Chapter 6: Searching

A Problem

Problem: search for the maximal natural number i for which $i^2 \leq N$.

$$
\begin{array}{l}
|[\\
\mathbf{con}\ N; \{N \geq 0\} \\
S \\
\{x = (\mathbf{max}\ i : 0 \leq i \wedge i^2 \leq N : i)\} \\
]|.
\end{array}
$$

Note the post-condition may also be formulated as

$$\{x = (\mathbf{min}\ i : \underline{0 \leq i} \wedge \underline{(i+1)^2 > N} : i)\}$$

Searching for i meeting a certain condition ...

Linear Searching

Consider the following programming problem.

$$
\begin{array}{l}
|[\\
\textbf{var } x : int; \\
\{(\exists i : 0 \leq i : b.i)\} \\
\textit{Linear Search} \\
\{x = (\textbf{min } i : 0 \leq i \wedge b.i : i)\} \\
]|.
\end{array}
$$

What is a possible invariant?

The post-condition can be rewritten as

$$R: \ 0 \leq x \wedge b.x \wedge (\forall i : 0 \leq i < x : \neg b.i)$$

So a possible invatiant is obtained by taking a conjunct:

$$P: \ 0 \leq x \wedge (\forall i : 0 \leq i < x : \neg b.i)$$

which is initialized by $x := 0$.

Investigation of $x := x + 1$ leads to

$$
\begin{array}{ll}
 & P(x := x + 1) \\
\equiv & \quad \{ \text{ definition of } P \ \} \\
 & 0 \leq x + 1 \wedge (\forall i : 0 \leq i < x + 1 : \neg b.i) \\
\Leftarrow & \quad \{ \text{ heading for } P \ \} \\
 & 0 \leq x \wedge (\forall i : 0 \leq i < x + 1 : \neg b.i) \\
\equiv & \quad \{ \text{ split off } i = x \ \} \\
 & 0 \leq x \wedge (\forall i : 0 \leq i < x : \neg b.i) \wedge \neg b.x \\
\equiv & \quad \{ \text{ definition of } P \ \} \\
 & P \wedge \neg b.x
\end{array}
$$

The final program for linear search:

$$
\begin{array}{l}
|[\\
\textbf{var } x : int; \\
\{(\exists i : 0 \le i : b.i)\} \\
x := 0; \\
\textbf{do } \neg b.x \to x := x + 1 \textbf{ od} \\
\{x = (\textbf{min } i : 0 \le i \wedge b.i : i)\} \\
]|.
\end{array}
$$

Does this program terminate? Why?

Remark:

If we know the upper bound, we may replace $x := x + 1$ by $x := x - 1$.

$$
\begin{array}{l}
|[\\
\textbf{con } N : \mathit{int};\ \ \textbf{var } x : \mathit{int}; \\
\{(\exists i : 0 \le i \le N : b.i)\} \\
x := N; \\
\textbf{do } \neg b.x \to x := x - 1 \textbf{ od} \\
\{x = (\textbf{min } i : 0 \le i \wedge b.i : i)\} \\
]|.
\end{array}
$$

Bounded Linear Search

The specification of the problem:

$$
\begin{array}{l}
|[\\
\mathbf{con}\ N : int\ \{N \geq 0\}; b : \mathbf{array}\ [0..N)\ \mathbf{of}\ \text{bool}; \\
\mathbf{var}\ x : int; \\
bounded\ linear\ search \\
\{x = (\mathbf{max}\ i : 0 \leq i \leq N \wedge (\forall j : 0 \leq j < i : \neg b.j) : i)\} \\
]|
\end{array}
$$

Could we use the invariant

$$0 \leq x \leq N \land (\forall i : 0 \leq i < x : \neg b.i)$$

to obtain the following program?

$$\begin{array}{l} x := 0 \\ \textbf{do } \neg b.x \land x \neq N \rightarrow x := x + 1 \textbf{ od} \end{array}$$

No, since N does not belong to the domain of b and $x = N$ is not excluded by the invariant.

If we could define $b.N$ as true, the post-condition would be written as

$$R: \; 0 \leq x \leq N \wedge (\forall i : 0 \leq i < x : \neg b.i) \wedge b.x$$

Our idea is to define as invariant:

$$\begin{aligned} P_0 &: \; 0 \leq x \leq N \wedge (\forall i : 0 \leq i < x : \neg b.i) \wedge b.y \\ P_1 &: \; x \leq y \leq N \end{aligned}$$

Then

$$P_0 \wedge P_1 \wedge x \neq y \rightarrow 0 \leq x < N,$$

so $b.x$ may occur in the statement of the repetition.

The final program:

$$
\begin{array}{l}
|[\\
\textbf{var } y : int; \\
x, y := 0, N; \\
\textbf{do } x \neq y \rightarrow \\
\qquad \textbf{if } \neg b.x \rightarrow x := x + 1 \\
\qquad [] \ b.x \rightarrow y := x \\
\qquad \textbf{fi} \\
\textbf{od} \\
]|
\end{array}
$$

Note that

$$P_0 \wedge P_1 \wedge x \neq y \wedge \neg b.x \Rightarrow (P_0 \wedge P_1)(x := x + 1)$$

$$P_0 \wedge P_1 \wedge x \neq y \wedge b.x \Rightarrow (P_0 \wedge P_1)(y := x)$$

Binary Search

Consider the problem:

$|[$

$\mathbf{con}\ N, A : int\ \{N \geq 1\};\ f : \mathbf{array}\ [0..N]\ \mathbf{of}\ int\ \{f.0 \leq A < f.N\};$

$\mathbf{var}\ x : int;$

binary search

$\{f.x \leq A < f.(x+1)\}$

$]|$

Could we do better than linear search?

From the post-condition:

$$R: \; f.x \leq A < f.(x+1)$$

we generalize $x+1$ to y, and define as invariants:

$$\begin{aligned} P_0: &\quad f.x \leq A < f.y \\ P_1: &\quad 0 \leq x < y \leq N \end{aligned}$$

which is established by $x, y := 0, N$. And we may choose guard as

$$x + 1 \neq y.$$

What is a suitable bound function?

A straightforward bound function is $y - x$. To decrease it, we may choose a h such that $x < h < y$, and both

$$x := h$$

and

$$y := h$$

will decrease $y - x$.

How to keep the invariants?

We investigate the effects of $x := h$ and $y := h$ on the invariants.

$$
\begin{array}{ll}
 & P_0(x := h) \\
\equiv & \quad \{ \text{ substitution } \} \\
 & f.h \leq A < f.y \\
\Leftarrow & \quad \{ \text{ definition of } P_0 \ \} \\
 & P_0 \wedge f.h \leq A
\end{array}
$$

$$
\begin{array}{ll}
 & P_0(y := h) \\
\equiv & \quad \{ \text{ substitution } \} \\
 & f.x \leq A < f.h \\
\Leftarrow & \quad \{ \text{ definition of } P_0 \ \} \\
 & P_0 \wedge A < f.h
\end{array}
$$

This leads to

$\{f.0 \leq A < f.N\}$
$|[$
$\textbf{var } y : \mathit{intl}$
$x, y := 0, N;$
$\{\mathit{invarinat{:}}\ P_0 \wedge P_1,\ \mathit{bound{:}}\ y - x\}$
$\textbf{do } x + 1 \neq y \rightarrow$
$\quad |[$
$\quad \textbf{var } h : \mathit{int};$
$\quad \underline{\mathit{establish}\ x < h < y};$
$\quad \textbf{if } f.h \leq A \rightarrow x := h$
$\quad [\,]\ A \leq f.h \rightarrow y := h$
$\quad \textbf{fi}$
$\quad]|$
$\textbf{od}$
$]|$

How to establish the underlined property?

So we obtain

$$
\begin{array}{l}
x, y := 0, N; \\
\mathbf{do}\ x + 1 \neq y \rightarrow \\
\qquad |[\\
\qquad \mathbf{var}\ h : int; \\
\qquad h = (x + y)\ \mathbf{div}\ 2; \\
\qquad \mathbf{if}\ f.h \leq A \rightarrow x := h \\
\qquad\ [\,]\ A \leq f.h \rightarrow y := h \\
\qquad \mathbf{fi} \\
\qquad]| \\
\mathbf{od}
\end{array}
$$

Two remarks:

- From the fact that $0 < h < N$, we know that $f.0$ and $f.N$ are not inspected.
- Precondition is only used for the initialization of x and y.

Square Root Problem

Consider writing a program for

$$
\begin{array}{l}
|[\\
\mathbf{con}\ N : \mathit{int}\ \{N \geq 0\}; \\
\mathbf{var}\ x : \mathit{int}; \\
\mathit{square\ root} \\
\{x^2 \leq N \wedge (x+1)^2 > N\}
\end{array}
$$

whose time complexity is $\mathcal{O}(\log N)$.

Idea: Let $f.x = x * x$, which satisfies $f.x \leq f.(x+1)$ and $0 \leq N \leq f.N$.
Applying the binary search yields:

$$
\begin{array}{l}
|[\\
\textbf{var } y : \mathit{int}; \\
x, y := 0, N+1; \\
\textbf{do } x+1 \neq y \rightarrow \\
\quad\quad |[\\
\quad\quad \textbf{var } h : \mathit{int}; \\
\quad\quad h = (x+y) \textbf{ div } 2; \\
\quad\quad \textbf{if } h*h \leq N \rightarrow x := h \\
\quad\quad \ [\,] \ N < h*h \rightarrow y := h \\
\quad\quad \textbf{fi} \\
\quad\quad]| \\
\textbf{od}
\end{array}
$$

The Binary Search Problem

Derive a program for

$$
\begin{array}{l}
|[\\
\textbf{con } N, A : \mathit{int}\ \{N \geq 1\};\ f : \textbf{array } [0..N] \textbf{ of } \mathit{int}\ \{f.0 \leq A < f.N\}; \\
\{\underline{(\forall i, j : 0 \leq i \leq j < N : f.i \leq f.j)}\} \\
\textbf{var } r : \mathit{bool}; \\
S \\
\{r \equiv (\underline{\exists i} : 0 \leq i < N : f.i = A)\} \\
]|
\end{array}
$$

Hint: Consider the post-condition as

$$R:\ 0 \leq x < N \wedge (f.x \leq A < f.(x+1) \vee A < f.0)$$

while virtually assuming $f.N = \infty$.

Program S for "the binary search":

$$
\begin{array}{l}
x, y := 0, N; \\
\mathbf{do}\ x + 1 \neq y \rightarrow \\
\qquad |[\\
\qquad \mathbf{var}\ h : int; \\
\qquad h = (x + y)\ \mathbf{div}\ 2; \\
\qquad \mathbf{if}\ f.h \leq A \rightarrow x := h \\
\qquad\ [\,]\ A \leq f.h \rightarrow y := h \\
\qquad \mathbf{fi} \\
\qquad]| \\
\mathbf{od}; \\
\underline{r := f.x = A}
\end{array}
$$

Exercises

[Problem 7-1] Derive a program for the following specification.

$$
\begin{array}{l}
|[\\
\textbf{con } N : int\ \{N \geq 0\}; \\
\textbf{var } r : bool; \\
S \\
\{r \equiv (\exists p :\ p \geq 0 : N = p^3)\} \\
]|
\end{array}
$$

[Problem 7-2] Derive for given N, $N \geq 0$, a program for the computation of the smallest integer x that satisfies $x^3 - 6x^2 + 9x \geq N$.

[Problem 7-3] Derive a program for the following specification.

$|[$

con $N : int\ \{N \geq 1\};\ A, B :$ **array** $[0..N]$ **of** int;

$\{A.0 \leq B.0 \land A.N \geq B.N\}$

var $r : int$;

S

$\{0 \leq r < N \land A.r \leq B.r \land A.(r+1) \geq B.(r+1)\}$

$]|$