probability puzzle, expected payoff for game

This was a question at a programming contest that finished recently at interviewstreet:

> Alice and Bob play a game. The operations at round i ($i \geq 1$) is as follows:
>
> - Alice pays Bob $2i-1$ dollars,
> - Alice tosses a biased coin,
> - If the result of the coin was heads for k consecutive rounds, the game stops, otherwise the game continues.
>
> Given k and the probablity p that the outcome of a toss is heads, your program should find the expected number of dollars Alice pays Bob, and also the expected number of rounds played.

I had gotten the first part of the problem, i.e the expected number of rounds of the game. I got it with the following code

```
if __name__ == '__main__':
    t = int(raw_input())
    while t :
        t -= 1
        temp = str(raw_input())
        p,k = temp.split(' ')
        p = float(p)
        k = int(k)
        num = k * (p**k)
        den = 1
        q = 1.0 - p
        for N in range(1,k+1):
            den = den - ((p**(N-1))*q)
            num = num + (N*(p**(N-1))*q)
            #print (N*(q**N))

        print int(num/den)
```

But the second part of the problem is still puzzling me, i.e the expected number of dollars Alice pays bob. How can expected payoff be calculated?

(probability)

edited May 5 '12 at 21:18
MJD
20.2k ● 4 ■ 32 ▲ 105

asked May 5 '12 at 18:23
rohanag
3 ▲ 1

1 Answer

Call N the number of rounds played, then the number of dollars paid by Alice is N^2. The usual one-step recursion yields

$$\mathrm{E}_i(s^N) = s(p\mathrm{E}_{i+1}(s^N) + (1-p)\mathrm{E}_0(s^N)),$$

for every $0 \leqslant i \leqslant k-1$, with $\mathrm{E}_k(s^N) = 1$. One is after $\mathrm{E}_0(N)$ and $\mathrm{E}_0(N^2)$ and the linear system above yields $\mathrm{E}_0(s^N) = u(s)$ with

$$u(s) = \frac{(1-ps)(ps)^k}{1-s+(1-p)s(ps)^k} = 1-(1-s)\,\frac{1-(ps)^k}{1-s+(1-p)s(ps)^k},$$

hence $u'(1) = \mathrm{E}_0(N)$, that is,

$$\mathrm{E}_0(N) = \frac{1-p^k}{(1-p)p^k}.$$

Similarly, $u''(1) = \mathrm{E}_0(N(N-1))$. This yields a slightly more complicated, but explicit, formula for $\mathrm{E}_0(N^2)$.

edited May 7 '12 at 16:10

answered May 6 '12 at 13:53

Did
100k ● 9 ■ 72 ▲ 175

Thank you for your answer. I don't quite understand, some of the notation, what is s, and how did you calculate u(s)? – rohanag May 9 '12 at 11:25

The argument s can be any number such that $|s|\leqslant 1$. One computes $u(s)$ by considering, for each such s, the affine system the vector $(\mathrm{E}_i(s^N))_{0\leqslant i\leqslant k}$ solves. – Did May 9 '12 at 13:23