

$$
\begin{array}{l}
|[ \\
\mathbf{var}\ p, x, y, N : \mathrm{int}; \\
\{N \geq 1\} \\
p, x, y := N - 1, 1, 1; \\
\{N = x * y + p\} \\
\mathbf{do}\ p \neq 0 \\
\quad \rightarrow \mathbf{if}\ p\ \mathbf{mod}\ x = 0 \rightarrow p, y := p - x, y + 1 \\
\qquad [ ]\ p\ \mathbf{mod}\ y = 0 \rightarrow x, p := x + 1, p - y \\
\qquad \mathbf{fi} \\
\mathbf{od} \\
\{x * y = N\} \\
]|.
\end{array}
$$

**Problem 2-5**

Prove

$$
\begin{array}{l}
|[ \\
\mathbf{con}\ N : \mathit{int}\ \{N \geq 0\}; \\
f :\ \mathbf{array}\ [0..N)\ \mathbf{of}\ \mathit{int}; \\
\mathbf{var}\ b : \mathit{bool}; \\
\quad |[ \\
\quad \mathbf{var}\ n : \mathit{int}; \\
\quad b, n := \mathit{false}, 0; \\
\quad \mathbf{do}\ n \neq N \rightarrow b := b \vee f.n = 0;\ n := n + 1\ \mathbf{od} \\
\quad ]| \\
\{b \equiv (\exists i : 0 \leq i < N :\ f.i = 0)\} \\
]|.
\end{array}
$$

**Problem 3-1**

Let $X[0..N)$ be an integer array. Express the following expressions in a natural language.

1. $b \equiv (\forall i : 0 \le i < N : X.i \ge 0)$
2. $r = (max\, p, q : 0 \le p \le q \le N \wedge (\forall i : p \le i < q : X.i \ge 0) : q - p)$
3. $r = (\# k : 0 \le k < N : (\forall i : 0 \le i < k : X.i < X.k))$
4. $b \equiv (\exists i : 0 < i < N : X.(i-1) < X.i)$
5. $r = (\# p, q : 0 \le p < q < N : X.p = 0 \wedge X.q = 0)$

6. $s = (max\, p, q : 0 \le p < q < N : X.p + X.q)$
7. $b \equiv (\forall p, q : 0 \le p \wedge 0 \le q \wedge p + q = N - 1 : X.p = X.q)$
8. $b = (\exists i : 0 \le i < N.X.i = 0)$

**Problem 4-1**

Solve the following problem.

$$
\begin{array}{l}
|[ \\
\textbf{con}\ N, X : int\ \{N \ge 0\};\ \ f : \textbf{array}\ [0..N)\ \textbf{of}\ int; \\
\textbf{var}\ r : int \\
S \\
\{r = (\Sigma i : 0 \le i < N : f.i * X^i)\} \\
]|.
\end{array}
$$

**Problem 4-2**

Solve the following problem.

$$
\begin{array}{l}
|[ \\
\textbf{con}\ N : int\ \{N \ge 1\};\ \ A : \textbf{array}\ [0..N)\ \textbf{of}\ int; \\
\textbf{var}\ r : int \\
S \\
\{r = (\textbf{max}\ p\ q : 0 \le p < q < N : A.p - A.q)\} \\
]|.
\end{array}
$$

**Problem 5-1**

Solve

$$
\begin{array}{l}
|[ \\
\textbf{con}\ N, X : int\ \{N \ge 0\};\ \ f : \textbf{array}\ [0..N)\ \textbf{of}\ int; \\
\textbf{var}\ r : bool \\
S \\
\{r \equiv (\exists i : 0 \le i < N : f.i = 0)\} \\
]|,
\end{array}
$$

by defining for $0 \le n \le N$

$$G.n \equiv (\exists i : n \le i < N : f.i = 0)$$

and deriving a suitable recurrence relation for $G$.

**Problem 5-2**

An $h$-sequence is either a sequnce consisting of the single element 0 or it is a 1 followinged by two $h$-sequences. Syntactically, $h$-sequence may be defined by

$$h \; = \; 0 \mid 1 \; h \; h$$

Solve

$|[$
**con** $N : int \; \{N \geq 0\}; \; A :$ **array** $[0..2 * N + 1)$ **of** $[0..1]$;
**var** $r : \; bool$;
$S$
$\{r \equiv \; A$ is an $h$-sequence$\}]|$.

**Problem 5-3**

Derive a program to solve the following problem.

$|[$
**con** $N : int \; \{N \geq 0\}$;
$X, Y, Z, W :$ **array** $[0..N)$ **of** $int$;
**var** $r : int$;
$S$
$\{r = \{\# i, j, k, l : 0 \leq i, j, k, l < N : X.i + Y.j + Z.k + W.l = 0\}\}$
$]|$.

**Problem 6-1**

Derive a program that has time complexity $\mathcal{O}(\log N)$ for

$|[$
**con** $N : int \; \{N \geq 1\}; f :$ **array** $[0..N]$ **of** $int \; \{f.0 < f.N\}$;
**var** $x : int$;
$S$
$\{0 \leq x < N \wedge f.x < f.(x + 1)\}$
$]|$

by introducing variable $y$ and invariants

$$P_0 : \quad f.x < f.y$$
$$P_1 : \quad 0 \leq x < y \leq N$$

**Problem 6-2**

Derive an $\mathcal{O}(\log N)$ algorithm for *square root*:

$|[$
**con** $N : int \; \{N \geq 0\}$;
**var** $x : int$;
*square root*
$\{x^2 \leq N \wedge (x + 1)^2 > N\}$
$]|$

by introducing variables $y$ and $k$ and invariants:

$$\begin{array}{ll} P_0: & x^2 \leq N \wedge (x+y)^2 > N \\ P_1: & y = 2^k \wedge 0 \leq k \end{array}$$

**Problem 6-3**

Solve

$$\begin{array}{l} |[ \\ \textbf{con } A, B, N : \textit{int} \; \{N \geq 0\}; \\ \textbf{var } x : \textit{int}; \\ S \\ \{x = (\Sigma i : 0 \leq i \leq N : A^{N-i} * B^i)\} \\ ]| \end{array}$$

**Problem 6-4**

Solve

$$\begin{array}{l} |[ \\ \textbf{con } N : \textit{int} \; \{N \geq 0\}; \\ \textbf{var } x : \textit{int}; \\ \textit{Fibolucci} \\ \{x = (\Sigma i : 0 \leq i \leq N : fib.i * fib.(N-i)\} \\ ]| \end{array}$$

where $fib$ is defined by

$$\begin{array}{lcl} fib.0 & = & 0 \\ fib.1 & = & 1 \\ fib.(n+2) & = & fib.n + fib.(n+1). \end{array}$$

**Problem 7-1**

Derive a program for the following specification.

$$\begin{array}{l} |[ \\ \textbf{con } N : \textit{int} \; \{N \geq 0\}; \\ \textbf{var } r : \textit{bool}; \\ S \\ \{r \equiv (\exists p : \; p \geq 0 : N = p^3)\} \\ ]| \end{array}$$

**Problem 7-2**

Derive for given $N$, $N \geq 0$, a program for the computation of the smallest integer $x$ that satisfies $x^3 - 6x^2 + 9x \geq N$.

**Problem 7-3**

Derive a program for the following specification.

$|[$
**con** $N : int$ $\{N \geq 1\}$; $A, B :$ **array** $[0..N]$ **of** $int$;
$\{A.0 \leq B.0 \wedge A.N \geq B.N\}$
**var** $r : int$;
$S$
$\{0 \leq r < N \wedge A.r \leq B.r \wedge A.(r+1) \geq B.(r+1)\}$
$]|$

**Problem 8-1: The Starting Pit Location Problem**

Given are $N+1$ pits located along a circular racetrack. The pits are numbered clockwise from 0 up to and including $N$. At pit $i$, there are $p.i$ gallons of petrol available. To race from pit $i$ to its clockwise neighbor one needs $q.i$ gallons of petrol. One is asked to design a linear algorithm to determine a pit from which it is possible to race a complete lap, starting with an empty fuel tank. To guarantee the existence of such a starting pit, we assume that

$$(\Sigma i : 0 \leq i \leq N : p.i) = (\Sigma i : 0 \leq i \leq N : q.i).$$

**Problem 8-2**

Derive an $O(N)$ solution to the following problem.

$|[$
**con** $N : int$ $\{N \geq 0\}$; $X :$ **array** $[0..N)$ **of** $int$;
**var** $r : int$;
*all equal*
$\{r = (\mathbf{max}\ p, q : 0 \leq p \leq q \leq N \wedge (\forall i, j : p \leq i \leq j < q : X.i = X.j) :\ q - p)\}$
$]|$

**Problem 9-1**

Derive an efficient program $S$ satisfying the following specification:

$|[$
**con** $N : int$ $\{N \geq 0\}$; **var** $r : bool$;
$S$
$\{r \equiv (\exists x, y : 0 \leq x \wedge 0 \leq y : N = 2^x + 3^y)\}$
$]|$

**Problem 9-2**

Derive an efficient program $S$ satisfying the following specification:

$|[$
**con** $M, N : int$ $\{M \geq 0 \wedge N \geq 0\}$;
$f :$ **array** $[0..M) \times [0..N)$ **of** int;
$\{f$ is ascending in both arguments$\}$
**var** $r : bool$;
$S$
$\{\#i, j : 0 \leq i < M \wedge 0 \leq j < N : f.i.j = 0\}]|$