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MUSICAL SCALES AND THE GENERALIZED CIRCLE OF FIFTHS

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This paper deals with the way the diatonic set (the white keys on the piano) is embedded in the chromatic scale (all the keys on the piano). To illustrate the problem, consider the chords CDF and EFA (the reader who happens to be temporarily without piano may find Fig. 1 helpful). If we ignore the black keys, these chords have the same structure; the second note is one key higher than the first, and the third note is two keys higher than the second. When actually played on the piano, the chords sound quite different, due to the embedding of the diatonic in the chromatic. From C to D is two semitones (a semitone is the distance between adjacent notes in the chromatic scale), and from D to F is three, whereas E to F is one and F to A is four. The problem

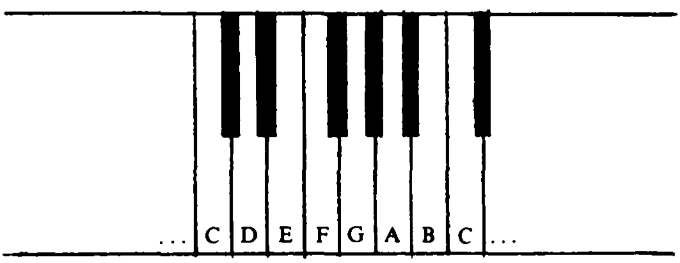

FIG. 1. Piano keyboard.

John Clough: Before coming to SUNY at Buffalo, I taught at the Oberlin College Conservatory of Music and in the School of Music at the University of Michigan. At all three places I have enjoyed the colleagueship of mathematicians who were willing to help me work through various problems in the application of mathematics to music: Edward Wong and Samuel Goldberg at Oberlin, Bernard Galler at Michigan, John Myhill and Gerald Myerson at Buffalo. Though trained only as a musician, in occasional flights of fancy I consider a second career in my first love—mathematics.

Gerald Myerson: I received my Ph.D. in Mathematics under the direction of Don Lewis at the University of Michigan in 1977. I have been on the faculty at the University of Buffalo, the University of British Columbia, and the University of Texas. I play two musical instruments: the phonograph and the cassette deck.

we pose here is, among all the chords of a given structure, how many "different" chords are there? (We remark that a different problem in the enumeration of chords was addressed recently in this MONTHLY [5].)

We first make our language more precise. The white-key set $\{A, B, C, D, E, F, G\}$ will be referred to as the *diatonic set*. (In fact, it is one of many diatonic sets similarly embedded in the chromatic scale; the results we obtain for the white-key set apply to all diatonic sets.) Each element of this set is a class of notes; thus, "C", for example, represents middle C, high C, low C, etc. A *chord* is a non-empty subset of the diatonic set. Juxtaposition will generally be used in preference to set notation; thus, CDF rather than $\{C, D, F\}$. A *line* is a finite non-empty sequence of distinct elements of the diatonic set. Hyphenation will be used for lines; thus, C–D–F is a line. The distinction between lines and chords is that between permutations and combinations; the chords CDF and DCF are identical, the lines C–D–F and D–C–F are distinct. Both concepts are significant in music theory. We will restrict our attention to lines, for which the mathematics is somewhat neater; we leave it to the reader to work out the theory of chords.

An *interval* is a two-note line. The *diatonic length* of an interval is the number of steps (in the diatonic) required to go from the first note of the line to the second—here, and always, we go "up" the scale. For example, the diatonic length of C–F is 3; that of F–C is 4. Notice that this is off by one from the traditional terminology of music theory, in which C–F is a "fourth". The line C–D–F can be thought of as the sequence of intervals C–D, D–F; similarly for any line.

Given two lines, we say they are related by a transposition in the diatonic if the sequences of diatonic lengths of their intervals are identical. This is easily seen to be an equivalence relation on the set of all lines. The equivalence classes will be called *genera*. For example, the genus containing C–D–F, which we denote $\langle$C–D–F$\rangle$, is

$$\{C\text{–}D\text{–}F, D\text{–}E\text{–}G, E\text{–}F\text{–}A, F\text{–}G\text{–}B, G\text{–}A\text{–}C, A\text{–}B\text{–}D, B\text{–}C\text{–}E\}.$$

Note that C–E–F is not in this genus; its sequence of lengths is 2, 1, which is not the same as 1, 2.

The *chromatic length* of an interval is the length of the interval measured in semitones; we shall use absolute value signs for chromatic length. Thus, $|C\text{–}F| = 5$. Given two lines in the same genus, we say they are related by a transposition in the chromatic if the sequences of chromatic lengths of their intervals are identical. This relation partitions each genus into subsets which we call *species*. For example, $\langle$C–D–F$\rangle$ partitions into the three species, $\{$C–D–F, D–E–G, G–A–C, A–B–D$\}$, $\{$E–F–A, B–C–E$\}$, and $\{$F–G–B$\}$. The lines in the first species all have intervals of chromatic lengths 2, 3; those of the second species, 1, 4; the lone line of the third species, 2, 4.

We see that the genus containing the *three*-note line C–D–F partitions into *three* species. This is not a coincidence, as the following theorem shows.

THEOREM 1. *Given any k, $1 \leqslant k \leqslant 7$, and any k-note line, the genus containing that line contains exactly k species.*

A brute-force proof can be carried out by examining all 13,699 lines, grouping them in their 1,957 genera, and counting the species within each genus. A more elegant proof, which leads to significant generalizations, is based on the construct known in music theory as the "circle of fifths." (See Fig. 2.)

The labels inside the upper semi-circle are diatonic lengths; the other labels are chromatic lengths. Measurement is clockwise.

Consider again the line C–D–F. The other lines in $\langle$C–D–F$\rangle$ are obtained by cycling around the upper semi-circle, since the diatonic distances on that semi-circle are constant. (See Fig. 3.)

The three species in this genus arise from the three possible locations of the short interval B–F (known in music theory as the "diminished fifth"); this interval can be included in the second interval of the three-note line (as in the first four diagrams), or in the first interval (next two

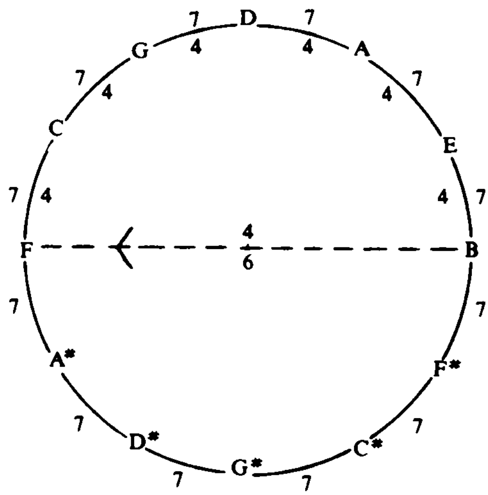

FIG. 2. The circle of fifths.

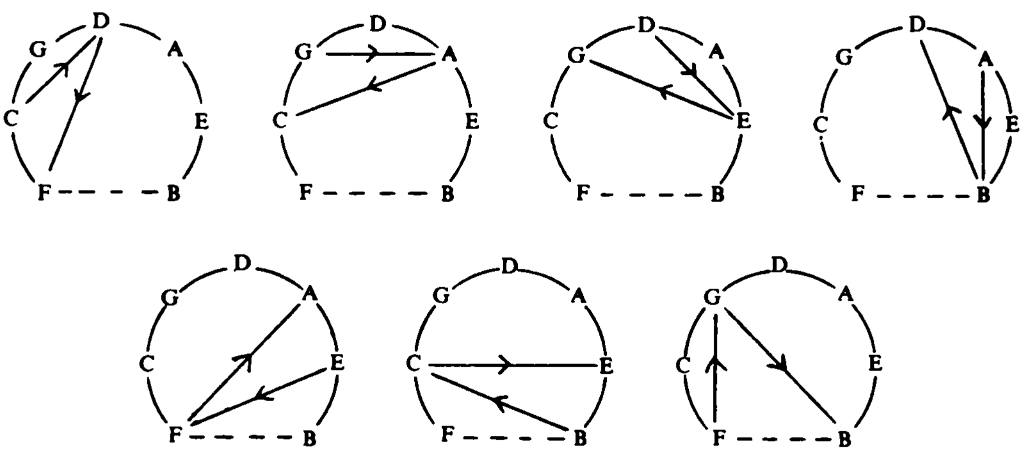

FIG. 3. The structure containing C–D–F.

diagrams), or in neither (last diagram). More generally, for any k, $1 \leqslant k \leqslant 7$, and any k-note line, the other lines in the genus are obtained by cycling through, and the species are distinguished by the position of B–F, for which there are k possibilities. This proves Theorem 1.

We can get somewhat more out of the circle of fifths. We have seen that the species in ⟨C–D–F⟩ contain 1, 2, and 4 lines. Now the distances from C to D, D to F, and F to C, when measured in fifths, are 2, 4, and 1, and the reason for the equality of these two sets of numbers is clear from Fig. 3. We state the general principle, which we call SM ("Structure yields Multiplicity"), in the following corollary to Theorem 1.

COROLLARY 1. *The numbers of lines in the species contained in the genus of a given line are given by the distances between each note in the line and the next note in the line, in the sense of clockwise travel around the circle of fifths. We include the distance from the last note to the first, and we measure distances in fifths.*

We leave it to the reader to define genus and species for chords, and to prove that given any k, $1 \leqslant k < 7$, and any k-note chord, the genus containing the chord contains exactly k species.

Myhill's property and its consequences. We now generalize. Instead of the usual chromatic set of twelve notes, we consider an abstract (but finite) chromatic set of c notes. By a *scale* we mean an ordered pair consisting of such a chromatic set together with a distinguished subset called the diatonic set. We let d be the cardinality of the diatonic set, and we label its elements $D_0, D_1, \ldots, D_{d-1}$. It is clear how we define line, interval, diatonic and chromatic lengths, genus, and species. We will make use later of the following simple property of lengths.

LEMMA 1. *Given k, $0 < k < d$, the sum of the chromatic lengths of the intervals of diatonic length k is ck.*

Proof. The number of semitones in the chromatic scale is, by definition, c, and each of these semitones is contained in precisely k of the intervals of diatonic length k.

A scale is said to have property CV ("Cardinality equals Variety") if for every k with $1 \leqslant k \leqslant d$, and for any k-note line, the genus containing that line contains exactly k species. This is the property of the usual scale asserted by Theorem 1. We wish to determine conditions under which a scale has property CV.

We define the *spectrum* of an interval I to be the set of all chromatic lengths of intervals in $\langle I \rangle$. If a scale has CV, then in particular every interval has a two-element spectrum. Our colleague John Myhill, to whom we are indebted for bringing us together to work on the problems of this paper, conjectured that the converse is true. We shall say that a scale has MP (Myhill's Property) if every interval has a two-element spectrum. We shall prove that MP implies CV by constructing, for any scale with MP, a "generalized circle of fifths."

Consider the spectrum of the interval $D_0 - D_1$. If this is a set of two consecutive integers, we say the scale is *rounded*. The usual scale is an example of a rounded scale. Given any scale in which the spectrum of $D_0 - D_1$ is a two-element set (in particular, any scale with MP), there is at least one corresponding rounded scale obtained by deleting non-diatonic notes, keeping equal chromatic lengths equal, as illustrated in Fig. 4. The deletion process preserves genera, since the diatonic set is unaffected. It preserves species, since species membership rests on equality of chromatic lengths. Thus, a rounded scale so obtained has MP (or CV) if and only if the original scale does. For the remainder of this section, we assume all scales are rounded.

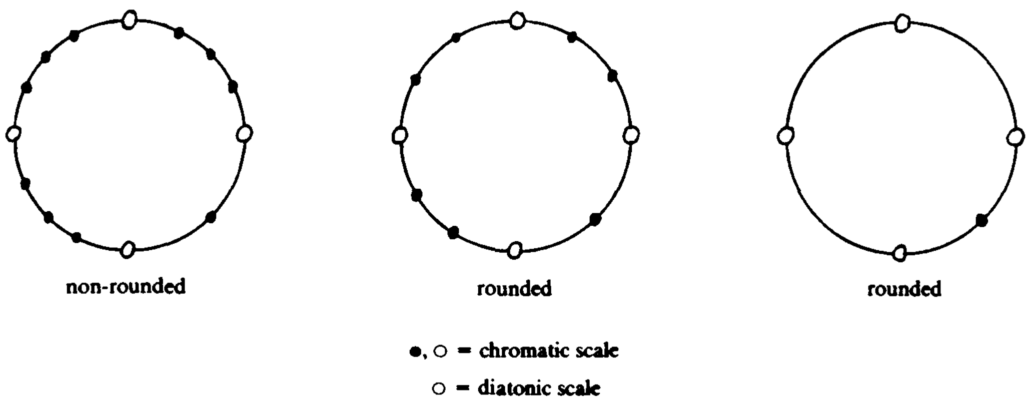

FIG. 4. Rounding a scale.

We say that a scale has CP (Consecutivity Property) if each interval has a spectrum consisting of consecutive integers.

LEMMA 2. *Every rounded scale has* CP.

Proof. Choose k, $0 < k < d$, and consider the interval $D_0 - D_k$. If the spectrum of this interval contains only one integer, the lemma is trivially true. Otherwise, there exist i and j with

$j - i = k$ and $|D_i - D_j| \neq |D_{i+1} - D_{j+1}|$, where the subscripts are to be reduced $(\text{mod } d)$, if necessary, to bring them into the range $\{0, 1, \ldots, d-1\}$. But

$$|D_{i+1} - D_{j+1}| - |D_i - D_j| = |D_j - D_{j+1}| - |D_i - D_{i+1}| = \pm 1,$$

since the scale is rounded. Thus no two consecutive terms in the sequence $|D_0 - D_k|$, $|D_1 - D_{k+1}|, \ldots, |D_{d-1} - D_{k-1}|$ differ by more than one, and the elements of the spectrum are consecutive.

LEMMA 3. *If a scale with parameters c and d has* MP, *then* $(c, d) = 1$.

Proof. Suppose to the contrary a scale has MP and $(c, d) = r > 1$. Consider the genus $\langle D_0 - D_{d/r} \rangle$. There are d intervals in this genus, of total chromatic length $(d/r)c$ (by Lemma 1). Thus we have d integers summing to $(c/r)d$, and c/r is an integer. These d integers cannot all be c/r—if they were, the spectrum of $D_0 - D_{d/r}$ would have only the one element c/r, violating MP. Thus at least one of the integers exceeds c/r, and at least one falls short. To satisfy CP then, c/r must be in the spectrum, but then the spectrum has at least three elements, violating MP.

The following number-theoretical lemma is crucial to the construction of the generalized circle of fifths.

LEMMA 4. *Let* $(c, d) = 1$; *then there exists an integer* c', $0 \leqslant c' < d$, *such that* $cc' \equiv -1$ $(\text{mod } d)$.

We omit the proof; a more general theorem is proved in the early chapters of nearly every introductory number theory textbook.

LEMMA 5. *Let a scale have* MP. *Let* c' *be as in the preceding lemma. Then with one exception the intervals of diatonic length* c' *all have chromatic length* $d' = (cc' + 1)/d$; *the exception has chromatic length* $d' - 1$.

Note that, for the usual scale, $c = 12$, $d = 7$, $c' = 4$, $d' = 7$, and the exceptional interval is the diminished fifth, B–F, with chromatic length $d' - 1 = 6$.

Proof. By Lemma 1, the sum of the chromatic lengths of the d intervals of diatonic length c' is cc'. By definition, $cc' = dd' - 1$, so we have d integers summing to $dd' - 1$. By MP there are exactly two distinct integers among these d integers, and by Lemma 2 they are consecutive; hence, $d - 1$ of these integers are d', and the other is $d' - 1$.

We can now label the diatonic set in such a way that

$$|D_0 - D_{c'}| = |D_{c'} - D_{2c'}| = \cdots = |D_{(d-2)c'} - D_{(d-1)c'}| = d', \quad |D_{(d-1)c'} - D_{dc'}| = d' - 1,$$

the subscripts being read $(\text{mod } d)$. Thus we have constructed a generalized circle of fifths (Fig. 5).

THEOREM 2. MP *implies* CV.

Proof. The argument from the circle of fifths given for Theorem 1 above goes over to the generalized circle of fifths.

Property SM, of Corollary 1, is also relevant to the more abstract setting of this section. With the understanding that "fifths" is interpreted as "generalized fifths," we have

COROLLARY 2. MP *implies* SM.

Existence and uniqueness of scales with CV and given parameters. We now turn to the construction of scales with CV. Given parameters c, d with $(c, d) = 1$ we show that there exists a scale with CV with those parameters; moreover, it is essentially unique. We consider unicity first.

THEOREM 3. *Let S and* S^* *be scales with* CV *and with parameters c and d. Let their chromatic*

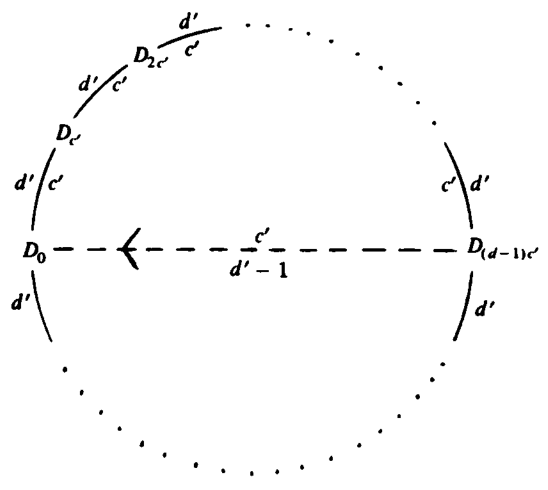

FIG. 5. Generalized circle of fifths.

sets be

$$C = \{C_0, C_1, \ldots, C_{c-1}\} \quad \textit{and} \quad C^* = \{C_0^*, C_1^*, \ldots, C_{c-1}^*\},$$

respectively, and let their diatonic sets be D and D^*, respectively. Then there exists an integer j such that C_i is in D if and only if C_{i+j}^* is in D^*; here, the subscripts are to be interpreted (modulo c).

Proof. Since the scales share parameters c and d, they also share c' and d'. By the construction of the generalized circle of fifths, there are notes C_k in C and C_h^* in C^* such that

$$D = \{C_k, C_{k+d'}, \ldots, C_{k+(d-1)d'}\} \quad \text{and} \quad D^* = \{C_h^*, C_{h+d'}^*, \ldots, C_{h+(d-1)d'}^*\}.$$

Now simply let $j = h - k$.

Concerning the existence of scales with CV and given parameters, we first show how to construct the usual scale. Here $c = 12$ and $d = 7$. Write down the multiples of $\frac{12}{7}$: $0, 1\frac{5}{7}, 3\frac{3}{7}, 5\frac{1}{7}, 6\frac{6}{7}, 8\frac{4}{7}, 10\frac{2}{7}, 12, \ldots$. Then erase the fractions, leaving $0, 1, 3, 5, 6, 8, 10, 12, \ldots$. Interpret this sequence as the positions of the white keys, and the omitted integers $2, 4, 7, 9, 11, \ldots$ as the positions of the black keys, and you obtain a scale with CV; if you identify position 0 with the note B, you recover the usual (C-major) scale:

0	1	2	3	4	5	6	7	8	9	10	11	12	...
B	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B	...

FIG. 6

We will prove that this procedure works quite generally. First we need to quote another well-known lemma from elementary number theory.

LEMMA 6. *If r, s, t are integers, r divides st, and $(r, s) = 1$, then r divides t.*

THEOREM 4. *Given c and d with $(c, d) = 1$, let $a_k = \left[\frac{kc}{d}\right]$, $k = 0, \pm 1, \pm 2, \ldots$. Then the integers a_k are the positions of the notes of the diatonic set in a scale with* CV *with parameters c and d.*

Proof. It suffices to show that the scale so constructed has MP, that is, for every j, $1 \leqslant j < d$, the set $\{a_{k+j} - a_k : k = 0, \pm 1, \pm 2, \ldots\}$ has cardinality two. Since, for all x, $x - 1 < [x] \leqslant x$, and since $a_{k+j} - a_k = \left[\frac{(k+j)c}{d}\right] - \left[\frac{kc}{d}\right]$, we have

$$\frac{(k+j)c}{d} - 1 - \frac{kc}{d} < a_{k+j} - a_k < \frac{(k+j)c}{d} - \frac{kc}{d} + 1.$$

Thus for fixed j there is an open interval (in the mathematical sense) of length 2 containing the spectrum of the interval (in the musical sense) of diatonic length j. Such a mathematical interval contains at most two integers, so the spectrum is at most a two-element set. Now suppose the spectrum is a one-element set. Then there is an integer, call it e, such that, for all k, $a_{k+j} - a_k = e$. Then

$$de = \sum_{n=1}^{d} (a_{k+nj} - a_{k+(n-1)j}) = a_{k+dj} - a_k = \left[\frac{(k+dj)c}{d}\right] - \left[\frac{kc}{d}\right] = cj,$$

so d divides cj. By Lemma 6, d divides j. But $1 \leqslant j < d$ by assumption, yielding a contradiction. Thus, the spectrum is a two-element set, so the scale has MP.

Concluding remarks. While the connection between mathematics and music goes back almost as far as the beginnings of the two disciplines themselves, the features of musical structure studied in this paper have not heretofore been subjected to rigorous analysis. (The reader will, however, find the contents of [1], [2], [3], and [4] to be relevant to our discussions.) We hope that our results will stimulate further research on mathematics and music theory.

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MORE TREES AND POWER SUMS

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1. Introduction. A recent article in the MONTHLY by A. H. Stone [8] gives an engaging presentation of the formula

$$n^{n-3} = \frac{1}{2}\sum_{r=1}^{n-1}\binom{n-2}{r-1} r^{r-2}(n-r)^{n-r-2} \quad \text{for } n \geq 2 \tag{1}$$

as a plausible generalization of two well-known equations, $5^2 = 4^2 + 3^2$ and $6^3 = 5^3 + 4^3 + 3^3$. The fundamental combinatorial technique Stone uses to prove (1) is that of counting a particular set in two ways, equating the results. Here we use this same technique of double counting to give

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James E. Simpson: I am a native of Chicago, with a Ph. D. from Yale University (1961). My research area was originally in analysis, but after a year of close association with Professor M. Behzad in 1972 I have floated gradually into combinatorics and graph theory. My major project this year is to finish an introductory text in discrete mathematics.