Program for Class Discussion

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Problem 2-1:

Show that

$$\{P_0\}S\{Q_0\} \text{ and } \{P_1\}S\{Q_1\}$$

implies

$$\{P_0 \wedge P_1\}S\{Q_0 \wedge Q_1\} \text{ and } \{P_0 \vee P_1\}S\{Q_0 \vee Q_1\}.$$

Speaker: 48076134

Problem 2-2

Prove

$$\begin{array}{l} |[\\ \textbf{var } x, y : \mathit{int}; \\ \{x = A \wedge y = B\} \\ x := x - y;\ y := x + y;\ x := y - x \\ \{x = B \wedge y = A\} \\ ]|. \end{array}$$

Speaker: 37086868

Problem 2-3

Determine the weakest P such that

$$\begin{array}{l} |[\\ \textbf{var } x : \mathit{int}; \\ \{P\} \\ x := x + 1; \\ \textbf{if } x > 0 \ \rightarrow\ x := x - 1 \\ \ [] \ x < 0 \rightarrow x := x + 2 \\ \ [] \ x = 1 \rightarrow \mathit{skip} \\ \textbf{fi} \\ \{x \geq 1\} \\ ]|. \end{array}$$

Speaker: 48086234, 48086226

Problem 2-4

Prove the correctness of the following program.

$$
\begin{array}{l}
|[\\
\textbf{var } x, y, z : \mathit{int}; \\
\{\mathit{true}\} \\
\textbf{do } x < y \rightarrow x := x + 1 \\
\ [] \ y < z \rightarrow y := y + 1 \\
\ [] \ z < x \rightarrow z := z + 1 \\
\textbf{od} \\
\{x = y = z\} \\
]|
\end{array}
$$

Speaker: 48086219

Problem 2-5

The following problem may be used to compute (non-deterministically) natural numbers x and y such that $x * y = N$. Prove:

$$
\begin{array}{l}
|[\\
\textbf{var } p, x, y, N : \text{int}; \\
\{N \geq 1\} \\
p, x, y := N - 1, 1, 1; \\
\{N = x * y + p\} \\
\textbf{do } p \neq 0 \\
\quad \rightarrow \textbf{if } p \textbf{ mod } x = 0 \rightarrow p, y := p - x, y + 1 \\
\qquad [] \ p \textbf{ mod } y = 0 \rightarrow x, p := x + 1, p - y \\
\qquad \textbf{fi} \\
\textbf{od} \\
\{x * y = N\} \\
]|.
\end{array}
$$

Speaker: 48086229

Problem 2-6

Prove

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 0\}; \\
f : \ \textbf{array } [0..N) \textbf{ of } \mathit{int}; \\
\textbf{var } b : \mathit{bool}; \\
\quad |[\\
\quad \textbf{var } n : \mathit{int}; \\
\quad b, n := \mathit{false}, 0; \\
\quad \textbf{do } n \neq N \rightarrow b := b \vee f.n = 0; \ n := n + 1 \textbf{ od} \\
\quad]| \\
\{b \equiv (\exists i : 0 \leq i < N : \ f.i = 0)\} \\
]|.
\end{array}
$$

Speaker: 37086386

Problem 3

Let $X[0..N)$ be an integer array. Express the following expressions in a natural language.

1. $b \equiv (\forall i : 0 \leq i < N : X.i \geq 0)$
2. $r = (max\ p, q : 0 \leq p \leq q \leq N \wedge (\forall i : p \leq i < q : X.i \geq 0) : q - p)$
3. $r = (\#k : 0 \leq k < N : (\forall i : 0 \leq i < k : X.i < X.k))$
4. $b \equiv (\exists i : 0 < i < N : X.(i-1) < X.i)$
5. $r = (\#p, q : 0 \leq p < q < N : X.p = 0 \wedge X.q = 0)$
6. $s = (max\ p, q : 0 \leq p < q < N : X.p + X.q)$
7. $b \equiv (\forall p, q : 0 \leq p \wedge 0 \leq q \wedge p + q = N - 1 : X.p = X.q)$
8. $b = (\exists i : 0 \leq i < N.X.i = 0)$

Speaker: 48086213

Problem 4-1

Solve the following problem.

$|[$
con $N, X : int\ \{N \geq 0\};\ f :$ **array** $[0..N)$ **of** $int;$
var $r : int$
S
$\{r = (\Sigma i : 0 \leq i < N : f.i * X^i)\}$
$]|.$

Speaker: 48086216, 48086230

Problem 4-2

Solve the following problem.

$|[$
con $N : int\ \{N \geq 1\};\ A :$ **array** $[0..N)$ **of** $int;$
var $r : int$
S
$\{r = (\textbf{max}\ p\ q : 0 \leq p < q < N : A.p - A.q)\}$
$]|.$

Speaker: 48087201

Problem 5-1

Solve

$|[$
con $N, X : int\ \{N \geq 0\};\ f :$ **array** $[0..N)$ **of** $int;$
var $r : bool$
S
$\{r \equiv (\exists i : 0 \leq i < N : f.i = 0)\}$
$]|,$

by defining for $0 \leq n \leq N$

$$G.n \equiv (\exists i : n \leq i < N : f.i = 0)$$

and deriving a suitable recurrence relation for G.

Speaker: 48086406

Problem 5-2

An h-sequence is either a sequnce consisting of the single element 0 or it is a 1 followinged by two h-sequences. Syntactically, h-sequence may be defined by

$$h \ = \ 0 \mid 1 \ h \ h$$

Solve

$|[$
con $N : int \ \{N \geq 0\}$; $A :$ **array** $[0..2 * N + 1)$ **of** $[0..1]$;
var $r :$ *bool*;
S
$\{r \equiv \ A$ is an h-sequence$\}]|$.

Speaker: 48086225

Problem 5-3

Derive a program to solve the following problem.

$|[$
con $N : int \ \{N \geq 0\}$;
$X, Y, Z, W :$ **array** $[0..N)$ **of** int;
var $r : int$;
S
$\{r = \{\#i, j, k, l : 0 \leq i, j, k, l < N : X.i + Y.j + Z.k + W.l = 0\}\}$
$]|$.

Speaker: 48086227

Problem 6-1

Derive a program that has time complexity $\mathcal{O}(\log N)$ for

$|[$
con $N : int \ \{N \geq 1\}; f :$ **array** $[0..N]$ **of** $int \ \{f.0 < f.N\}$;
var $x : int$;
S
$\{0 \leq x < N \wedge f.x < f.(x + 1)\}$
$]|$

by introducing variable y and invariants

$$\begin{array}{ll} P_0 : & f.x < f.y \\ P_1 : & 0 \leq x < y \leq N \end{array}$$

Speaker: 48086215

Problem 6-2

Derive an $\mathcal{O}(\log N)$ algorithm for *square root*:

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 0\}; \\
\textbf{var } x : \mathit{int}; \\
\mathit{square\ root} \\
\{x^2 \leq N \wedge (x+1)^2 > N\} \\
]|
\end{array}
$$

by introducing variables y and k and invariants:

$$
\begin{array}{ll}
P_0 : & x^2 \leq N \wedge (x+y)^2 > N \\
P_1 : & y = 2^k \wedge 0 \leq k
\end{array}
$$

Speaker: 48086217

Problem 6-3

Solve

$$
\begin{array}{l}
|[\\
\textbf{con } A, B, N : \mathit{int}\ \{N \geq 0\}; \\
\textbf{var } x : \mathit{int}; \\
S \\
\{x = (\Sigma i : 0 \leq i \leq N : A^{N-i} * B^i)\} \\
]|
\end{array}
$$

Speaker: 48067210

Problem 6-4

Solve

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 0\}; \\
\textbf{var } x : \mathit{int}; \\
\mathit{Fibolucci} \\
\{x = (\Sigma i : 0 \leq i \leq N : fib.i * fib.(N-i)\} \\
]|
\end{array}
$$

where fib is defined by

$$
\begin{array}{lcl}
fib.0 & = & 0 \\
fib.1 & = & 1 \\
fib.(n+2) & = & fib.n + fib.(n+1).
\end{array}
$$

Speaker: 48086201

Problem 7-1

Derive a program for the following specification.

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 0\}; \\
\textbf{var } r : \mathit{bool}; \\
S \\
\{r \equiv (\exists p :\ p \geq 0 : N = p^3)\} \\
]|
\end{array}
$$

Speaker: 48086202

Problem 7-2

Derive for given N, $N \geq 0$, a program for the computation of the smallest integer x that satisfies $x^3 - 6x^2 + 9x \geq N$.
Speaker: 48086227, 31086830

Problem 7-3

Derive a program for the following specification.

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 1\};\ A, B : \textbf{array } [0..N] \textbf{ of } \mathit{int}; \\
\{A.0 \leq B.0 \wedge A.N \geq B.N\} \\
\textbf{var } r : \mathit{int}; \\
S \\
\{0 \leq r < N \wedge A.r \leq B.r \wedge A.(r+1) \geq B.(r+1)\} \\
]|
\end{array}
$$

Speaker: 31086827, 48086123

Problem 8-1: The Starting Pit Location Problem

Given are $N + 1$ pits located along a circular racetrack. The pits are numbered clockwise from 0 up to and including N. At pit i, there are $p.i$ gallons of petrol available. To race from pit i to its clockwise neighbor one needs $q.i$ gallons of petrol. One is asked to design a linear algorithm to determine a pit from which it is possible to race a complete lap, starting with an empty fuel tank. To guarantee the existence of such a starting pit, we assume that

$$(\Sigma i : 0 \leq i \leq N : p.i) = (\Sigma i : 0 \leq i \leq N : q.i).$$

Speaker: 31086830

Problem 8-2

Derive an $O(N)$ solution to the following problem.

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int}\ \{N \geq 0\};\ X : \textbf{array } [0..N) \textbf{ of } \mathit{int}; \\
\textbf{var } r : \mathit{int}; \\
\mathit{all\ equal} \\
\{r = (\textbf{max } p, q : 0 \leq p \leq q \leq N \wedge (\forall i, j : p \leq i \leq j < q : X.i = X.j) :\ q - p)\} \\
]|
\end{array}
$$

Speaker: 48086435