INTEREST RATE CURVE CONSTRUCTION UNDER COLLATERALIZATION - THEORY AND PRACTICE

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Abstract. Technical documents recording theory and practices of interest rate curve construction, under collateralization.

1. Discounting Under Collateralization

1.1. Notations.

d, f, c	Currencies d, f, c
r^d, r^f, r^c	Instantaneous risk-free rates for currency d, f, c
e^d, e^f, e^c	Instantaneous standard collateral rates for currency d, f, c
$e^{d\|f}$	Instantaneous collateral rates for d-payment under f-collateral. In particular we have $e^{f\|f} = e^f$
$D(t, T; x)$	Discount factor from t to T, with (stochastic) rate $x(t)$: $D(t, T; x) = exp(-\int_t^T x(s)ds)$
$V^d(T), V^f(T)$	The d-currency (f-currency) payoff at time T
$V^{d\|d}(t), V^{d\|f}(T)$	The d-currency value of the payoff $V^d(T)$ at time t, under d-collateral(f-collateral)
$\chi^{fd}(t)$	Spot FX from currency f to currency d
$X^{fd\|d}(t; T)$	FX Forward at time t, to convert currency f to currency d at T, under the assumption that the corresponding FX Swap is collateralized against d currency with standard collateral rate e^d
$E_t^d[\cdot], E_t^f[\cdot]$	t - risk neutral expectation for currency d (respectively currency f)
$P(t, T; e^{d\|f})$	$P(t, T; e^{d\|f}) = E_t^d[D(t, T; e^{d\|f})]$
$\mu^{fd\|d}(t)$	The drift of FX spot $\chi^{fd}(t)$ under $E_t^d(\cdot)$, i.e. $\mu^{fd\|d}(t) = \frac{E_t^d[\chi^{fd}(t)]}{dt}$

1.2. Pricing Formula under Collateralization.

1.2.1. *Pricing with Effective Collateral Rate.*

In the following we would assume that the price of a d-payoff under c-collateral is given by

$$V^{d|c}(t) = E_t^d[D(t, T; e^{d|c})V(T)] \tag{1.1}$$

where $e^{d|c}$ is the effective collateral rate. In particular under $e^{d|d} = e^d$ and we have

$$V^{d|d}(t) = E_t^d[D(t, T; e^{d|d})V(T)] = E_t^d[D(t, T; e^d)V(T)] \tag{1.2}$$

We will then show the relationship between different effective collateral rates.

1.2.2. ***FX Forward under Collateralization.***

Lets consider a FX Swap in which the investor borrows from the counterparty in f currency, while lending in d currency to the same counterparty. We assume that the trade is collateralized against d currency with collateral rate $e^d(t)$. At the contract inception t, investor borrows one unit of f currency and lend out an equivalent amount of d currency, while at the maturity T one unit of f currency is exchanged back against a given quantity $X^{fd|d}(t;T)$. Then we have:

$$V^{d|d}_{FXSwap}(t) = E^d_t[(\chi^{fd}(T) - X^{fd|d}(t;T))D(t,T;e^{d|d})] \tag{1.3}$$

Using the usual assumption that $V^{d|d}_{FXSwap}(t) = 0$ at contract inception, we then get:

$$X^{fd|d}(t,T) = \frac{E^d_t[\chi^{fd}(T)D(t,T;e^{d|d})]}{E^d_t[D(t,T;e^{d|d})]} \tag{1.4}$$

Similarly consider a FX Swap in which the investor borrows from the counterparty in f currency, while lending in d currency to the same counterparty, and we assume the trade is collateralized against c currency. Then we have

$$V^{d|c}_{FXSwap}(t) = E^d_t[(\chi^{fd}(T) - X^{fd|c}(t;T))D(t,T;e^{d|c})] \tag{1.5}$$

With the usual assumption that $V^{d|c}_{FXSwap}(t) = 0$ at the contract inception, we then get

$$X^{fd|c}(t,T) = \frac{E^d_t[\chi^{fd}(T)D(t,T;e^{d|c})]}{E^d_t[D(t,T;e^{d|c})]} \tag{1.6}$$

1.2.3. ***Collateralized Foreign Measure.***

Consider a d-payoff under f-collateralization, then we should have:

$$V^{d|f}(0) = E^d_0[V^d(T)D(0,T;e^{d|f})] \tag{1.7}$$

On the other hand, remembering the fact that $e^{f|f} = e^f$, we should also have:

$$\frac{V^{d|f}(0)}{\chi^{fd}(0)} = E^f_0\left[\frac{V^d(T)}{\chi^{fd}(T)}D(0,T;e^f)\right] = E^f_0\left[\frac{V^d(T)}{\chi^{fd}(T)}D(0,T;e^{f|f})\right] \tag{1.8}$$

From these, we can derive the following:

$$\begin{aligned}
V^{d|f}(0) &= \chi^{fd}(0)E^f_0\left[\frac{V^d(T)}{\chi^{fd}(T)}D(0,T;e^{f|f})\right] \\
&= \chi^{fd}(0)E^d_0\left[\frac{V^d(T)}{\chi^{fd}(T)}D(0,T;e^{f|f})\frac{dQ^f}{dQ^d}\right] \\
&= E^d_0\left[V^d(T)D(0,T;e^{d|f})\frac{\chi^{fd}(0)}{\chi^{fd}(T)}D(0,T;e^{f|f} - e^{d|f})\frac{dQ^f}{dQ^d}\right] \\
&= E^d_0\left[V^d(T)D(0,T;e^{d|f})\right]
\end{aligned}$$

Therefore we have

$$\frac{\chi^{fd}(0)}{\chi^{fd}(T)}D(0,T;e^{f|f} - e^{d|f})\frac{dQ^f}{dQ^d} = 1 \tag{1.9}$$

which then gives the expresson for the Radon-Nikodym derivative

$$\frac{dQ^f}{dQ^d} = D(0,T;e^{d|f} - e^{f|f})\frac{\chi^{fd}(T)}{\chi^{fd}(0)} \tag{1.10}$$

Note that $\frac{dQ^f}{dQ^d}$ is a martingale under Q^d, we should have

$$\mu^{fd|d}(t) = e^{d|f}(t) - e^{f|f}(t) \tag{1.11}$$

and hence we have

$$\frac{dQ^f}{dQ^d} = D(0,T;e^{d|f} - e^{f|f})\frac{\chi^{fd}(T)}{\chi^{fd}(0)} = D(0,T;\mu^{fd|d})\frac{\chi^{fd}(T)}{\chi^{fd}(0)} \tag{1.12}$$

1.2.4. ***FX Drift and Collateral Basis***.

Finally we consider a f-payoff under c-collateralization. We have

$$V^{f|c}(t) = E_t^f[V(T)D(t,T;e^{f|c})] \tag{1.13}$$

and at the same time also

$$V^{f|c}(t)\chi^{fd}(t) = E_t^d[\chi^{fd}(T)V(T)D(t,T;e^{d|c})] \tag{1.14}$$

Using the formula for $\frac{dQ^f}{dQ^d}$ we can then derive that

$$e^{f|c} = e^{d|c} - \mu^{fd} \tag{1.15}$$

and rearranging to get

$$\mu^{fd} = e^{d|c} - e^{f|c} \tag{1.16}$$

The above discussion shows the following results:

Proposition 1.
The FX drift $\mu^{fd|d}(t) = \frac{E_t^d[\chi^{fd}(t)]}{dt} = e^{d|c} - e^{f|c}$ *is independent of the collateral currency* c.

Using the result of the above proposition, we then have:

$$\mu^{fd|d}(t) = e^{d|c} - e^{f|c} = e^{d|g} - e^{f|g} \tag{1.17}$$

and rearranging we have

$$e^{d|c} - e^{d|g} = e^{f|c} - e^{f|g} \tag{1.18}$$

Hence we have the following results:

Proposition 2. *We can define the collateral basis as*

$$s^{c,g} = e^{d|c} - e^{d|g} = e^{f|c} - e^{f|g} \tag{1.19}$$

which is independent of the payment currency.

1.2.5. ***Collateralized T-Forward Measure***.
Using the effective collateral rate $e^{d|f}$ we can define the collateralized T-forward measure:

$$\frac{dQ^{d|f,T}}{dQ^d} = \frac{D(0,T;e^{d|d})P(T,T;e^{d|f})}{P(0,T;e^{d|f})} = \frac{D(0,T;e^{d|d})}{P(0,T;e^{d|f})} \tag{1.20}$$

and the last equality is because we have $P(T,T;e^{d|f}) = 1$. Also we should have

$$E_t^d\left[\frac{dQ^{d|f,T}}{dQ^d}\right] = \frac{D(0,t;e^{d|d})P(t,T;e^{d|f})}{P(0,T;e^{d|f})} \tag{1.21}$$

With this definition of collateralized T-Forward measure, we can have the following general formula:

$$\begin{aligned}
V^{d|f}(t) &= E^d\left[V^d(T)D(t,T;e^{d|f})\right] \\
&= E^d\left[V^d(T)D(t,T;e^{d|f})\frac{dQ^{d|c,T}}{dQ^d}\Big/\frac{dQ^{d|c,T}}{dQ^d}\right] \\
&= E_t^d\left[\frac{dQ^{d|c,T}}{dQ^d}\right]E_t^{d|c,T}\left[\frac{V^d(T)D(t,T;e^{d|f})}{\frac{dQ^{d|c,T}}{dQ^d}}\right] \\
&= \frac{D(0,t;e^{d|d})P(t,T;e^{d|c})}{P(0,T;e^{d|c})}E_t^{d|c,T}\left[V^d(T)D(t,T;e^{d|f})\frac{P(0,T;e^{d|c})}{D(0,T;e^{d|d})}\right] \\
&= P(t,T;e^{d|c})E_t^{d|c,T}\left[\frac{V^d(T)D(t,T;e^{d|f})}{D(t,T;e^{d|d})}\right] \\
&= P(t,T;e^{d|c})E_t^{d|c,T}\left[V^d(T)D(t,T;e^{d|f}-e^{d|d})\right] \\
&= P(t,T;e^{d|c})E_t^{d|c,T}\left[V^d(T)D(t,T;s^{f,d})\right]
\end{aligned}$$

Proposition 3. *Changing to T-Forward Measure: Consider a general d-currency payoff $V^d(T)$, then we have:*

$$\begin{aligned}
V^{d|f}(t) &= E^d\left[V^d(T)D(t,T;e^{d|f})\right] \\
&= P(t,T;e^{d|c})E_t^{d|c,T}\left[V^d(T)D(t,T;e^{d|f}-e^{d|d})\right] \qquad (1.22)\\
&= P(t,T;e^{d|d})E_t^{d|d,T}\left[V^d(T)D(t,T;e^{d|f}-e^{d|d})\right] \qquad (1.23)
\end{aligned}$$

And in particular we have:

$$V^{d|d}(t) = E_t^{d|d,T}\left[V^d(T)\right] \tag{1.24}$$

Moreover, with this definition of collateralized T-Forward measure, we have the following result:

Proposition 4. *We can then express the FX forward from Equation(1.4) and Equation(1.6) as follows:*

$$X^{fd|d}(t,T) = \frac{E_t^d[\chi^{fd}(T)D(t,T;e^{d|d})]}{E_t^d[D(t,T;e^{d|d})]} = E_t^{d|d,T}\left[\chi^{fd}(T)\right] \tag{1.25}$$

$$X^{fd|c}(t,T) = \frac{E_t^d[\chi^{fd}(T)D(t,T;e^{d|c})]}{E_t^d[D(t,T;e^{d|c})]} = E_t^{d|c,T}\left[\chi^{fd}(T)\right] \tag{1.26}$$

1.2.6. ***FX Forward Invariance under Collateral.***
Lets consider the FX Forward $X^{fd|c}(t;T)$ under c-collateral, which can be written as

$$X^{fd|c}(t;T) = \frac{E_t^d[\chi^{fd}(T)D(t,T;e^{d|c})]}{E_t^d[D(t,T;e^{d|c})]} \tag{1.27}$$

Using the Radon-Nikodym derivative $\frac{dQ^{d|c,T}}{dQ^d}$ we can then derive that

$$\begin{aligned}
E_t^d[\chi^{fd}(T)D(t,T;e^{d|c})] &= P(t,T;e^{d|d})E_t^{d|d,T}\left[\chi^{fd}(T)D(t,T;s^{c,d})\right] \\
&= P(t,T;e^{d|d})\left\{Cov_t^{d|d,T}\left[\chi^{fd}(T),D(t,T;s^{c,d})\right] + E^{d|d,T}\left[\chi^{fd}(T)\right]E^{d|d,T}\left[D(t,T;s^{c,d})\right]\right\}
\end{aligned}$$

At the same time we have

$$E_t^d\left[D(t,T;e^{d|c})\right] = P(t,T;e^{d|d})E_t^{d|d,T}\left[D(t,T;s^{c,d})\right] \tag{1.28}$$

By using the fact $P(t,T;e^{d|d} = E_t^d[D(t,T;e^{d|d})]$, we then have

$$\begin{aligned}
X^{fd|c}(t;T) &= \frac{E_t^d[\chi^{fd}(T)D(t,T;e^{d|c})]}{E_t^d[D(t,T;e^{d|c})]} \\
&= \frac{P(t,T;e^{d|d})\left\{Cov_t^{d|d,T}\left[\chi^{fd}(T),D(t,T;s^{c,d})\right] + E^{d|d,T}\left[\chi^{fd}(T)\right]E^{d|d,T}\left[D(t,T;s^{c,d})\right]\right\}}{P(t,T;e^{d|d})E_t^{d|d,T}\left[D(t,T;s^{c,d})\right]} \\
&= E^{d|d,T}\left[\chi^{fd}(T)\right] + Convexity \\
&= X^{fd|d}(t,T) + Convexity
\end{aligned}$$

Where the convexity term is given by:

$$Convexity = \frac{Cov_t^{d|d,T}\left[\chi^{fd}(T),D(t,T;s^{c,d})\right]}{E_t^{d|d,T}\left[D(t,T;s^{c,d})\right]} \tag{1.29}$$

These are summarized as follows:

Proposition 5. *The FX Forward $X^{fd|c}(t,T)$ under the c-collateral can be written as*

$$X^{fd|c}(t,T) = X^{fd|d}(t,T) + Convexity \tag{1.30}$$

where the convexity correction is given by:

$$Convexity = \frac{Cov_t^{d|d,T}\left[\chi^{fd}(T),D(t,T;s^{c,d})\right]}{E_t^{d|d,T}\left[D(t,T;s^{c,d})\right]} \tag{1.31}$$

Moreover, if the collateral basis $s^{(c,d)}$ is independent of the FX spot $\chi^{fd}(T)$ then the FX forward is invariant under collaterals. In other words we have

$$X^{fd|d}(t,T) = X^{fd|d}(t,T) \tag{1.32}$$

if $s^{(c,d)}$ is independent of the FX spot $\chi^{fd}(T)$.

Sectioning commands. The first one is the

`\section{The Most Important Features}`

command. Below you shall find examples for further sectioning commands:

1.3. **Subsection.** Subsection text.

1.3.1. *Subsubsection.* Subsubsection text.
Paragraph. Paragraph text.

Subparagraph. Subparagraph text.

Select a part of the text then click on the button Emphasize (H!), or Bold (Fs), or Italic (Kt), or Slanted (Kt) to typeset *Emphasize*, **Bold**, *Italics*, *Slanted* texts.

You can also typeset Roman, Sans Serif, SMALL CAPS, and `Typewriter` texts.

You can also apply the special, mathematics only commands $\mathbb{BLACKBOARD}$ $\mathbb{BOLD}$, $\mathcal{CALLIGRAPHIC}$, and $\mathfrak{fraktur}$. Note that blackboard bold and calligraphic are correct only when applied to uppercase letters A through Z.

You can apply the size tags – Format menu, Font size submenu – tiny, scriptsize, footnotesize, small, normalsize, large, Large, LARGE, huge and Huge.

You can use the `\begin{quote} etc. \end{quote}` environment for typesetting short quotations. Select the text then click on Insert, Quotations, Short Quotations:

> The buck stops here. *Harry Truman*
>
> Ask not what your country can do for you; ask what you can do for your country. *John F Kennedy*
>
> I am not a crook. *Richard Nixon*
>
> I did not have sexual relations with that woman, Miss Lewinsky. *Bill Clinton*

The Quotation environment is used for quotations of more than one paragraph. Following is the beginning of *The Jungle Books* by Rudyard Kipling. (You should select the text first then click on Insert, Quotations, Quotation):

> It was seven o'clock of a very warm evening in the Seeonee Hills when Father Wolf woke up from his day's rest, scratched himself, yawned and spread out his paws one after the other to get rid of sleepy feeling in their tips. Mother Wolf lay with her big gray nose dropped across her four tumbling, squealing cubs, and the moon shone into the mouth of the cave where they all lived. "*Augrh*" said Father Wolf, "it is time to hunt again." And he was going to spring down hill when a little shadow with a bushy tail crossed the threshold and whined: "Good luck go with you, O Chief of the Wolves; and good luck and strong white teeth go with the noble children, that they may never forget the hungry in this world."
>
> It was the jackal—Tabaqui the Dish-licker—and the wolves of India despise Tabaqui because he runs about making mischief, and telling tales, and eating rags and pieces of leather from the village rubbish-heaps. But they are afraid of him too, because Tabaqui, more than any one else in the jungle, is apt to go mad, and then he forgets that he was afraid of anyone, and runs through the forest biting everything in his way.

Use the Verbatim environment if you want LaTeX to preserve spacing, perhaps when including a fragment from a program such as:

```
#include <iostream>         // < > is used for standard libraries.
void main(void)             // ''main'' method always called first.
{
 cout << ''This is a message.'';
                            // Send to output stream.
}
```

(After selecting the text click on Insert, Code Environments, Code.)

1.4. **Mathematics and Text.** It holds [1] the following

Theorem 1. *(The Currant minimax principle.) Let T be completely continuous selfadjoint operator in a Hilbert space H. Let n be an arbitrary integer and let $u_1, \dots, u_{n-1}$ be an arbitrary system of $n-1$ linearly independent elements of H. Denote*

$$\max_{\substack{v \in H, v \neq 0 \\ (v,u_1)=0,\dots,(v,u_n)=0}} \frac{(Tv,v)}{(v,v)} = m(u_1,\dots,u_{n-1}) \tag{1.33}$$

Then the n-th eigenvalue of T is equal to the minimum of these maxima, when minimizing over all linearly independent systems $u_1, \dots u_{n-1}$ in H,

$$\mu_n = \min_{u_1,\dots,u_{n-1} \in H} m(u_1,\dots,u_{n-1}) \tag{1.34}$$

The above equations are automatically numbered as equation (1.33) and (1.34).

1.5. **List Environments.** You can create numbered, bulleted, and description lists (Use the Itemization or Enumeration buttons, or click on the Insert menu then chose an item from the Enumeration submenu):

(1) List item 1
(2) List item 2
    (a) A list item under a list item.
        However, the typeset style for this level is different.
    (b) Just another list item under a list item.
        (i) Third level list item under a list item.
            (A) Fourth and final level of list items allowed.

- Bullet item 1
- Bullet item 2
    – Second level bullet item.
        ∗ Third level bullet item.
            · Fourth (and final) level bullet item.

Description List: Each description list item has a term followed by the description of that term. Double click the term box to enter the term, or to change it.

Bunyip: Mythical beast of Australian Aboriginal legends.

1.6. **Theorem-like Environments.** The following theorem-like environments (in alphabetical order) are available in this style.

Acknowledgement 1. *This is an acknowledgement*

Algorithm 1. *This is an algorithm*

Axiom 1. *This is an axiom*

Case 1. *This is a case*

Claim 1. *This is a claim*

Conclusion 1. *This is a conclusion*

Condition 1. *This is a condition*

Conjecture 1. *This is a conjecture*

Corollary 1. *This is a corollary*

Criterion 1. *This is a criterion*

Definition 1. *This is a definition*

Example 1. *This is an example*

Exercise 1. *This is an exercise*

Lemma 1. *This is a lemma*

Proof. This is the proof of the lemma. □

Notation 1. *This is notation*

Problem 1. *This is a problem*

Proposition 6. *This is a proposition*

Remark 1. *This is a remark*

Solution 1. *This is a solution*

Summary 1. *This is a summary*

Theorem 2. *This is a theorem*

Proof of the Main Theorem. This is the proof. □

This text is a sample for a short bibliography. You can cite a book by making use of the command `\cite{KarelRektorys}`: [1]. Papers can be cited similarly: [2]. If you want multiple citations to appear in a single set of square brackets you must type all of the citation keys inside a single citation, separating each with a comma. Here is an example: [2, 3, 4].

References

[1] Rektorys, K., *Variational methods in Mathematics, Science and Engineering*, D. Reidel Publishing Company, Dordrecht-Hollanf/Boston-U.S.A., 2th edition, 1975

[2] Bertóti, E.: *On mixed variational formulation of linear elasticity using nonsymmetric stresses and displacements*, International Journal for Numerical Methods in Engineering., **42**, (1997), 561-578.

[3] Szeidl, G.: *Boundary integral equations for plane problems in terms of stress functions of order one*, Journal of Computational and Applied Mechanics, **2**(2), (2001), 237-261.

[4] Carlson D. E.: *On Günther's stress functions for couple stresses*, Quart. Appl. Math., **25**, (1967), 139-146.