Chapter 7: Formalizing Programming Principles

A Useful Technique: Slope Search

Programming Problem:

Given an array

$$f : [0..M] \times [0..N] \to \mathcal{Z}$$

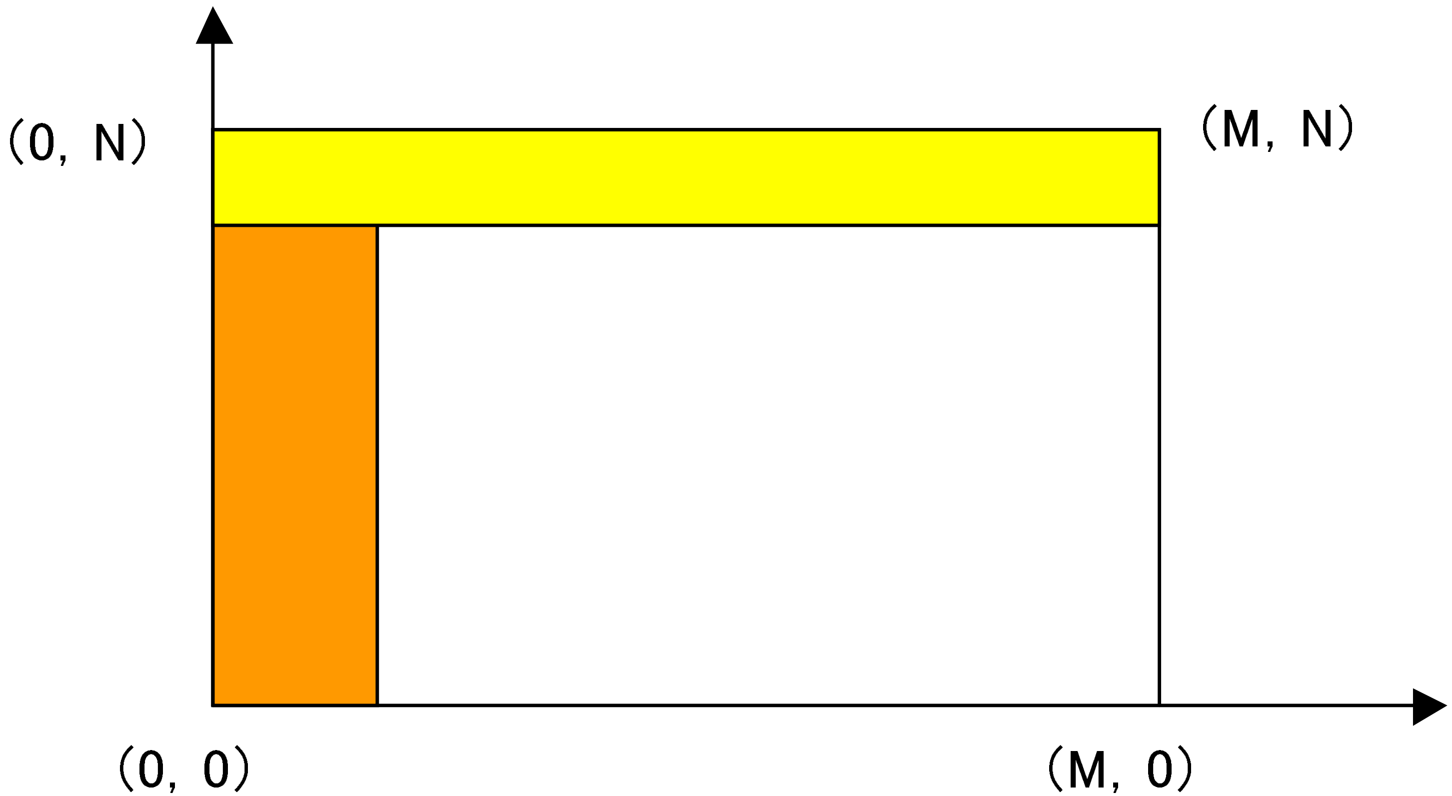
which is ascending in both arguments, and assuming X occurs in f, derive a program that establishes for integer a and b such that

$$f.a.b = X.$$

Examples of f: $f.a.b = a^2 + b^3$; $f.a.b = a \ \mathbf{max} \ b$

The post condition is:

$$R: \; 0 \leq a \leq M \;\wedge\; 0 \leq b \leq N \;\wedge\; f.a.b = X$$



Choosing (0,N) for inspection:

$$\begin{array}{lcl} f.0.N > X & \Rightarrow & (\forall i : 0 \leq i \leq M : f.i.N > X) \\ f.0.N < X & \Rightarrow & (\forall j : 0 \leq j \leq N : f.0.j < X) \end{array}$$

To be formal, we define (I, J) be the point satisfying

$$R: \; 0 \leq I \leq M \; \wedge \; 0 \leq J \leq N \; \wedge \; f.I.J = X$$

and choose the following invariant:

$$P: \; 0 \leq a \leq I \; \wedge \; J \leq b \leq N$$

which is established by $a, b := 0, N$.

$$
\begin{array}{ll}
 & f.a.b < X \\
\Rightarrow & \quad \{\ f \text{ is ascending in its second argument, } J \leq b\ \} \\
 & f.a.J < X \\
\Rightarrow & \quad \{\ f.I.J = X\ \} \\
 & a \neq I \\
\equiv & \quad \{\ a \leq I\ \} \\
 & a + 1 \leq I
\end{array}
$$

So

$$P \wedge f.a.b < X \Rightarrow P(a := a + 1)$$

$$
\begin{array}{ll}
 & f.a.b > X \\
\Rightarrow & \quad \{\ f \text{ is ascending in its second argument, } a \leq I\ \} \\
 & f.I.b > X \\
\Rightarrow & \quad \{\ f.I.J = X\ \} \\
 & b \neq J \\
\equiv & \quad \{\ a \leq I\ \} \\
 & J \leq b - 1
\end{array}
$$

So

$$P \wedge f.a.b > X \Rightarrow P(b := b - 1)$$

This yields the following slope searching solution:

$$
\begin{array}{l}
a, b := 0, N; \\
\textbf{do } f.a.b < X \rightarrow a := a + 1 \\
\ [] \ f.a.b > X \rightarrow b := b - 1 \\
\textbf{od}
\end{array}
$$

It has time complexity of $\mathcal{O}(M + N)$. A similar program can be obtained by choosing $(M, 0)$ as starting point.

Notice the tail invariant:

$$
\begin{array}{l}
(\exists i, j : 0 \leq i < M \wedge 0 \leq j < N : f.i.j = X) \\
\quad \equiv \ (\exists i, j : a \leq i < M \wedge 0 \leq j < b : f.i.j = X)
\end{array}
$$

Application 1: Searching

Given an array

$$f : [0..M] \times [0..N] \to \mathcal{Z}$$

which is ascending in both arguments, determine whether value X occurs in f.

$|[$
$\textbf{con}\ M, N, X : \mathit{int};$
$f :\ \textbf{array}\ [0..M) \times [0..N)\ \textbf{of}\ \mathit{int};$
$\{f$ is ascending in both arguments$\}$
$\textbf{var}\ r : \mathit{bool};$
S
$\{r \equiv (\exists i, j : 0 \leq i < M \wedge 0 \leq j < N : f.i.j = X)\}$
$]|$

Following the approach of "slope searching", we define a "tail" invariant:

$$G.a.b \equiv (\exists i, j : a \le i < M \wedge 0 \le j < b : f.i.j = X)$$

So:

$$R : \ r \equiv G.0.N$$

$$P_0 : \ r \vee G.a.b \equiv G.0.N$$
$$P_1 : \ 0 \le a \le M \wedge 0 \le b \le N$$

It is not difficult to prove that

$$P \wedge (a = M \vee b = 0 \vee r) \ \Rightarrow R$$

What is the guard?

Investigating an increase of a by 1.

$$
\begin{array}{ll}
 & G.a.b \\
\equiv & \quad \{ \text{ definition of } G \text{ } \} \\
 & (\exists i,j : a \le i < M \wedge 0 \le j < b : f.i.j = X) \\
\equiv & \quad \{ \text{ split off } i = a \text{ } \} \\
 & G.(a+1).b \vee (\exists j : 0 \le j < b : f.a.j = X) \\
\equiv & \quad \{ \text{ assuming } f.a.(b-1) < X,\ f \text{ is ascending in its second argument } \} \\
 & G.(a+1).b \vee \mathit{false} \\
\equiv & \quad \{ \text{ predicate calculus } \} \\
 & G.(a+1).b
\end{array}
$$

Hence,

$$f.a.(b-1) < X \;\Rightarrow\; (G.a.b = G.(a+1).b)$$

Similarly(?), we have

$$f.a.(b-1) > X \;\Rightarrow\; (G.a.b = G.a.(b-1))$$

And for the case $f.a.(b-1) = X$, we have

$$P_0 \wedge (f.a.(b-1) = X) \;\Rightarrow\; P_0(r := \mathit{true})$$

Our final program:

$$
\begin{array}{l}
|[\\
\textbf{var } a, b : \mathit{int}; \\
a, b, r := 0, N, \mathit{false}; \\
\textbf{do } a \neq M \wedge b \neq 0 \wedge \neg r \rightarrow \\
\qquad \textbf{if } f.a.(b-1) < X \rightarrow a := a + 1; \\
\qquad [\,]\ f.a.(b-1) > X \rightarrow b := b - 1; \\
\qquad [\,]\ f.a.(b-1) = X \rightarrow r := \mathit{true} \\
\qquad \textbf{fi} \\
\textbf{od} \\
]|
\end{array}
$$

The time complexity is $\mathcal{O}(M + N)$. How to show this algorithm is optimal?

Application 2: Decomposition in a sum of two squares

Derive a program for the computation of the number of ways in which a natural number N can be written as the sum of two squares.

$$
\begin{array}{l}
|[\\
\textbf{con}\ N : int\ \{N \geq 0\}; \\
\textbf{var}\ r : int; \\
S \\
\{r = (\#x, y : 0 \leq x \leq y : x^2 + y^2 = N)\} \\
]|
\end{array}
$$

Since $x^2 + y^2$ is increasing in both arguments, we define

$$G.a.b = (\#x, y : a \leq x \leq y \leq b : x^2 + y^2 = N)$$

Hence

$$\begin{aligned} R: &\quad r = G.0.\sqrt{N} \\ P_0: &\quad r + G.a.b = G.0.\sqrt{N} \\ P_1: &\quad 0 \leq a \leq b \leq \sqrt{N} \end{aligned}$$

The invariants can be established by:

$$a, b, r := 0, \sqrt{N}, 0$$

Investigate the effect of increase of a by 1.

$$
\begin{array}{ll}
 & G.a.b \\
= & \quad \{ \text{ definition of } G \text{ } \} \\
 & (\#x,y : a \le x \le y \le b : x^2 + y^2 = N) \\
= & \quad \{ \text{ split } x = a \text{ } \} \\
 & G.(a+1).b + (\#y.a \le y \le b : a^2 + y^2 = N) \\
= & \quad \{ \text{ } a \le b \text{ } \} \\
 & G.(a+1).b + 0, \quad \text{if } a^2 + b^2 < N \\
 & G.(a+1).b + 1, \quad \text{if } a^2 + b^2 = N
\end{array}
$$

Investigate the effect of decrease of b by 1.

$$
\begin{array}{ll}
 & G.a.b \\
= & \quad \{ \text{ definition of } G \ \} \\
 & (\#x, y : a \le x \le y \le b : x^2 + y^2 = N) \\
= & \quad \{ \text{ split } y = b \ \} \\
 & G.a.(b-1) + (\#x.a \le x \le b : x^2 + b^2 = N) \\
= & \quad \{ \ a \le b \ \} \\
 & G.a.(b-1) + 0, \quad \text{if } a^2 + b^2 > N \\
 & G.a.(b-1) + 1, \quad \text{if } a^2 + b^2 = N
\end{array}
$$

Our final program:

$$
\begin{array}{l}
|[\\
\textbf{var } a, b : \mathit{int}; \\
r, a := 0, 0; b := 0;\ \textbf{do } b * b < N \rightarrow b := b + 1 \textbf{ od}; \\
\textbf{do } a \leq b \rightarrow \\
\qquad \textbf{if } a * a + b * b < N \rightarrow a := a + 1 \\
\qquad [\,]\ a * a + b * b > N \rightarrow b := b - 1 \\
\qquad [\,]\ a * a + b * b = N \rightarrow r, a := r + 1, a + 1 \\
\qquad [\,]\ a * a + b * b = N \rightarrow r, b := r + 1, b - 1 \\
\quad \textbf{fi} \\
\textbf{od} \\
]|
\end{array}
$$

The time complexity is $\mathcal{O}(\sqrt{N})$.

Application 3: Minimal distance

Derive a program for the computation of the minimal distance of two ascending sequences.

$$
\begin{array}{l}
|[\\
\textbf{con } M, N : \mathit{int}\ \{M \geq 0 \wedge N \geq 0\}; \\
f :\ \textbf{array } [0..M)\ \textbf{of } \mathit{int}\ \{f \text{ is ascending}\}; \\
g :\ \textbf{array } [0..N)\ \textbf{of } \mathit{int}\ \{g \text{ is ascending}\}; \\
\textbf{var } r : \mathit{int}; \\
S \\
\{r = (\textbf{min } x, y : 0 \leq x < M \wedge 0 \leq y < N : |f.x - g.y|)\} \\
]|
\end{array}
$$

Note that $f.x - g.y$ is increasing in x and decreasing in y, but $|f.x - g.y|$ does not have this property. Anyway, slope search is still possible.

Define $G.a.b$ by

$$G.a.b = (\mathbf{min}\ x, y : a \le x < M \land b \le y < N : |f.x - g.y|)$$

So

$$\begin{aligned} R: &\quad r = G.0.0 \\ P_0: &\quad r\ \mathbf{min}\ G.a.b = G.0.0 \\ P_1: &\quad 0 \le a \le M \land 0 \le b \le N \end{aligned}$$

The invariants can be established by

$$a, b, r = 0, 0, \infty$$

Furthermore

$$P_0 \land (a = M \lor b = N) \ \Rightarrow\ R$$

$$
\begin{array}{ll}
 & G.a.b \\
= & \quad \{ \text{ definition of } G \text{ } \} \\
 & (\mathbf{min}\ x, y : a \le x < M \wedge b \le y < N : |f.x - g.y|) \\
= & \quad \{ \text{ split off } x = a \text{ } \} \\
 & G.(a+1).b\ \mathbf{min}\ (\mathbf{min}\ y : b \le y < N : |f.a - g.y|) \\
= & \quad \{ \text{ assuming } g.b \ge f.a,\ g \text{ is increasing } \} \\
 & G.(a+1).b\ \mathbf{min}\ (\mathbf{min}\ y : b \le y < N : g.y - f.a) \\
= & \quad \{ \ g \text{ is increasing } \} \\
 & G.(a+1).b\ \mathbf{min}\ (g.b - f.a)
\end{array}
$$

Hence,

$$g.b \ge f.a \ \Rightarrow\ G.a.b = G.(a+1).b\ \mathbf{min}\ (g.b - f.a)$$

Symmetrically,

$$f.a \ge g.b \ \Rightarrow\ G.a.b = G.a.(b+1)\ \mathbf{min}\ (f.a - g.b)$$

Exercises

[Problem 9-1]: Derive an efficient program S satisfying the following specification:

$$
\begin{array}{l}
|[\\
\textbf{con } N : int \; \{N \geq 0\}; \textbf{var } r : bool; \\
S \\
\{r \equiv (\exists x, y : 0 \leq x \wedge 0 \leq y : N = 2^x + 3^y)\} \\
]|
\end{array}
$$

[Problem 9-2] Derive an efficient program S satisfying the following specification:

$$
\begin{array}{l}
|[\\
\textbf{con } M, N : \mathit{int} \ \{M \geq 0 \wedge N \geq 0\}; \\
f : \textbf{array } [0..M) \times [0..N) \textbf{ of } \mathrm{int}; \\
\{f \text{ is ascending in both arguments}\} \\
\textbf{var } r : \mathit{bool}; \\
S \\
\{\#i, j : 0 \leq i < M \wedge 0 \leq j < N : f.i.j = 0\}]|
\end{array}
$$

About the Final Report

- Solve 8 problems freely selected from the exercises in the lecture notes (see the problem set for a summary of all the problems).
- Submit your report to my post-box in the first floor of Engineering Building 6 no later than 17pm, June 16, 2006. Never forget writing your name, student identification number, and your department.
- You will be asked to present one of your solutions in class. Never be absent from the class.