Conjugate Bayesian Online Ranking for Thurstone-Mosteller Model via Unified Skew-Normal Distributions

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Abstract

Pairwise ranking methods are widely studied for inferring rankings from pairwise comparison data, yet most are tailored for static settings, neglecting practical constraints of data collection. In response, active sampling strategies within the Bayesian framework have emerged to optimize pairwise data labeling. However, existing approaches rely on approximated posteriors, leading to substantial estimation errors and inefficient sampling. Addressing this gap, we present a novel approach utilizing the closed-form posterior derived from the unified skew-normal (SUN) distribution. We observe that the posterior under the Thurstone-Mosteller (TM) model, a classical ranking model, follows the SUN distribution with Gaussian priors. Demonstrating under mild assumptions, this closed-form posterior yields an optimal ranking based on the expected Kendall's τ correlation coefficient. Moreover, we establish its convergence to the true ranking under certain conditions. Benefiting from closed-form posterior, we integrate our framework into the knowledge gradient policy for active sampling. We address computational challenges using Mean Field Variational Inference. Extensive experimental validation on simulated and real-world datasets showcases the efficacy and efficiency of our approach.

Keywords: Thurstone-Mosteller model, Bayesian conjugate framework, active ranking, pairwise comparisons, unified skew-normal distribution, knowledge gradient, mean field variational inference.

1 Introduction

Ranking inference over a set of items is important in various applications, including recommendation systems, web search algorithms, and competitive arenas like sports and gaming competitions (Baltrunas et al. [2010], Zehlike et al. [2022]; Lempel and Moran [2005], Lamberti et al. [2008]; Pelechrinis et al. [2016], Minka et al. [2018];Chen et al. [2022], Fan et al. [2022]). In particular, many works have been proposed in pairwise ranking, employing paired comparison data where workers are presented with each pair to choose which one is better. Typical examples include Jiang et al. [2011], Emerson [2013], Rajkumar and Agarwal [2014], Negahban et al. [2012], Azari Soufiani et al. [2013], Chen et al. [2022].

While these methods exhibit strong performance on static datasets, it is crucial to acknowledge that real-world data collection often faces constraints imposed by budget or time limitations. To address this challenge, numerous active sampling strategies have been proposed Li et al. [2018], Mikhailiuk et al. [2021], Xu et al. [2018], Chen et al. [2013, 2016], with the goal of optimizing the efficiency of collecting samples in terms of ranking accuracy. Among these methods, most of them are based on Bayesian inference. For instance, Li et al. [2018] introduced

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the *Hybrid Minimum Spanning Tree* (Hybrid-MST), selecting a new pair via the maximization of the expected information gain (EIG) based on the asymptotic distribution of Maximum Likelihood Estimation. Furthermore, Chen et al. [2016] aimed to maximize the expected Kendall's τ correlation coefficient, a direct measure of ranking accuracy, under the Bradley-Terry-Luce (BTL) model with a Dirichlet Prior. To determine the optimal sampling sequence, they introduced the *approximated knowledge gradient* (AKG) method, leveraging moment matching to approximate the posterior at each iteration.

A significant challenge faced by these methods lies in the occurrence of approximation errors, which can severely impact sampling efficiency and ranking accuracy. For instance, the asymptotic distribution of Hybrid-MST in Li et al. [2018] may not hold in early iterations, especially when the sample size is small. Additionally, the approximated posterior distribution via moment matching approach employed in AKG Chen et al. [2016] may deviate from the ground-truth posterior since the target function for approximation is not the true posterior after more than one sample, leading to non-ignorable estimation errors. To illustrate this, consider the following example.

Example 1. *Consider the ranking of $N = 3$ items with scoring parameters θ_1, θ_2, and θ_3. We examine the estimation error of three methods:* AKG *Chen et al. [2016], utilizing an approximated posterior;* Hybrid-MST *Li et al. [2018], utilizing asymptotic distribution; and our proposed method SUN. The error is defined as:*

$$\text{Error}_t = \sum_{i<j} \left| \Pr(\theta_i - \theta_j > 0 | X_t, Y_t) - \widehat{\Pr}(\theta_i - \theta_j > 0 | X_t, Y_t) \right|, \tag{1}$$

where, at the t-th iteration, X_t is the design matrix of pairwise comparisons, and Y_t is generated by the Thurstone-Mostelle (TM) or Bradley-Terry-Luce (BTL) models. $\Pr(A)$ *denotes the probability of event A from the ground-truth posterior, and* $\widehat{\Pr}(A)$ *denotes the probability of event A from the distribution used in each method. To ensure fair comparison, we maintain the same X_t and Y_t across all methods. The average error over 25 trials under both models is shown in Fig. 1. Our method consistently yields the lowest errors, while AKG and Hybrid-MST may exhibit larger errors.*

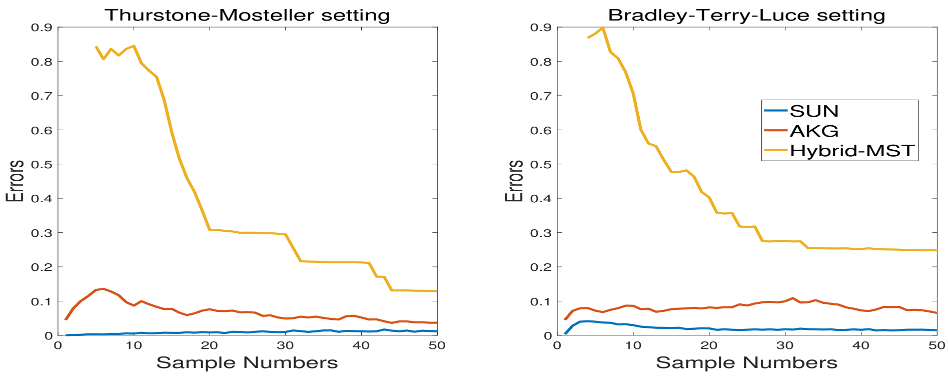

Figure 1: Estimation errors of posterior. Left: the TM model; right: the BTL model. x-axis corresponds to the number of paired comparison samples, while the y-axis represents the error.

In this paper, we resolve this problem by providing the closed-form posterior in the Bayesian inference framework. Inspired by recent findings in Durante [2019], we establish that the posterior distribution under the commonly utilized Thurstone-Mosteller (TM) model, coupled with a Gaussian prior, conforms to the *unified skew-normal* (SUN) distribution Arellano-Valle and Azzalini [2006]. By incorporating this novel posterior into Bayesian inference, we demonstrate that, under mild conditions, the estimated ranking, based on the expected Kendall's τ correlation coefficient, is optimal. Furthermore, we establish the convergence of the estimated ranking towards the true ranking, showcasing the effectiveness of our framework.

Subsequently, we integrate our framework into the knowledge gradient procedure and apply it in the active ranking scenario, consistently utilizing the ground-truth posterior throughout the entire process. At each iteration, we obtain the posterior and select the next pair for comparison, aiming to maximize the expected increment of Kendall's τ correlation coefficient. To alleviate the computational burden in later iterations with a large sample size, we introduce an approximate version of our algorithm through Mean Field Variational Inference. The utility and efficiency of our methods are demonstrated on both synthetic datasets and two real-world applications: reading difficulty and image/video quality assessment.

Contributions. We summarize our contributions below:

- We introduce a novel Bayesian conjugate framework, where we can derive the closed-form posterior under the classical Thurstone-Mosteller model.
- We establish the convergence result of our estimated ranking.
- We incorporate our framework into the knowledge gradient procedure and apply it to the active sampling scenario, which greatly enhances the sampling efficiency and ranking accuracy.
- We approximate the posterior by employing the Mean Field Variational Inference, which can significantly address computational burdens at later iterations in active sampling.

Organization. The remainder of this paper is organized as follows: In Section 2 we review related works. Section 3 introduces our Bayesian inference framework based on the SUN-distributed posterior. Section 4 proposes our active sampling strategy that exploits the Bayesian framework in the knowledge gradient method. Section 5 introduces an approximate version of our algorithm. Sections 6 and 7 test our active sampling strategy in simulated and real-world datasets, respectively. Finally, Section 8 draws conclusions. For computational details, proofs, please refer to the appendix.

2 Related Work

Unified Skew-Normal distribution. The unified skew-normal (SUN) distribution, initially introduced by Arellano et al. Arellano-Valle and Azzalini [2006], represents a comprehensive extension of the skew-normal distribution. This distribution constructs its probability density function by multiplying a Gaussian density function by the cumulative distribution function of a Gaussian distribution. Notably, this construction accommodates variations in dimensionality for the two components, allowing for dimensions that differ and exceed 1. Recent years have seen researchs into the properties of the SUN distribution. Azzalini et al. Azzalini and Bacchieri [2010] have provided expressions for its mean and distribution function, while Arellano et al. Arellano-Valle and Azzalini [2022] have offered the form of its higher moments. Gupta et al. Gupta et al. [2013] have demonstrated that the distribution family is closed under linear transformations, marginal and conditional distributions. The SUN distribution also has found applications in diverse fields, including clinical trials Azzalini and Bacchieri [2010], time series models Zarrin et al. [2019], small area estimation Diallo and Rao [2018], and the analysis of spatial data Allard and Naveau [2007], Hosseini et al. [2011]. Of particular relevance to our framework is its role as a conjugate prior for various models, such as Probit regression Durante [2019], Multinomial probit models Fasano et al. [2022].

In Our Work, we discover that the SUN distribution serves as a conjugate prior for the classical Thurstone-Mosteller model, significantly bolstering its Bayesian foundation. Based on this discovery, we propose our Bayesian inference method on the ground-truth posterior to estimate the ranking, generated through the maximization of the expectation of Kendall's τ over the ground-truth posterior. We prove that the estimated ranking is optimal under mild conditions and establish its convergent properties. Our work enriches the application of the SUN distribution.

Pairwise Ranking and Active Sampling. The task of recovering rankings from pairwise comparisons has been widely studied, including the Bradley-Terry-Luce model Bradley and Terry [1952], Luce [2005], the Thurstone-Mosteller model Thurstone [1927], Mallows models Mallows [1957], Tang [2019], Trueskill model Herbrich et al. [2006], HodgeRank Jiang et al. [2011], Borda Count Emerson [2013], Plackett-Luce model Plackett [1975], Azari Soufiani et al. [2013], SVM-based rank aggregation Rajkumar and Agarwal [2014], Iterative Ranking algorithm Negahban et al. [2012], Generalized Method-of-Moments Azari Soufiani et al. [2013], Spectral MLE Chen and Suh [2015], and Divide-and-conquer rank algorithm Chen et al. [2022]. Notably, all these methods are designed in an offline manner, where the comparison data has been collected and provided for modeling.

In practical situations, task requesters often encounter constraints related to budget and time limitations when it comes to data collection. Therefore, recently many active sampling methods have been proposed and most of these methods are based on the Bayesian framework. For instance, Pfeiffer et al. [2012] and Li et al. [2018] estimated the posterior distribution of item scores by leveraging the asymptotic distribution of Maximum Likelihood Estimation. However, these methods may encounter substantial errors at the onset of the sampling process, especially when the sample size is small. Moreover Mikhailiuk et al. [2021] tackled the Thurstone-Mosteller model with a Gaussian prior but encountered challenges in deriving a closed-form posterior, resorting to an approximation via the message-passing method. Similarly, Xu et al. Xu et al. [2018] proposed an approach to optimize the expected information gain (EIG) under the Thurstone-Mosteller setting, estimating the ranking through ridge regression. However, the inherent bias in ridge regression introduces estimation errors in item scores. Notably, Chen et al. Chen et al. [2013] introduced the Crowd-BT method under the Bradley-Terry-Luce (BTL) model, optimizing EIG based on the approximated posterior via moment matching. Furthermore, Chen et al. [2016] extended this approach to optimize the expectation of Kendall's τ correlation coefficient over the approximated posterior distribution via moment matching. However, both of these works employing moment matching methods may suffer from accumulated estimation errors over time. This is attributed to potential deviations of the target distribution for approximation from the ground-truth posterior as the sample size increases.

In Contrast, our framework leverages the known form of the ground-truth posterior, the SUN distribution, generated from the classical Thurstone-Mosteller model with a Gaussian prior. Thanks to the closed-form, our active sampling strategy relies on the ground-truth posterior throughout the entire process, enhancing both theoretical robustness and estimation accuracy. For the approximated version of our approach, we ensure that the target distribution for approximation remains consistent with the ground-truth posterior throughout the entire process, guaranteeing dependable approximation performance.

3 Bayesian Ranking with Closed-Form Posteriors

In this section, we introduce our framework based on Bayesian inference for pairwise ranking. We first show that under the Thurstone-Mosteller model (Sec. 3.1), the score vector has a closed-form posterior, i.e., unified skew-normal distribution (Sec. 3.2). We then infer the ranking by optimizing the expectation of Kendall's τ objective function (Sec. 3.3). Finally, we establish the convergence result for our method in Sec. 3.5.

3.1 Basic Setup

We address the rank aggregation problem, aiming to rank N items labeled as $1, 2, \ldots, N$. Each item i ($i = 1, \ldots, N$) is linked to an underlying latent score θ_i. A ranking π is defined as a mapping $\pi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$, i.e., a permutation of N items, and the ground-truth ranking, denoted as π^*, is established in accordance with the underlying scores $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)^\top$:

$$\pi^*(i) > \pi^*(j) \text{ if and only if } \theta_i > \theta_j. \tag{2}$$

To infer the ranking, pairwise comparison data are collected through crowdsourcing platforms, where workers compare a pair of items such as (i, j), and the corresponding comparison outcome Y_{ij} is recorded. A value of $Y_{ij} = 1$ indicates a preference for item i over item j, while $Y_{ij} = -1$ denotes the opposite preference. We assume comparison outcomes are independent and identically distributed (i.i.d.) and adhere to the widely accepted Thurstone-Mosteller (TM) model Thurstone [1927], which can be expressed as:

$$\Pr(Y_{ij} = 1|\boldsymbol{\theta}) = \Phi\left(\frac{\theta_i - \theta_j}{\sqrt{2}}\right),\ \Pr(Y_{ij} = -1|\boldsymbol{\theta}) = \Phi\left(\frac{\theta_j - \theta_i}{\sqrt{2}}\right), \forall i, j = 1, ..., N \text{ and } i \neq j, \tag{3}$$

where Φ denotes the cumulative distribution function (CDF) of the standard normal variable. To infer $\boldsymbol{\theta}$ for ranking, we adopt the Bayesian framework, which is flexible in the extension to the online ranking scenario, as elaborated in the subsequent section. Specifically, with the prior $\boldsymbol{\theta} \sim p_\alpha(\boldsymbol{\theta})$ and the collected T samples, we can compute the posterior of $\boldsymbol{\theta}$ using the compared pairs $\mathcal{S}_T := \{(i_t, j_t)\}_{t=1}^T$ and outcomes $\mathcal{Y}_T(\mathcal{S}_T) := \{y_t\}_{t=1}^T$, where y_t is an abbreviation of Y_{i_t, j_t}.

$$p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \alpha) \propto \prod_{t=1}^{T} \Phi\left(\frac{(\theta_{i_t} - \theta_{j_t})y_t}{\sqrt{2}}\right) p_\alpha(\boldsymbol{\theta}). \tag{4}$$

We select the TM model due to its ability to yield a closed-form posterior when appropriately setting $p_\alpha(\boldsymbol{\theta})$. Specifically, choosing $p_\alpha(\boldsymbol{\theta})$ as isotropic independent Gaussian distributions, i.e., $\theta_i \sim \mathcal{N}(\mu_i^0, (\sigma_i^0)^2)$, Eq. (4) has:

$$p\left(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \alpha = (\boldsymbol{\mu}^0, \boldsymbol{\sigma}^0)\right) = \frac{\prod_{t=1}^T \Phi\left(\frac{(\theta_{i_t} - \theta_{j_t})y_t}{\sqrt{2}}\right) \prod_{i=1}^N \frac{1}{\sqrt{2\pi(\sigma_i^0)^2}} e^{-\frac{(\theta_i - \mu_i^0)^2}{2(\sigma_i^0)^2}}}{B(\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \boldsymbol{\mu}^0, \boldsymbol{\sigma}^0)}, \tag{5}$$

where $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0, ..., \mu_N^0)^\top$, $\boldsymbol{\sigma}^0 = (\sigma_1^0, \sigma_2^0, ..., \sigma_N^0)^\top$, and $B(\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \boldsymbol{\mu}^0, \boldsymbol{\sigma}^0)$ represents the normalizing constant. According to the recent results in Durante [2019], Eq. (5) corresponds to the *unified skew-normal* (SUN) distribution, and detailed derivations of SUN will be introduced in the subsequent section.

It's important to note that some earlier works, such as Chen et al. [2016], assumed the Bradley-Terry-Luce (BTL) model Bradley and Terry [1952], Luce [2005] for $Y_{i,j}$:

$$\Pr(Y_{ij} = 1|\boldsymbol{\theta}) = \frac{\theta_i}{\theta_i + \theta_j}, \quad \Pr(Y_{ij} = -1|\boldsymbol{\theta}) = \frac{\theta_j}{\theta_i + \theta_j}. \tag{6}$$

Additionally, the prior distribution for $\boldsymbol{\theta}$ was specified as Dirichlet. However, in contrast to our framework, the approach in Chen et al. [2016] lacks a closed-form posterior. While one can approximate the posterior using methods like moment matching as in Chen et al. [2016], this approach may be susceptible to noticeable errors, as demonstrated in Fig. 1. Furthermore, the closed-form nature of our posterior enables us to establish the convergent property of the estimated ranking. In other words, with a sufficient number of samples and some mild conditions, we can reliably recover the ground-truth ranking.

3.2 Unified Skew-Normal Posterior with Gaussian Priors

In this section, we reformulate our posterior distribution from Eq. (5) in the form of the SUN distribution. We begin by revisiting the construction of such a distribution, as detailed in Arellano-Valle and Azzalini [2022]. The process initiates with the consideration of the following $(N+T)$-dimensional multivariate normal distribution:

$$\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix} \sim \mathcal{N}_{N+T}\left(\mathbf{0}_{N+T}, \Omega^*\right), \quad \Omega^* := \begin{pmatrix} \bar{\Omega} & \Delta \\ \Delta^\top & \Gamma \end{pmatrix}, \tag{7}$$

where $\tilde{\boldsymbol{\theta}}$ denotes a T-dimensional random vector with $\tilde{\boldsymbol{\theta}} = (\theta_{N+1}, \ldots, \theta_{N+T})^\top$, and Ω^* denotes the full-rank correlation matrix of $\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}$. Let $\omega \in \mathbb{R}^{N\times N}$ be a diagonal matrix, and ξ, γ be N-dimensional, T-dimensional constant vectors respectively. Then, $Z = \xi + \omega(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} + \gamma > 0)$ [1] $\sim \mathrm{SUN}_{N,T}(\xi, \Omega, \Delta, \gamma, \Gamma)$, which is characterized by the following density function at z:

$$\phi_N(z;\xi,\Omega)\frac{\Phi_T\left(\gamma + \Delta^\top\bar{\Omega}^{-1}\omega^{-1}(z-\xi);\Gamma - \Delta^\top\bar{\Omega}^{-1}\Delta\right)}{\Phi_T\left(\gamma;\Gamma\right)}, \tag{8}$$

where $\Omega = \omega\bar{\Omega}\omega$, and $\phi_N(z;\xi,\Omega)$ represents the probability density function (PDF) of a N-dimensional Gaussian distribution at z with mean ξ and variance-covariance matrix Ω. Additionally, the term $\Phi_T\left(\gamma + \Delta^\top\bar{\Omega}^{-1}\omega^{-1}(z-\xi);\Gamma - \Delta^\top\bar{\Omega}^{-1}\Delta\right)$ (*resp.* $\Phi_T(\gamma;\Gamma)$) stands for the CDF of a T-dimensional Gaussian distribution evaluated at $\gamma + \Delta^\top\bar{\Omega}^{-1}\omega^{-1}(z-\xi)$ (*resp.* γ) with mean $\mathbf{0}_T$ ($\mathbf{0}_T$ is a T-dimensional vector with all elements set to 0) and variance-covariance matrix $\Gamma - \Delta^\top\bar{\Omega}^{-1}\Delta$ (*resp.* Γ).

Drawing from Arellano-Valle and Azzalini [2022], it is established that the CDF and cumulant generating function of the SUN distribution possess closed forms. Furthermore, the SUN distribution exhibits the property that an affine transformation of a SUN-distributed variable also follows the SUN distribution. These results are summarized in the following Lemma.

Lemma 1 (Eq. (3)-(8) in Arellano-Valle and Azzalini [2022])**.** *For the SUN distribution $Z := \xi + \omega(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} + \gamma > 0) \sim \mathrm{SUN}_{N,T}(\xi, \Omega, \Delta, \gamma, \Gamma)$ derived from Eq. (7), we have*

- *(**CDF of** Z): The CDF $F_Z(z) := \Pr(Z \le z)$ takes the form of*

$$F_Z(z) = \frac{\Phi_{N+T}(\tilde{z};\tilde{\Omega})}{\Phi_T(\gamma;\Gamma)}, \text{ where } \tilde{z} = \begin{pmatrix} \frac{z-\xi}{\omega} \\ \gamma \end{pmatrix} \text{ and } \tilde{\Omega} = \begin{pmatrix} \bar{\Omega} & -\Delta \\ -\Delta^\top & \Gamma \end{pmatrix}. \tag{9}$$

- *(**Cumulant Generating Function**): Denote $K_Z(t)$ as the cumulant generating function of Z, we have*

$$K_Z(t) = \log M(t) = \xi^\top t + 2^{-1}t^\top\Omega t + \log\Phi_T\left(\gamma + \Delta^\top\omega t;\Gamma\right) - \log\Phi_T(\gamma;\Gamma). \tag{10}$$

Then, the mean of Z, denoted as $\mu_z := \mathbb{E}_Z(Z) = \left.\frac{\mathrm{d}K_Z(t)}{\mathrm{d}t}\right|_{t=0}$, has the following form

$$\mu_z = \xi + \frac{\omega\Delta}{\Phi_T(\gamma;\Gamma)}\nabla\Phi_T(\Gamma),\ \left(\nabla\Phi_T(\Gamma)\right)_j = \begin{cases} \phi\left(\gamma_j\right) & \text{if } T = 1, \\ \phi\left(\gamma_j\right)\Phi_{T-1}\left(\gamma_{-j} - \Gamma_{-j}\gamma_j;\tilde{\Gamma}_{-j}\right) & \text{if } T > 1. \end{cases} \tag{11}$$

Here, $\phi(\gamma_j)$ is the PDF of the standard normal distribution evaluated at γ_j. $\left(\nabla\Phi_T(\Gamma)\right)_j$ denotes the j-th element of the T-dimensional gradient vector. γ_{-j} denotes a vector derived from γ by excluding its j-th element. The term Γ_{-j} represents the j-th column of the matrix Γ with its j-th element omitted. $\tilde{\Gamma}_{-j} = \Gamma_{-j,-j} - \Gamma_{-j}\Gamma_{-j}^\top$, where $\Gamma_{-j,-j}$ is a $(T-1)\times(T-1)$ matrix obtained by eliminating the j-th row and column from Γ.

[1] $\tilde{\boldsymbol{\theta}} + \gamma > 0$ indicates that the inequality sign holds for each element of the T components.

- ***(Affine Transformation)**: For $b \in \mathbb{R}^d$ and $D \in \mathbb{R}^{N\times d}$ with full rank,*

$$b + D^\top Z \sim \mathrm{SUN}_{d,T}(b + D^\top \xi, D^\top \Omega D, \Delta_D, \gamma, \Gamma), \tag{12}$$

where $\Delta_D = \mathrm{Diag}(D^\top \Omega D)^{-1/2} D^\top \omega \Delta$, and $\mathrm{Diag}(\cdot)$ specifies a diagonal matrix formed by the diagonal elements of a square matrix within the bracket.

After introducing the SUN distribution, we proceed to derive the posterior in Eq. (5). We begin by modeling the prior of $\boldsymbol{\theta}$ as an isotropic Gaussian with zero mean and an identity variance-covariance matrix:

$$\boldsymbol{\theta}_N \sim \mathcal{N}_N(\mathbf{0}_N, I_N),\ \mathrm{Var}(\theta_i) = 1 \text{ for each } i. \tag{13}$$

This choice reflects a lack of preferences in the absence of observations. Then we denote

$$\tilde{\boldsymbol{\theta}} = D_T\boldsymbol{\theta} + \varepsilon,\ \text{ where }\ \varepsilon \sim \mathcal{N}_T(\mathbf{0}_T, I_T) \text{ is independent of } \boldsymbol{\theta}. \tag{14}$$

Here, $D_T = \mathrm{diag}(\boldsymbol{y}_T)X_T$, where $\mathrm{diag}(\boldsymbol{a})$ specifies a diagonal matrix expanded by the vector $\boldsymbol{a}$. Here, X_T and $\boldsymbol{y}_T$ represent the matrix form of $\mathcal{S}_T$ and the vector form of $\mathcal{Y}_T(\mathcal{S}_T)$, respectively. Specifically, $X_T \in \mathbb{R}^{T\times N}$ has its k-th row ($k = 1, \ldots, T$) defined according to the k-th comparison pair (i_k, j_k): for $l = 1, \ldots, N$, $X_T(k,l)$ [2] $= \frac{1}{\sqrt{2}}$ if $l = i_k$, $= -\frac{1}{\sqrt{2}}$ if $l = j_k$, and $= 0$ otherwise. Additionally, $\boldsymbol{y}_T := (y_1, \ldots, y_T)^\top$. We can express the probability $\Pr\left(\tilde{\theta}_t > 0|\boldsymbol{\theta}\right)$ as $\Pr\left(\varepsilon_t > -y_t\frac{\theta_{i_t}-\theta_{j_t}}{\sqrt{2}}|\boldsymbol{\theta}\right) = \Phi\left(\frac{y_t(\theta_{i_t}-\theta_{j_t})}{\sqrt{2}}\right)$. Note that each element in ε is independent, then we can have

$$p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0) \propto \Pr(\tilde{\boldsymbol{\theta}} > 0|\boldsymbol{\theta})p(\boldsymbol{\theta}) = \prod_{t=1}^{T} \Phi\left(\frac{(\theta_{i_t} - \theta_{j_t})y_t}{\sqrt{2}}\right) \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta_i^2}{2}}.$$

Therefore, compared to Eq. (5), we find $p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) = p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0)$. Furthermore, Eq. (14) implies $(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}) \sim \mathcal{N}_T(D_T\boldsymbol{\theta}, I_T)$. Given that the SUN variable derivation above Eq. (8) originates from the correlation matrix, we firstly establish the following lemma.

Lemma 2. *Let $\boldsymbol{\theta}$ be defined as in Eq. (13), and $\varepsilon \sim \mathcal{N}_T(\mathbf{0}_T, I_T)$ be independent of $\boldsymbol{\theta}$. Consider a $T \times N$ design matrix D_T, where $D_T(k,l)$ equals a_i if $l = i_k$, $-a_i$ if $l = j_k$, and 0 otherwise, for $k = 1, \ldots, T$ and $l = 1, \ldots, N$, with $a_i \in \mathbb{R}$, i_k, $j_k \in \{1, \ldots, N\}$. If $\tilde{\boldsymbol{\theta}} = D_T\boldsymbol{\theta} + \varepsilon$, then the distribution of $p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0)$ remains invariant whether derived from the variance-covariance matrix or the correlation matrix of the joint distribution $\begin{pmatrix}\boldsymbol{\theta}\\ \tilde{\boldsymbol{\theta}}\end{pmatrix}$.*

The proof of this Lemma can be found in A.1. From Lemma 2, $p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0)$ can be derived from the joint distribution of $\begin{pmatrix}\boldsymbol{\theta}\\ \tilde{\boldsymbol{\theta}}\end{pmatrix}$, with mean and correlation matrix represented below:

$$\begin{pmatrix}\boldsymbol{\theta}\\ \tilde{\boldsymbol{\theta}}\end{pmatrix} \sim \mathcal{N}_{N+T}\left(\begin{array}{c}\mathbf{0}_N\\ \mathbf{0}_T\end{array}, \left(\begin{array}{cc} I_N & \Delta_T\\ \Delta_T^\top & \Gamma_T\end{array}\right)\right), \tag{15}$$

where $\Delta_T = D_T^\top s_T^{-1}$, $\Gamma_T = s_T^{-1}(D_T D_T^\top + I_T)s_T^{-1}$, in which $s_T := \mathrm{diag}(\sqrt{2}\mathbf{1}_T)$ ($\mathbf{1}_T$ is a T-dimensional vector with all elements set to 1). With $\xi = \mathbf{0}_N$, $\gamma = \mathbf{0}_T$, $\omega = I_N$, based on the derivation of SUN variable above Eq. (8), we obtain:

$$(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) \overset{d}{=} (\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0) \sim \mathrm{SUN}_{N,T}\left(\mathbf{0}_N, I_N, \Delta_T, \mathbf{0}_T, \Gamma_T\right). \tag{16}$$

According to Eq. (8), its density function can be expressed as:

$$p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) = \phi_N(\boldsymbol{\theta}; \mathbf{0}_N, I_N)\frac{\Phi_T\left(\Delta_T^\top\boldsymbol{\theta}; \Gamma_T - \Delta_T^\top\Delta_T\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)}. \tag{17}$$

[2] $X(k,l)$ indicates the entry in the k-th row and l-th column of matrix X.

3.3 Bayesian Inference of Ranking

With posterior derived in Eq. (16), we introduce the Bayesian method to estimate π^*. Ideally, we expect our estimated π to maximize the Kendall's τ rank correlation coefficient Kendall [1938], which was commonly adopted in ranking inference Chen et al. [2013]:

$$\begin{aligned}\tau\left(\pi,\pi^*\right) &\equiv \frac{|\{(i,j): i<j, (\pi(i)-\pi(j))(\pi^*(i)-\pi^*(j))>0\}|}{N(N-1)/2} \\ &= \frac{2}{N(N-1)}\sum_{i\neq j}\mathbf{1}_{\{\pi(i)>\pi(j)\}}\mathbf{1}_{\{\theta_i>\theta_j\}}.\end{aligned} \tag{18}$$

However, directly optimizing Eq. (18) is intractable as $\boldsymbol{\theta}$ are unknown. Therefore, we adopt the Bayesian framework by optimizing the following objective used in Chen et al. [2016]:

$$\pi_T \in \arg\max_{\pi} \mathbb{E}[\tau(\pi,\pi^*)|S_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N]. \tag{19}$$

According to Chen et al. [2016], we have:

$$\pi_T \in \arg\max_{\pi} C_T(\pi) \equiv \sum_{i\neq j}\mathbf{1}_{\{\pi(i)>\pi(j)\}}\mathrm{Pred}_T(i>j), \tag{20}$$

where we denote the prediction probability of item i surpassing item j, based on the posterior distribution, as:

$$\mathrm{Pred}_T(i>j) := \Pr(\theta_i>\theta_j|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), 0_N, I_N). \tag{21}$$

By noting that each pair in $C_T(\pi)$ is uniquely assigned to either $\mathrm{Pred}_T(i>j)$ or $\mathrm{Pred}_T(j>i)$, we have the following upper bound for $C_T(\pi)$:

$$C_T(\pi) = \sum_{\pi(i)>\pi(j)} \mathrm{Pred}_T(i>j) \leq \sum_{i<j}\max\left(\mathrm{Pred}_T(i>j), \mathrm{Pred}_T(j>i)\right). \tag{22}$$

This upper bound can be achieved by a certain π if it exists and satisfies: $\pi(i)>\pi(j)$ only if $\mathrm{Pred}_T(i>j)\geq \mathrm{Pred}_T(j>i)$, i.e., $\mathrm{Pred}_T(i>j)\geq 0.5$. To compute $\mathrm{Pred}_T(i>j)$, we firstly denote: $\Delta_T(ij)$ as a $1\times T$ matrix given by

$$\Delta_T(ij) := \Delta_T(i,:) - \Delta_T(j,:). \tag{23}$$

Here $\Delta_T(i,:)$ and $\Delta_T(j,:)$ are respectively the i-th and j-th rows of Δ_T. To compute $\mathrm{Pred}_T(i>j)$, we have the following Proposition.

Proposition 1. $\mathrm{Pred}_T(i>j)$ *defined in Eq.* (21) *takes the form of:*

$$\mathrm{Pred}_T(i>j) = \frac{\Phi_{T+1}(\mathbf{0}_{T+1};\tilde{\Omega}_{ij})}{\Phi_T(\mathbf{0}_T;\Gamma_T)}, \quad \text{where} \quad \tilde{\Omega}_{ij} = \begin{pmatrix} 1 & \frac{\Delta_T(ij)}{\sqrt{2}} \\ \frac{\Delta_T(ij)^\top}{\sqrt{2}} & \Gamma_T \end{pmatrix}. \tag{24}$$

The proof of Proposition 1 can be found in A.2. However, simply determining π_T in Eq. (20) via $\{\mathrm{Pred}_T(i>j)\}_{(i\neq j)}$ necessitates a well-defined ranking, implying the absence of a loop that is defined below:

Definition 1. *We say that a loop exists among* $i_1,\ldots,i_K$ *for some integer* $K\geq 3$ *if, for each* $1\leq r\leq K-1$, $\mathrm{Pred}_T(i_r>i_{r+1})\geq 0.5$, *with at least one occurrence of ">" being satisfied and additionally,* $\mathrm{Pred}_T(i_K>i_1)>0.5$.

In the presence of such a loop, $C_T(\pi)$ fails to reach its upper bound. In the following, we provide an algorithm to check whether a loop exists among $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$.

Algorithm 1 Ranking Verification.

Input $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$, where $i, j \in \{1, \dots, N\}$.
Output A ranking π if there is no loop, otherwise, **None**.

Set the candidate set $S = \{1, 2, \dots, N\}$.
while $|S| \geq 1$ **do**
  if there exists $s \in S$ such that $\text{Pred}_T(s > k) \geq 0.5$ for all $k \in S$ and $k \neq s$, or $|S \cap \{s\}| = 1$. **then**
    Set $\pi(s) = |S|$, and remove s from S.
  else
    Break the loop and return **None**.
  end if
end while

If loops exist, in the worst case, we would need to compare $N!$ possible rankings to achieve the maximized $C_T(\pi)$, which is NP-hard. Fortunately, our empirical observations suggest that loops do not exist. This observation can be attributed to the always-satisfied sufficient condition in our empirical study, specifically, the non-symmetric condition, which is introduced below:

Condition 1. *If Δ_T defined below Eq. (15) has $\Delta_T(ij)^\top \neq \mathbf{0}_T$, where $\Delta_T(ij)$ is defined in Eq. (23), then we assume the following holds for all $s \in (0, +\infty)$:*

$$\Phi_T\left(\frac{\Delta_T(ij)^\top s}{2}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right) \neq \Phi_T\left(\frac{-\Delta_T(ij)^\top s}{2}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right), \tag{25}$$

where Γ_T is obtained as shown below Eq. (15).

To empirically demonstrate this condition, we first introduce a univariate SUN-distributed random variable:

$$\theta_{ij} \sim A_{ij}^\top \boldsymbol{\theta}, \tag{26}$$

where A_{ij} is an N-dimensional vector with its i-th element set to 1, j-th element set to -1, and other elements set to 0. Additionally, $\boldsymbol{\theta}$ follows the distribution given in Eq. (16).

The non-symmetric condition asserts that the PDF of θ_{ij} will not yield identical values at points symmetric about the y-axis. We demonstrate one example of θ_{ij} in our empirical study as shown in Fig. 2. The blue line represents the density of θ_{ij}, while the orange line represents the density of $-\theta_{ij}$, with $\theta_{ij} \in \{0, +\infty\}$. The fact that the blue line and orange line do not intersect in $\mathbb{R}^+$ demonstrates the satisfaction of the non-symmetric condition. This condition can be tested and is always satisfied, thanks to the inherent skewness of the SUN distribution. Under this condition, the following theorem guarantees the absence of loops.

Theorem 1. *If $\boldsymbol{\theta}$ follows the posterior distribution as described in Eq. (16), then for any pair (i, j) that satisfies Condition 1, we have $\text{sign}(\mathbb{E}[\theta_i - \theta_j]) = \text{sign}\left(\text{Pred}_T(i > j) - 0.5\right)$.*

The proof of this theorem can be found in A.3. This result implies that the ranking given by $\text{Pred}_T(i > j)$ is equivalent to that via the posterior mean $\mathbb{E}_T(\theta_{i-j}) := \mathbb{E}\left[\theta_i - \theta_j\right]$, where $\boldsymbol{\theta}$ follows the distribution as described in Eq. (16). Furthermore, if Condition 1 is satisfied for all pairs, then the ranking sorted by posterior mean will optimize $C_T(\pi)$, as summarized below:

Corollary 1. *If $\boldsymbol{\theta}$ follows the posterior distribution as described in Eq. (16), and Condition 1 is satisfied for all pairs, then the ranking π_{pos} sorted by $\mathbb{E}(\boldsymbol{\theta})$ will optimize the expectation of Kendall's τ in Eq. (19).*

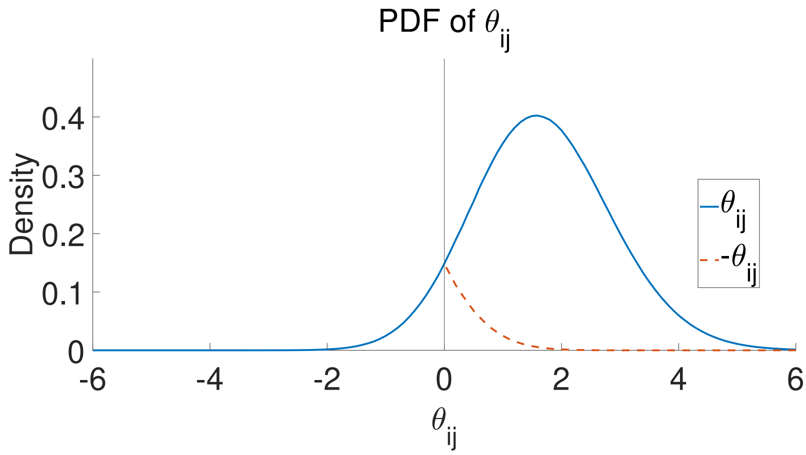

Figure 2: The x-axis represents the value of θ_{ij}, while the y-axis represents its density.

The proof of this corollary can be found in A.3. This result implies that if the non-symmetric condition holds for all pairs, there are no loops in the ranking induced by $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$, enabling us to find the optimal ranking using $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$. However, if a loop exists, determining the optimal ranking is NP-hard as discussed in Chen et al. [2016]. In such cases, we resort to the posterior mean, denoted as μ_T, for estimation. Leveraging the sorting of μ_T is optimal under the non-symmetric condition and widely applied in ranking scenarios, as demonstrated in Xu et al. [2018]. We derive the form of its posterior mean from Eq. (11) in Lemma. 1, with the distribution of $\boldsymbol{\theta}$ given in Eq. (16).

$$\mu_T = \frac{\Delta_T \nabla \Phi_T(\Gamma_T)}{\Phi_T(\mathbf{0}_T; \Gamma_T)}, \; (\nabla \Phi_T(\Gamma))_j = \begin{cases} \phi(0) & \text{if } T = 1, \\ \phi(0)\Phi_{T-1}(\mathbf{0}_{T-1}, \Gamma_{-j,-j} - \Gamma_{-j}\Gamma_{-j}^{\top}) & \text{otherwise.} \end{cases} \tag{27}$$

We summarize our ranking estimation in Alg. 2.

Algorithm 2 SUN Ranking

Input Observed data $\mathcal{S}_T = \{(i_t, j_t)\}_{t=1}^T$, $\mathcal{Y}_T(\mathcal{S}_T) = \{y_t\}_{t=1}^T$.
Output Estimated Ranking π_{pos}.
1: Obtain the PDF of the posterior distribution specified in Eq. (17) from the observed data.
2: Calculate $\text{Pred}_T(i > j)$ with Eq. (24) $\forall\ i < j$. Set $\text{Pred}_T(j > i) = 1 - \text{Pred}_T(i > j)$.
3: Run Alg. 1 and check if there exists a loop.
4: **if** return a ranking π **then**
5: Set $\pi_{pos} = \pi$.
6: **else**
7: Obtain the posterior mean μ_T using Eq. (27). Set π_{pos} based on the sorting of μ_T.
8: **end if**

3.4 Monte Carlo Estimation of $\text{Pred}_T(i > j)$ and μ_T

As the computation of $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$ with Eq. (24) requires to calculate $(T+1)$-dimensional multivariate normal CDFs for $N(N-1)/2$ times, Alg. 2 can be expensive when N is large. To address this issue, we implement Monte Carlo integration to compute $\{\text{Pred}_T(i > j)\}_{(i \neq j)}$ or posterior mean. The key idea is based on the orthogonal decomposition of the SUN distribution into a multivariate normal distribution and a multivariate truncated normal distribution Arellano-Valle and Azzalini [2022].

Proposition 2 (Eq. (6) in Arellano-Valle and Azzalini [2022]). *If the posterior distribution* $(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, \boldsymbol{I}_N)$ *has the probability density function in Eq. (17), then it can be orthogonally decomposed into the following form:*

$$(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) \overset{d}{=} \boldsymbol{V}_0 + D_T^\top(D_T D_T^\top + I_T)^{-1} s_T \boldsymbol{V}_1, \qquad (\boldsymbol{V}_0 \perp \boldsymbol{V}_1) \tag{28}$$

where D_T *and* s_T *are defined below Eq. (14) and Eq. (15) separately.* $\boldsymbol{V}_0$ *follows a* N*-dimensional multivariate normal distribution with mean* $\mathbf{0}_N$ *and variance-covariance matrix* $I_N - D_T^\top(D_T D_T^\top + I_T)^{-1} D_T$. $\boldsymbol{V}_1$ *comes from a* T*-dimensional multivariate truncated normal distribution, with mean* $\mathbf{0}_T$*, variance-covariance matrix* $s_T^{-1}(D_T D_T^\top + I_T)s_T^{-1}$ *and truncation below* $\mathbf{0}_T$*.*

According to Eq. (28), we can approximate $\mathrm{Pred}_T(i > j)$ and μ_T via Monte Carlo integration as summarized below:

Algorithm 3 Monte Carlo Integration to estimate $\mathrm{Pred}_T(i > j)$ and μ_T

Input Observed data $\mathcal{S}_T = \{(i_t, j_t)\}_{t=1}^T$, $\mathcal{Y}_T(\mathcal{S}_T) = \{y_t\}_{t=1}^T$, and the number of samples n.
Output Estimated $\mathrm{Pred}_T(i > j)$ and μ_T.

1: Calculate D_T and s_T, introduced below Eq. (14) and Eq. (15) separately.
2: Sample n independent samples $v_0^{(1)}, ..., v_0^{(n)}$ from an N-dimensional multivariate normal distribution with mean $\mathbf{0}_N$ and variance-covariance matrix $I_N - D_T^\top(D_T D_T^\top + I_T)^{-1} D_T$.
3: Sample n independent samples $v_1^{(1)}, ..., v_1^{(n)}$ from a T-dimensional multivariate truncated normal distribution with mean $\mathbf{0}_T$, variance-covariance matrix $s_T^{-1}(D_T D_T^\top + I_T)s_T^{-1}$, and truncation below $\mathbf{0}_T$.
4: Obtain $\boldsymbol{\theta}^{(1)}, ..., \boldsymbol{\theta}^{(n)}$, where $\boldsymbol{\theta}^{(s)} = v_0^{(s)} + D_T^\top(D_T D_T^\top + I_T)^{-1} s_T v_1^{(s)}$.
5: For all i, j with $i \neq j$, estimate $\mathrm{Pred}_T(i > j) \approx \frac{1}{n}\sum_{s=1}^n \mathbf{1}_{\{\theta_i^{(s)} > \theta_j^{(s)}\}}$.
6: Estimate $\mu_T \approx \frac{1}{n}\sum_{s=1}^n \boldsymbol{\theta}^{(s)}$.

3.5 Convergence of SUN Ranking Algorithm

In this section, we demonstrate the convergent result of Alg. 2, which highlights the effectiveness of our Bayesian framework in recovering the ground-truth ranking. Since the comparison model TM in Eq. (3) only depends on the difference between item scores, we assume a reference alternative $\theta_r = 0$, with $r \in \{1, \ldots, N\}$. Then, the solution of the Maximum Likelihood Estimator (MLE) with the likelihood function being TM has a unique and bounded solution as shown in Noothigattu et al. [2020]. We then impose conditions on the MLE and obtain the convergent result of our SUN Ranking algorithm.

Theorem 2. *Suppose the ground-truth model is TM in Eq. (3), with* T *i.i.d. samples, and the MLE of* $\boldsymbol{\theta}$*, with the likelihood function as TM in Eq. (3), is consistent and asymptotic normal. Then, we have* $\lim_{T\to\infty} \pi_T \to \pi^*$*, where* π_T *is given by Alg. 2, indicating its dependence on* T*.*

The proof of this theorem can be found in A.4. This theorem states that once the MLE of $\boldsymbol{\theta}$ is consistent and asymptotically normal, the ranking proposed via our SUN Ranking algorithm will also converge to the ground-truth ranking. However, one advantage of our SUN-based framework over MLE lies in the active sampling scenario, where we have knowledge of the posterior distribution throughout the entire process. In contrast the asymptotic distribution from MLE may not hold true at early steps in active sampling, as depicted in Example 1.

Although Theorem 2 requires assumptions on MLE, the convergent result can always be satisfied due to the mild conditions of the Bernstein–von Mises theorem, as discussed in Bochkina and Green [2014]. To illustrate this empirically, we conduct a simulation experiment. We consider $N = 5$ items to be ranked, with $\theta_i \sim U(0, 10)$ for each $i = 1, \ldots, 5$, and π^* defined according to $\boldsymbol{\theta}$. We generate paired samples and outcomes following Eq. (3). We run 10 independent trials and plot the median curve of Kendall's τ along with its error bar region whose lower bound is the 0.25th quantile and upper bound is the 0.75th quantile. As shown in Fig. 3, Kendall's τ approaches 1 as T exceeds 60, indicating the recovery of π^*.

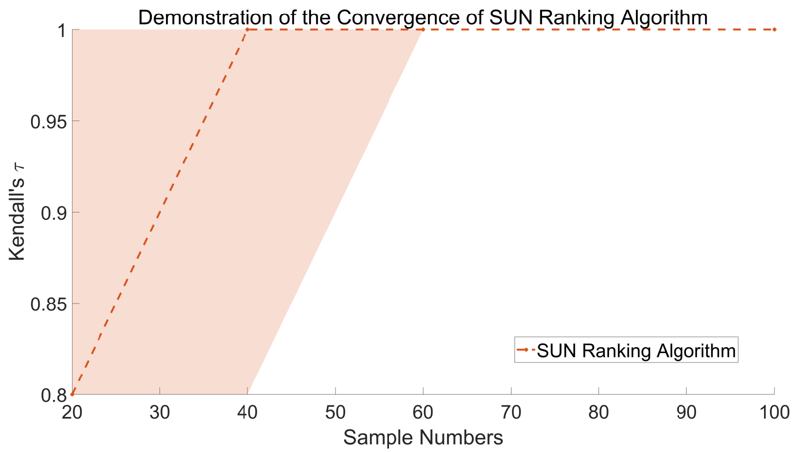

Figure 3: x-axis represents the number of samples; y-axis represents the Kendall's τ.

4 Bayesian Active Sampling

In the previous section, we considered the static ranking problem over T samples, which are provided beforehand. However, such an offline setting ignores the impact of the data collection process, which can affect the effectiveness of collected samples in recovering the true ranking. This issue is particularly pertinent in scenarios involving crowdsourcing, often constrained by a limited budget. In such cases, there is a desire to dynamically collect the most impactful pairs concerning Kendall's τ.

To determine the optimal sampling process, we in this section extend our Bayesian ranking method to the active learning setting. Similar to Sec. 3, we consider N items to be ranked under the TM model in Eq. (3), and the prior remains i.i.d. standard normal. With a sampling budget of at most T samples, the primary objective of our active learning algorithm is to determine a sequence of sampling pairs $\mathcal{S}_T$ to maximize the expectation of Kendall's τ over the posterior given $\mathcal{S}_T$ and $\mathcal{Y}_T(\mathcal{S}_T)$ (with $\mathcal{S}_0 = \emptyset$ and $\mathcal{Y}_0(\mathcal{S}_0) = \emptyset$):

$$\max_{\mathcal{S}_T} \mathbb{E}^{\mathcal{S}_T} \left(U(\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T)) \middle| \mathbf{0}_N, I_N \right), \tag{29}$$

where $U(\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T)) := C_T \left(\pi_T^{\{\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T)\}} \right)$ with $\pi_T^{\{\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T)\}} \in \arg\max_\pi C_T(\pi)$, indicating its dependence on $\mathcal{S}_T$ and $\mathcal{Y}_T(\mathcal{S}_T)$. $\mathbb{E}^{\mathcal{S}_T}$ denotes the expectation over the sample path $\mathcal{S}_T$ and the corresponding outcomes. Eq. (29) was introduced by Chen et al. [Chen et al., 2016], which formulated the objective into the Bayesian Markov Decision Process. Specifically, Eq. (29) can be equivalently written as:

$$\max_{\mathcal{S}_T} \mathbb{E}^{\mathcal{S}_T} \left(U(\mathcal{S}_0, \mathcal{Y}_0(\mathcal{S}_0)) + \sum_{t=0}^{T-1} \mathcal{I}\left(\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i_{t+1}, j_{t+1}, y_{t+1})\right) \middle| \mathbf{0}_N, I_N \right),$$

where $\mathcal{I}(\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i_{t+1}, j_{t+1}, y_{t+1})) := U(\mathcal{S}_{t+1}, \mathcal{Y}_{t+1}(\mathcal{S}_{t+1})) - U(\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t))$, and a newly collected sample at stage $t+1$ is denoted as $(i_{t+1}, j_{t+1}, y_{t+1})$, such that $\mathcal{S}_{t+1} = \mathcal{S}_t \cup \{(i_{t+1}, j_{t+1})\}$ and $\mathcal{Y}_{t+1}(\mathcal{S}_{t+1}) = \mathcal{Y}_t(\mathcal{S}_t) \cup \{y_{t+1}\}$. However, directly solving Eq. (29) can be intractable since the space of $\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}$ grows exponentially in t, making the precise resolution challenging through traditional methods like value or policy iteration, as detailed in Chen et al. [2016]. To address this issue, we propose the knowledge gradient (KG) policy in the subsequent section.

4.1 Knowledge Gradient Policy

In this section, we employ the KG policy Frazier et al. [2008], Powell [2010] to address the optimization problem defined in Eq. (29). It is worth noting that the KG policy within our framework distinguishes itself from Chen et al. [2013, 2016], by consistently utilizing the ground-truth posterior distribution throughout the entire process. This is in contrast to policies in Chen et al. [2013, 2016] that rely on an approximated posterior distribution, potentially leading to significant estimation errors, as illustrated in Example 1. These errors hinder the accuracy of the estimated ranking in practice, as demonstrated in Sec. 6 and Sec. 7.

Now, let us delve into our procedure. At the t-th stage, assuming we have gathered $\mathcal{S}_t$ and their corresponding outcomes $\mathcal{Y}_t(\mathcal{S}_t)$, the KG policy within our framework selects the next pair to maximize the expected increment of the expectation of Kendall's τ based on the accumulated data and the prior distribution, which can be expressed as:

$$\begin{aligned}(i_{t+1}, j_{t+1})_{KG} &\in \arg\max_{(i,j)} \mathbb{E}\left[\mathcal{I}\left(\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij})\right) | \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N\right] \\ &= \arg\max_{(i,j)} \mathbb{E}\left[C_{t+1}\left(\pi_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij})\}}\right) \middle| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N\right] \\ &\qquad - C_t\left(\pi_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}\right). \end{aligned} \tag{30}$$

The second equation in Eq. (30) is due to the tower property of conditional expectation. Expanding the expectation over Y_{ij} in Eq. (30) and given that $C_t\left(\pi_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}\right)$ is fixed at t-th stage with respect to any newly sampled pair (i,j), we obtain

$$\begin{aligned}(i_{t+1}, j_{t+1})_{KG} \in \arg\max_{(i,j)} &\left[\Pr(Y_{ij} = 1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) C_{t+1}\left(\pi_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=1)\}}\right)\right. \\ &\left. + \Pr(Y_{ij} = -1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) C_{t+1}\left(\pi_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=-1)\}}\right)\right], \end{aligned} \tag{31}$$

where $\Pr(Y_{ij} = \pm 1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N)$ represents the probabilities of observing $Y_{ij} = \pm 1$, given the current data $\mathcal{S}_t$, $\mathcal{Y}_t(\mathcal{S}_t)$ and the prior distribution. To implement the KG policy in Eq. (31), we need to compute these probabilities. According to Corollary 2 in Durante [2019], we have

$$\Pr(Y_{ij} = \pm 1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) = \frac{\Phi_{t+1}\left(\mathbf{0}_{t+1}; \Gamma_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=\pm 1)\}}\right)}{\Phi_t\left(\mathbf{0}_t; \Gamma_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}\right)}. \tag{32}$$

Here, the elements within the braces of $\Gamma_t^{\{\cdot\}}$ or $\Gamma_{t+1}^{\{\cdot\}}$ represent the collected data from which the variance-covariance matrix Γ is derived, as illustrated below Eq. (15). To avoid any ambiguity, we state that $\Phi_t\left(\mathbf{0}_t; \Gamma_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}\right) = 1$ when $t = 0$. The derivation of Eq. (32) is provided in Appendix B.2.

It is worth noting that obtaining $\pi_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}$ can be computationally expensive due to potential loops. From an alternative perspective, in active sampling, proposing the estimated ranking after each sampling isn't necessary. Thus, utilizing the KG policy on the upper bound of $C_t(\cdot)$ is advantageous, as a larger upper bound may propose a superior ranking (based on Eq. (20)) and alleviate potential computational burden. Moreover, loops never occur in our empirical study, making this approach reasonable. The KG policy on the upper bound will be:

$$\begin{aligned}(i_{t+1}, j_{t+1})_{KG} \approx \arg\max_{(i,j)} &\left[\Pr(Y_{ij} = 1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) \max_{\pi} C_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=1)\}}(\pi)\right. \\ &\left. + \Pr(Y_{ij} = -1 \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) C_{t+1} \max_{\pi} C_{t+1}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=-1)\}}(\pi)\right], \end{aligned} \tag{33}$$

where $C_{t+1}^{\{\mathcal{S}_t,\mathcal{Y}_t(\mathcal{S}_t),(i,j,Y_{ij}=\pm1)\}}(\pi)$ indicates $C_{t+1}(\pi)$ from Eq. (20), with the posterior generated from $\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=\pm1)$, under an i.i.d. standard normal prior. The upper bound given in Eq. (22) can be easily obtained given the known form of the posterior.

After collecting all the data, we can propose the estimated ranking using Alg. 2, and our SUN Ranking Active algorithm is summarized in Alg. 4:

Algorithm 4 SUN Ranking Active

Input The budget T for sampling.

Output Estimated ranking $\pi_{pos}^{\{\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T)\}}$.

1: Initialize $\mathcal{S}_0=\emptyset$ and $\mathcal{Y}_0(\mathcal{S}_0)=\emptyset$.
2: **for** $t=0$ to $T-1$ **do**
3: Obtain data $\mathcal{S}_t\cup\{(i,j)\}$ and $\mathcal{Y}_t(\mathcal{S}_t)\cup\{Y_{ij}=\pm1\}$ for all possible pairs (i,j).
4: Calculate probabilities using parameters generated from inputted data in Eq. (32).
5: Obtain $\mathrm{Pred}_{t+1}(\mathrm{i}>\mathrm{j})$ via Monte Carlo Integration in Alg. 3.
6: Select (i_{t+1},j_{t+1}) via KG policy on upper bound in Eq. (33).
7: Record outcome y_{t+1}.
8: Renew data: $\mathcal{S}_{t+1}=\mathcal{S}_t\cup\{(i_{t+1},j_{t+1})\}$, $\mathcal{Y}_{t+1}(\mathcal{S}_{t+1})=\mathcal{Y}_t(\mathcal{S}_t)\cup\{y_{t+1}\}$.
9: **end for**
10: Estimate ranking $\pi_{pos}^{\{\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T)\}}$ using Alg. 2.

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5 SUN Ranking with Variational Inference

While our SUN Ranking algorithms excel in estimating the optimal ranking, they may suffer from considerable computational costs. These costs primarily arise from the computation of multivariate normal CDFs in Eq. (24), (27), (32). The dimension of these CDFs increases with the number of samples, imposing a considerable computational burden, as discussed in Durante [2019]. This becomes particularly pronounced in later iterations of the active learning scenario, where computational speed is crucial.

Although Alg. 3 alleviates the need for directly computing these CDFs, it introduces another computational challenge by requiring sampling from high-dimensional multivariate truncated normal distributions, as discussed in Fasano et al. [2022]. The dimension of these multivariate truncated normal distributions is, in fact, the number of collected samples, presenting a similar challenge to that encountered in the computation of CDFs, as mentioned earlier.

Previous work in Chen et al. [2016] utilized the moment matching method to approximate a target function. However, in our framework, this method still requires the computation of the first and second moments of a given SUN distribution, which is still faced with the computational challenge mentioned above. To address computational challenges, we employ the Mean-Field Variational Inference (MFVI) method Consonni and Marin [2007] to approximate the posterior distribution $(\boldsymbol{\theta}|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N)$ with a simple multivariate normal distribution.

To begin, recalling the matrix form of $\mathcal{S}_T$ and vector form of $\mathcal{Y}_T(\mathcal{S}_T)$, denoted as X_T and $\boldsymbol{y}_{\scriptscriptstyle T}$ respectively, as shown below Eq. (14), we utilize them to represent the collected data for simplicity. We introduce latent variables $\boldsymbol{z}=(z_1,\cdots,z_T)^\top$, where $(z_l|\boldsymbol{\theta})\sim\mathcal{N}(X_T(l,:)\boldsymbol{\theta},1)$ for each $l=1,\ldots,T$, with $X_T(l,:)$ representing the l-th row of X_T. The comparison outcome y_l therefore is defined by:

$$y_l=\begin{cases}1 & \text{if } z_l>0\\ -1 & \text{otherwise}\end{cases}. \tag{34}$$

The goal of the MFVI method is to find a proper mean-field distribution $q_{\text{MF}}(\boldsymbol{\theta},\boldsymbol{z})=q_{\text{MF}}(\boldsymbol{\theta})\prod_{l=1}^{T}q_{\text{MF}}(z_l)$ to minimize its Kullback–Leibler (KL) divergence Kullback and Leibler [1951] from $p(\boldsymbol{\theta},\boldsymbol{z}|\boldsymbol{y}_{\scriptscriptstyle T})$, i.e., $\mathrm{KL}(q_{\text{MF}}(\boldsymbol{\theta},\boldsymbol{z})\parallel p(\boldsymbol{\theta},\boldsymbol{z}|\boldsymbol{y}_{\scriptscriptstyle T}))$. It is worth noting that the specific form of the Evidence Lower Bound (ELBO) of $q_{\text{MF}}(\boldsymbol{\theta}, \boldsymbol{z})$ is given in B.3. As is well-known, maximizing ELBO is equivalent to minimizing the KL divergence Blei et al. [2017]. Given the prior of $\boldsymbol{\theta}$ as i.i.d. standard normal distributions, $p(z_l|\boldsymbol{\theta}, \boldsymbol{y}_T)$ and $p(\boldsymbol{\theta}|\boldsymbol{z}, \boldsymbol{y}_T)$ have the following distributions:

$$\begin{aligned} (z_l|\boldsymbol{\theta}, \boldsymbol{y}_T) &\sim \text{TN}(X_T(l,:)\boldsymbol{\theta}, 1, \mathbb{D}_{Z_l}), \quad \text{with } l = 1, \dots, T. \\ (\boldsymbol{\theta}|\boldsymbol{z}, \boldsymbol{y}_T) &\sim \mathcal{N}_N\left((I_N + X_T^\top X_T)^{-1} X_T^\top \boldsymbol{z}, (I_N + X_T^\top X_T)^{-1}\right). \end{aligned} \tag{35}$$

Here, $\text{TN}(X_T(l,:)\boldsymbol{\theta}, 1, \mathbb{D}_{Z_l})$ is the truncated normal distribution with mean $X_T(l,:)\boldsymbol{\theta}$, variance 1, and domain $\mathbb{D}_{Z_l} = \{z_l : y_{T,l} z_l > 0\}$, where $y_{T,l}$ is the l-th element of $\boldsymbol{y}_T$.

With the conditional distribution established, we then use the coordinate ascent variational algorithm Blei et al. [2017] to maximize the ELBO. At the s-th iteration, we update $q_{\text{MF}}^{(s)}(\boldsymbol{\theta})$ and $q_{\text{MF}}^{(s)}(\boldsymbol{z})$ as follows:

$$q_{\text{MF}}^{(s)}(\boldsymbol{\theta}) \propto \exp\left[\mathbb{E}_{q_{\text{MF}}^{(s-1)}(\boldsymbol{z})}\{\log p(\boldsymbol{\theta}|\boldsymbol{z}, \boldsymbol{y}_T)\}\right], \; q_{\text{MF}}^{(s)}(\boldsymbol{z}) \propto \exp\left[\mathbb{E}_{q_{\text{MF}}^{(s)}(\boldsymbol{\theta})}\{\log p(\boldsymbol{z}|\boldsymbol{\theta}, \boldsymbol{y}_T)\}\right]. \tag{36}$$

From Fasano et al. [2022], $q_{\text{MF}}(\boldsymbol{\theta})$ and $q_{\text{MF}}(z_l)$ are multivariate normal and truncated normal distributions, respectively. Then, according to Eq. (36) and (35), we have the following iterative algorithm Consonni and Marin [2007], Fasano et al. [2022]:

Algorithm 5 Mean-Field Variational Inference for SUN

Input X_T and y_T from $\mathcal{S}_T$, $\mathcal{Y}(\mathcal{S}_T)$ as defined below Eq. (14).
Output $q_{\text{MF}}(\boldsymbol{\theta}) \sim \mathcal{N}_N(\boldsymbol{\theta}_T^*, V_T)$.

1: Initialize $\bar{\boldsymbol{z}}^{(0)} \in \mathbb{R}^T$, $s = 0$.
2: **while** ELBO$\{q_{\text{MF}}^{(s)}(\boldsymbol{\theta}, \boldsymbol{z})\}$ in Eq. (92) is not convergent **do**
3: Let $s = s + 1$ and $\bar{\boldsymbol{\theta}}^{(s)} = (I_N + X_T^\top X_T)^{-1} X_T^\top \bar{\boldsymbol{z}}^{(s-1)}$.
4: Let the l-th element of $\bar{\boldsymbol{z}}^{(s)}$ being $\bar{z}_l^{(s)} = X_T(l,:)\bar{\boldsymbol{\theta}}^{(s)} + y_{T,l} \frac{\phi(X_T(l,:)\bar{\boldsymbol{\theta}}^{(s)})}{\Phi\left(\boldsymbol{y}_{T,l} X_T(l,:)\bar{\boldsymbol{\theta}}^{(s)}\right)}$, for $l = 1, ..., T$.
5: **end while**
6: Obtain the converged $\bar{\boldsymbol{\theta}}^{(s)}$ and denote it as $\bar{\boldsymbol{\theta}}_T^*$. Denote $V_T := (I_N + X_T^\top X_T)^{-1}$.

With $\mathcal{N}_N(\boldsymbol{\theta}_T^*, V_T)$ to approximate the ground-truth posterior distribution, we can estimate the ranking based on the approximated objective $\widehat{C_T(\pi)}$:

$$\widehat{\pi}_T \in \arg\max_{\pi} \widehat{C_T(\pi)} := \sum_{\pi(i) > \pi(j)} \widehat{\text{Pred}_T(i > j)}, \; \widehat{\text{Pred}_T(i > j)} := \int_{\theta_i > \theta_j} \phi_N\left(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}_T^*, V_T\right) d\boldsymbol{\theta}. \tag{37}$$

Indeed, we do not need to calculate $\widehat{\text{Pred}_T(i > j)}$ to obtain the corresponding ranking. The next theorem shows that the estimated ranking in Eq. (37) can be obtained according to the sorting of $\bar{\boldsymbol{\theta}}_T^*$.

Theorem 3. *If $\boldsymbol{\theta}$ follows a multivariate normal distribution generated from Alg. 5 with density function $\phi_N\left(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}_T^{(*)}, V_T\right)$, then:*

$$\begin{aligned} \Pi_{\bar{\boldsymbol{\theta}}_T^{(*)}} &\triangleq \{\pi \mid \pi \textit{ is a ranking with } \pi(i) > \pi(j) \textit{ only if } \bar{\boldsymbol{\theta}}_{T,i}^{(*)} \geq \bar{\boldsymbol{\theta}}_{T,j}^{(*)}, \textit{ for all } i,j \textit{ pairs with } i \neq j\} \\ &= \arg\max_{\pi} \widehat{C_T(\pi)}, \end{aligned} \tag{38}$$

where $\bar{\boldsymbol{\theta}}_{T,i}^{(*)}$ and $\bar{\boldsymbol{\theta}}_{T,j}^{(*)}$ indicate the i-th and j-th elements of the mean vector $\bar{\boldsymbol{\theta}}_T^{(*)}$.

For the proof of this theorem, please refer to A.5. The choice of $\mathcal{N}_N\left(\bar{\boldsymbol{\theta}}_T^{(*)}, V_T\right)$ to approximate the posterior distribution via the MFVI method is motivated by the favorable convergence property established for $\bar{\boldsymbol{\theta}}_T^{(*)}$ in Wang and Titterington [2012]. As discussed in Consonni and Marin [2007], with property 1, as the sample number T approaches infinity, $\bar{\boldsymbol{\theta}}_T^{(*)}$ converges locally to the ground-truth score $\boldsymbol{\theta}$ defined in Eq. (2) (i.e. $\bar{\boldsymbol{\theta}}_T^{(*)}$ converges to the ground-truth score $\boldsymbol{\theta}$ whenever the starting value is sufficiently near to it). According to Theorem. 3, we have that the proposed ranking from Eq. (37) will also converge locally to the ground-truth ranking π^*. In summary, our SUN Ranking algorithm employing the MFVI method is outlined below:

Algorithm 6 SUN Ranking Algorithm with Mean Field Variational Inference

---

Input X_T and y_T from $\mathcal{S}_T$, $\mathcal{Y}(\mathcal{S}_T)$ according to Eq. (14).
Output The proposed estimated ranking $\widehat{\pi}_T$.
1: Obtain the estimated distribution $\phi_N\left(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}_T^{(*)}, (I_N + X_T^\top X_T)^{-1}\right)$ from Alg. 5.
2: Obtain $\widehat{\pi}_T$ from the sorting of $\bar{\boldsymbol{\theta}}_T^{(*)}$.

We can also accerate our active learning framework using the MFVI method. Specifically, $\phi_N\left(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}_t^{(*)}, V_t\right)$ is denoted as the approximated distribution for $p(\boldsymbol{\theta}|\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N)$ through Alg. 5. We denote the estimated ranking $\widehat{\pi}_t^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)\}}$, which is generated from Alg. 6, emphasizing its dependence on the collected data $\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t)$. Additionally, $\Pr(y_{t+1} = s \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N)$, with $s = \pm 1$, can also be approximated. We define,

$$\begin{aligned} \hat{\Pr}(y_{t+1} = s \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) &:= \int \Phi\left(\frac{s(\theta_i - \theta_j)}{\sqrt{2}}\right) \phi_N\left(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}_t^{(*)}, V_t\right) d\boldsymbol{\theta} \\ &= \Phi\left(\frac{s(\bar{\boldsymbol{\theta}}_{t,i}^{(*)} - \bar{\boldsymbol{\theta}}_{t,j}^{(*)})}{\sqrt{2 + A_{ij}^\top V_t A_{ij}}}\right), \end{aligned} \tag{39}$$

where A_{ij} is introduced below Eq. (26). The integration result in Eq. (39) is due to Lemma 2.4.1 in Aziz [2011]. Then, Eq. (31) equipped with the MFVI method can be represented as:

$$(i_{t+1}, j_{t+1})_{KG} \approx \sum_{s=\pm 1} \underset{(i,j)}{\operatorname{argmax}} \hat{\Pr}(Y_{ij} = s \big| \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) \widehat{C_{t+1}}\left(\widehat{\pi_{t+1}}^{\{\mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i,j,Y_{ij}=s)\}}\right). \tag{40}$$

In summary, the active learning framework employing the MFVI method is outlined below:

Algorithm 7 SUN Ranking Active with Mean Field Variational Inference

---

Input The budget of sampling number T.
Output The estimated ranking $\widehat{\pi_T}$.
1: **for** $t = 0$ to $T - 1$ **do**
2: Initialize the compared pair set $\mathcal{S}_0 = \emptyset$ and the outcome set $\mathcal{Y}_0(\mathcal{S}_0) = \emptyset$.
3: Run Alg. 5 to obtain the estimated multivariate normal distribution with the input $\mathcal{S}_t \cup \{(i,j)\}$ and $\mathcal{Y}_t(\mathcal{S}_t) \cup \{Y_{ij}\}$ for each (i,j) with $i \neq j$ and $Y_{ij} = \pm 1$.
4: Select (i_{t+1}, j_{t+1}) using Eq. (40).
5: Record the outcome y_{t+1}.
6: Renew the collected data with $\mathcal{S}_{t+1} = \mathcal{S}_t \cup \{(i_{t+1}, j_{t+1})\}$, $\mathcal{Y}_{t+1}(\mathcal{S}_{t+1}) = \mathcal{Y}_t(\mathcal{S}_t) \cup \{y_{t+1}\}$.
7: **end for**
8: Obtain $\widehat{\pi}_T$ from Alg. 6, with the input $\mathcal{S}_T$ and $\mathcal{Y}_T(\mathcal{S}_T)$.

Remark 1. *As an approximation to Alg. 4, the primary aim of Alg. 7 is to reduce the computational costs when the sample size is large. Therefore, practically one can integrate Alg. 4 and Alg. 7 for implementation. Specifically, during early iterations with small sample numbers, one can employ Alg. 4; after reaching a large sample size, transitioning to Alg. 7 can be advantageous to alleviate the computational burden.*

6 Simulation Study

In this section, we demonstrate the effectiveness of our SUN Ranking Active algorithms through practical applications using synthetic datasets.

6.1 Experimental Setting

We explore two scenarios: the TM scenario, where the ground-truth comparison model is TM given in Eq. (3), and the BTL scenario, where the ground-truth comparison model is BTL given in Eq. (6). In both scenarios, we consider $N = 10$ items to be ranked. For the ground-truth scores $\boldsymbol{\theta}$, we generate $\theta_i \sim \text{Uniform}(0, 20)$ for each i in the TM scenario. In the BTL scenario, we generate $\boldsymbol{\theta}$ from a Dirichlet distribution, *i.e.*, $\boldsymbol{\theta} \sim \text{Dir}(\alpha)$ with $\alpha_i = 1$ for each i.

6.2 Implementation Details

We implemented two versions of our methods, namely SUN Ranking Active (Alg. 4) and SUN Ranking Active MFVI (Alg. 7). For the MFVI version, we set the threshold for the convergence of the ELBO to 10^{-2}, and the maximum number of iterations to 10^4. The experiments were conducted using MATLAB 2020a.

6.3 Compared Baselines

Our method is compared to the following approaches:

1. **Crowd-BT** Chen et al. [2013]: This method iteratively selects pairs to maximize the Expected Information Gain (EIG). It employs moment matching to approximate the posterior distribution and subsequently generates estimated rankings based on the sorting of the mean of the estimated posterior.

2. **AKG** Chen et al. [2016]: Similar to ours, this method iteratively selects pairs to optimize the expectation of Kendall's τ. Different from our method with a closed-form posterior, it approximates the posterior via moment matching in each iteration, and the estimated ranking is proposed from the approximated distribution.

3. **HodgeRank Active Supervised** Xu et al. [2018]: This method addresses a Maximum A Posterior (MAP) problem through ridge regression. It selects pairs for comparison by maximizing the EIG, with the final ranking estimated according to the posterior mean.

6.4 Kendall's τ Comparison

For each method, we repeat the experiment 25 times to mitigate the impact of randomness. The results are presented as the median of Kendall's τ across 25 runs, accompanied by the range spanning from the 0.25th to the 0.75th quantile values, as shown in Fig. 4.

Our SUN Ranking Active consistently outperforms other baselines for both the TM and BTL models. Specifically, under the TM model illustrated in the left panel of Fig. 4, our methods significantly outperform Crowd-BT and AKG, mainly because these two methods adopt the BTL model instead of the TM model. Comparing with HodgeRank Active Supervised, our algorithm's median curve converges more rapidly. This improved convergence rate can be attributed to our method's direct optimization of the expectation of Kendall's τ, which measures ranking accuracy, rather than the EIG adopted by HodgeRank Active Supervised. Besides, the 0.25th quantile of our SUN Ranking Active is no less than the median of HodgeRank Active Supervised after convergence. This enhanced Kendall's τ performance can be attributed to the fact that our algorithm is firmly anchored in the ground-truth posterior, while HodgeRank Active Supervised relies on ridge regression, which may introduce bias in item score estimation.

Under the BTL model illustrated in the right panel of Fig. 4, our method consistently outperforms HodgeRank Active Supervised, demonstrating the robustness of our approach. Surprisingly, even when considering the BTL model, our methods exhibit superior performance compared to Crowd-BT and AKG, both grounded in the BTL model. This unexpected result can be attributed to the fact that both AKG and Crowd-BT rely on moment matching to approximate a target distribution, which deviates from the ground-truth posterior with more than one sample. As the number of samples grows, the cumulative effect of such an approximation may lead to significant deviations between the approximated posterior and the ground-truth posterior. In contrast, our method directly derives rankings based on the closed-form posterior distribution.

Besides, it is interesting to note that our SUN Ranking Active MFVI, although an approximated algorithm, still achieves better or comparable Kendall's τ value than other methods. This can be attributed to the fact that the target distribution for approximation is the known form of the ground-truth posterior throughout the entire process, and such an approximated distribution has a minimum KL divergence between the target distribution in the mean field family, which ensures its robustness and performance empirically.

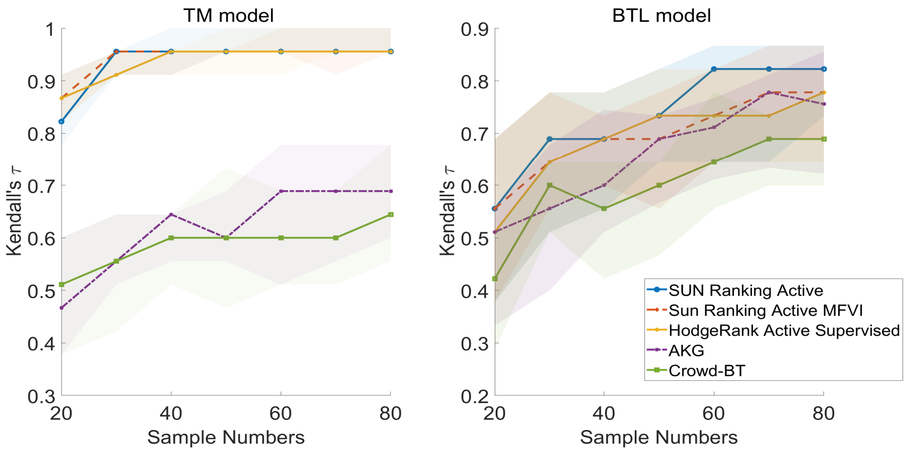

Figure 4: Error bar figures of the five methods, the x-axis represents the number of samples, and the y-axis represents Kendall's τ. In the left panel, the ground-truth comparison model is TM, while in the right panel, it is BTL.

6.5 Computational Costs

To illustrate the computational costs, we report the computation time for a trial of each method, as shown in Tab. 1. Despite the superior performance of our SUN Ranking Active in terms of Kendall's τ, it may incur higher computational costs, primarily due to the computation of high-dimensional multivariate normal CDFs or sampling from high-dimensional multivariate truncated normal distributions, especially when the sample number T is large. In contrast, Crowd-BT and AKG, with posterior approximation, can be much more efficient since their rankings are based on a simple estimated posterior throughout the entire process. Additionally, HodgeRank Active Supervised, although it may introduce bias in item scores estimation, possesses a closed-form posterior that is a simple multivariate normal distribution and can also enjoy computational efficiency.

It is worth noting that SUN Ranking Active MFVI outperforms the other methods in terms of computational speed in the simulated setting, making it a suitable choice when computational efficiency is a priority. On the other hand, our SUN Ranking Active method requires more time due to its higher computational demands. This trade-off suggests that if one seeks more accurate estimated rankings and is willing to address the computational burden through techniques such as parallel computing, then SUN Ranking Active becomes a valuable choice.

Table 1: Computational time for one trial (seconds)

Algorithms	$T = 10$	$T = 20$	$T = 50$
AKG	0.097	0.168	0.376
Crowd-BT	0.750	0.770	0.873
HodgeRank Active Supervised	0.740	0.759	0.788
SUN Ranking Active	1.519	4.756	82.451
SUN Ranking Active MFVI	**0.096**	**0.150**	**0.362**

7 Real-World Experiments

In this section, we evaluate our method on two real-world applications: *reading difficulty evaluation* and *quality assessment*. Due to computational efficiency, we implement SUN Ranking Active MFVI (Alg. 7) as our method. The compared baseline methods are the same in Sec. 6.3.

7.1 Datasets and Settings

Reading Difficulty. The dataset was proposed in Chen et al. [2016], which was extracted from CrowdFlower[3]. It contains 491 documents, each assigned with a reading difficulty score θ_i $(i = 1, ..., 491)$ ranging from 1 to 12. A higher score means greater difficulty in reading. The dataset contains a total of 7,898 pairwise comparison samples, where for each pair workers determined which document is more challenging to read. For simplicity, we choose the first 50 documents ($N = 50$) and the corresponding 692 pairwise comparison samples within these documents. To evaluate the effectiveness of an estimated ranking π, we calculate the accuracy defined in Chen et al. [2016]:

$$\text{Accuracy}(\pi) = \frac{2}{N(N-1)} \sum_{\pi(i)>\pi(j)} \mathbf{1}_{\{\theta_i \geq \theta_j\}}. \tag{41}$$

Video Quality Assessment (VQA) and Image Quality Assessment (IQA). The VQA dataset Xu et al. [2011] was extracted from the LIVE dataset Live:2008. It contains 10 distinct reference images, with each reference featuring 16 unique distorted versions ($N = 16$), comprising a total of 43,266 pair-wise comparisons. The Image Quality Assessment (IQA) dataset, extracted from IVC:2005 and Live:2008, comprises a total of 43,266 pair-wise comparisons. This dataset includes 15 reference images, each accompanied by 16 different distortions ($N = 16$).

Given the absence of ground-truth scores in these datasets, we employ the HodgeRank estimator Jiang et al. [2011], which has been widely used Ghadiyaram and Bovik [2015], Li et al. [2016], Xu et al. [2018] to estimate the global ranking from paired comparison data, due to its effectiveness and theoretically robustness in establishing the global ranking. Similar to the *reading difficulty* dataset, we use the accuracy to evaluate the estimated ranking. It is worthwhile to note that the accuracy defined in Eq. (41) becomes Kendall's τ when the ground-truth scores differ between elements, as is VQA and IQA.

7.2 Implementation Details

For our SUN Ranking Active MFVI, we set the convergence threshold of ELBO at 10^{-2}, and the maximum number of iteration steps to 10^4.

[3] http://www.crowdflower.com

7.3 Results Analysis

Similar to simulation experiments, we report the median, together with the error bar from 25th quantile to 75th quantile over 25 independent trials for each dataset. As shown in Fig. 5, 7, 6, our methods are better or comparable to other methods.

Specifically, in the reading difficulty dataset (refer to Fig. 5), our SUN Ranking Active MFVI still demonstrates superior performance over other methods. Notably, the 0.25th quantile achieves a higher Kendall's τ value than the median value obtained by other methods. This result demonstrates the utility of our SUN Ranking Active MFVI method.

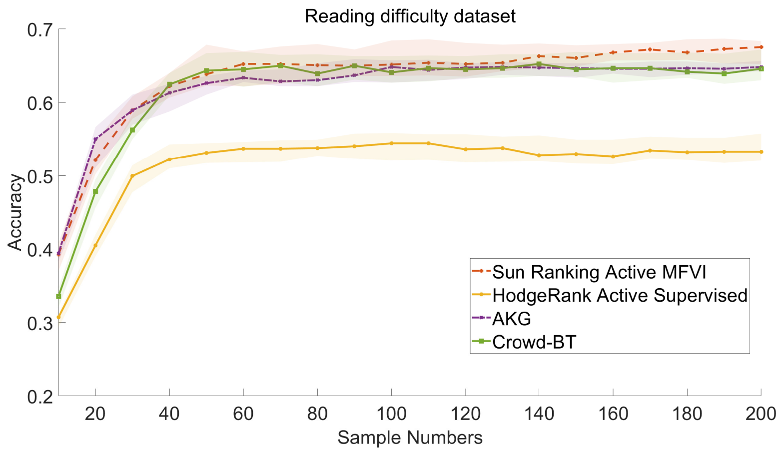

Figure 5: Error bar figure of the four methods in the reading difficulty dataset. The x-axis represents the number of samples, and the y-axis represents Accuracy, as defined in Eq. (41).

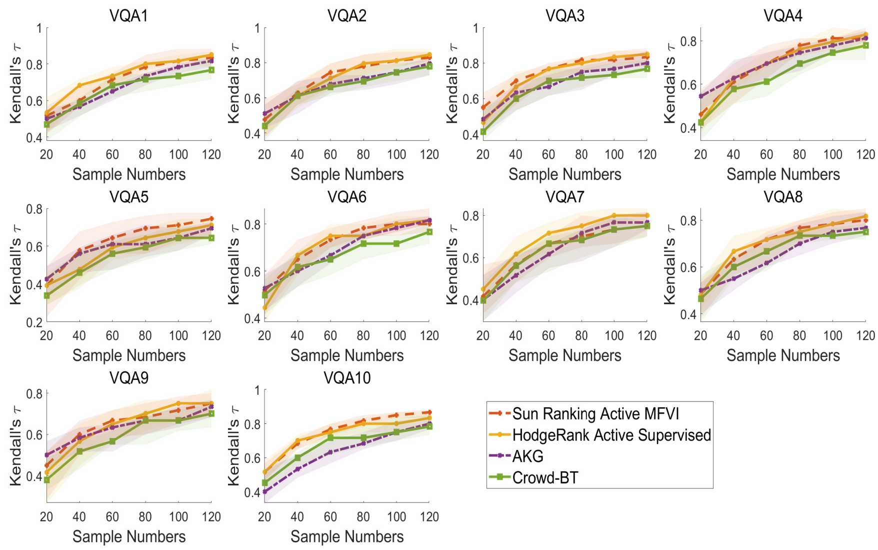

Figure 6: Error bar figures of the four methods in VQA datasets. The x-axis represents the number of samples, and the y-axis represents Kendall's τ. Each figure with its title represents the corresponding datasets in VQA.

In the VQA dataset (refer to Fig. 6) and the IQA dataset (refer to Fig. 7), the median curve of our SUN Ranking Active MFVI is comparable to that of HodgeRank Active Supervised, and surpassing that of AKG and Crowd-BT. It's worth noting that the ground-truth score

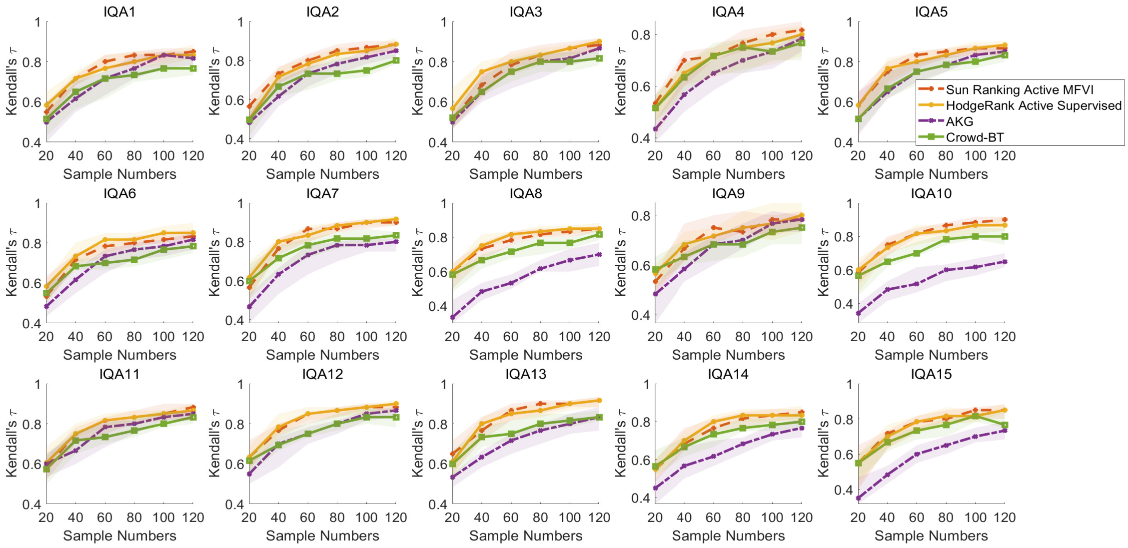

Figure 7: Error bar figures of the four methods in IQA datasets. The x-axis represents the number of samples, and the y-axis represents Kendall's τ. Each figure with its title represents the corresponding datasets in IQA.

is estimated via the HodgeRank estimator, which is more aligned with the objective of the HodgeRank Active Supervised method for active learning. This result further emphasizes the robustness of our SUN Ranking Active approach.

8 Conclusions and Discussions

In conclusion, our paper presents a pioneering Bayesian conjugate framework built upon the Thurstone-Mosteller model, featuring a posterior distribution conforming to the unified skew-normal distribution. We propose a ranking criterion that optimizes the expectation of Kendall's τ over the ground-truth posterior, enabling accurate ranking inferences. Under certain conditions, the resulting ranking converges to the ground-truth ranking. By maximizing this objective and leveraging the known form of the posterior distribution, we extend our algorithm from offline to an active sampling strategy using the knowledge gradient method. Additionally, we present an approximated version of our framework using Mean Field Variational Inference, significantly improving computational speeds. Empirically, our methods outperform the compared baselines in terms of ranking accuracy and computational costs, underscoring the utility and efficiency of our approach.

While our work represents a significant advancement, several promising directions for future research remain. For example, it is desired to establish sufficient and necessary conditions for the absence of loops when determining the optimal ranking in terms of the expectation of Kendall's τ, which may require the evaluation of complex and non-closed-form multivariate normal cumulative distribution functions. Furthermore, it is interesting to extend our framework by incorporating variations in annotator reliability on crowdsourcing platforms while preserving conjugate properties. This extension would enhance the applicability of our model to real-world scenarios with diverse annotator expertise. Additionally, in the context of Variational Inference, exploring other families of distributions beyond multivariate normal could be pursued to enhance approximation accuracy without compromising computational efficiency.

Appendix A Proofs

A.1 Proof of Lemma 2 for the Derivation of SUN Distribution

Proof. of Lemma 2. From Eq. (13) and $\tilde{\boldsymbol{\theta}} = D_T\boldsymbol{\theta} + \varepsilon$, the joint distribution of $\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}$ is given by

$$\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix} \sim \mathcal{N}_{N+T}\left(\begin{array}{c} \mathbf{0}_N \\ \mathbf{0}_T \end{array}, \left(\begin{array}{cc} I_N & D_T^\top \\ D_T & D_T D_T^\top + I_T \end{array} \right)\right).$$

According to properties of the multivariate normal distribution, the conditional distribution $p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta})$ takes the form

$$p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}) \sim \mathcal{N}_T\left(D_T\boldsymbol{\theta}, I_T\right).$$

Thus, $\Pr(\tilde{\boldsymbol{\theta}} > 0|\boldsymbol{\theta})$ is given by

$$\Pr(\tilde{\boldsymbol{\theta}} > 0|\boldsymbol{\theta}) = \Phi_T\left(D_T\boldsymbol{\theta}, I_T\right) = \Phi_T\left(s_T^{-1}D_T\boldsymbol{\theta}, s_T^{-1}s_T^{-1}\right), \tag{42}$$

where $s_T = \operatorname{diag}\left((D_T(1,:)D_T(1,:)^\top + 1)^{\frac{1}{2}}, \ldots, (D_T(T,:)D_T(T,:)^\top + 1)^{\frac{1}{2}}\right)$, with $D_T(i,:)$ representing the i-th row of D_T for $i = 1, \ldots, T$.

On the other hand, the joint distribution of $\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}$, with the correlation matrix form, is

$$\begin{pmatrix} \boldsymbol{\theta} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix} \sim \mathcal{N}_{N+T}\left(\begin{array}{c} \mathbf{0}_N \\ \mathbf{0}_T \end{array}, \left(\begin{array}{cc} I_N & \Delta_T \\ \Delta_T^\top & \Gamma_T \end{array} \right)\right),$$

where $\Delta_T = D_T^\top s_T^{-1}$, $\Gamma_T = s_T^{-1}(D_T D_T^\top + I_T)s_T^{-1}$. In this scenario,

$$p(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}) \sim \mathcal{N}_T\left(\Delta_T^\top\boldsymbol{\theta}, \Gamma_T - \Delta_T^\top\Delta_T\right) = \mathcal{N}_T\left(s_T^{-1}D_T\boldsymbol{\theta}, s_T^{-1}s_T^{-1}\right).$$

Hence, from the above equation, we also obtain

$$\Pr(\tilde{\boldsymbol{\theta}} > 0|\boldsymbol{\theta}) = \Phi_T\left(s_T^{-1}D_T\boldsymbol{\theta}, s_T^{-1}s_T^{-1}\right),$$

which is identical to Eq. (42). Since the distribution of $p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0)$ is given by

$$p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}} > 0) \propto \Pr(\tilde{\boldsymbol{\theta}} > 0|\boldsymbol{\theta})p(\boldsymbol{\theta}) = \Phi_T\left(s_T^{-1}D_T\boldsymbol{\theta}, s_T^{-1}s_T^{-1}\right)p(\boldsymbol{\theta}), \tag{43}$$

we conclude the proof. □

A.2 Proof of Proposition 1 for the CDF form of $\mathrm{Pred}_T(i > j)$

In this section, we begin by exploring the expectation of the TM model for item i being preferred over item j with a tuning parameter $M > 0$. This expectation is computed over the posterior distribution given in Eq. (17). The resulting expression is as follows:

$$\int \Phi\left(M(\theta_i - \theta_j)\right) p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N)d\boldsymbol{\theta}. \tag{44}$$

The following lemma states the relationship between Eq. (44), and $\mathrm{Pred}_T(i > j)$.

Lemma 3. *As $M \to +\infty$, the integration in Eq. (44) will converge to $\mathrm{Pred}_T(i > j)$ which is defined in Eq. (21), i.e. we have*

$$\lim_{M\to+\infty} \int \Phi\left(M(\theta_i - \theta_j)\right) p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N)d\boldsymbol{\theta} = \mathrm{Pred}_T(i > j). \tag{45}$$

Proof. of Lemma 3. According to the dominated convergence theorem and the fact that $|\Phi(M(\theta_i-\theta_j))|\le 1$, we have

$$\begin{aligned}
\lim_{M\to+\infty}\int \Phi\left(M(\theta_i-\theta_j)\right)p(\boldsymbol{\theta}|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N)d\boldsymbol{\theta}&\\
&=\int \lim_{M\to+\infty}\Phi\left(M(\theta_i-\theta_j)\right)p(\boldsymbol{\theta}|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N)d\boldsymbol{\theta}\\
&=\mathbb{E}\left[\lim_{M\to+\infty}\Phi(M(\theta_i-\theta_j))|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N\right]\\
&\overset{(1)}{=}\mathbb{E}[\mathbf{1}_{\{\theta_i-\theta_j>0\}}|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N]\\
&=\Pr(\theta_i>\theta_j|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),0_N,I_N)=\mathrm{Pred}_T(i>j),
\end{aligned}\tag{46}$$

where "(1)" is due to that $\lim_{M\to\infty}\Phi(M(\theta_i-\theta_j))=1$ if $\theta_i-\theta_j>0$, $=0.5$ if $\theta_i-\theta_j=0$ and $=0$ otherwise, which ends our proof. □

With the above lemma established, we can prove Proposition 1.

Proof. of Proposition 1. Expanding the form of the posterior distribution as shown in Eq. (17), the integration in Eq. (44) can be represented as:

$$\begin{aligned}
(44)&=\int \Phi\left(M(\theta_i-\theta_j)\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N)\frac{\Phi_T\left(\Delta_T^\top\boldsymbol{\theta};\Gamma_T-\Delta_T^\top\Delta_T\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}d\boldsymbol{\theta}\\
&=\int \phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N)\frac{\Phi_{T+1}\left(\begin{pmatrix}\Delta_T^\top\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix};\Omega_{\text{new}}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}d\boldsymbol{\theta},
\end{aligned}\tag{47}$$

where D_{ij} is an N-dimensional vector with its i-th element set to M, j-th element set to $-M$, and other elements set to 0. Besides, $\Omega_{\text{new}}=\begin{pmatrix}\Gamma_T-\Delta_T^\top\Delta_T & \mathbf{0}_T\\ \mathbf{0}_T^\top & 1\end{pmatrix}$. Recall the derivation of Γ_T and Δ_T below Eq. (15), we have $\Gamma_T-\Delta_T^\top\Delta_T=s_T^{-1}s_T^{-1}$, with s_T also defined below Eq. (15). Then we have

$$\Phi_{T+1}\left(\begin{pmatrix}\Delta_T^\top\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix};\Omega_{\text{new}}\right)=\Phi_{T+1}\left(\begin{pmatrix}\Delta_T^\top\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix};\begin{pmatrix}s_T^{-1}s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & 1\end{pmatrix}\right),\tag{48}$$

which is a multivariate normal CDF with a diagonal variance-covariance matrix. Given that a univariate normal CDF with mean 0, variance b , and evaluated a satisfies: $\Phi(a;b)=\Phi(\frac{a}{\sqrt{b}})$, we will have

$$(48)=\Phi_{T+1}\left(\begin{pmatrix}D_T\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix};\begin{pmatrix}I_T & \mathbf{0}_T\\ \mathbf{0}_T^\top & 1\end{pmatrix}\right)=\Phi_{T+1}\left((D_{\text{new}}^{\{D_T,(i,j)\}})\boldsymbol{\theta};I_{\text{T+1}}\right),\tag{49}$$

where $D_{\text{new}}^{\{D_T,(i,j)\}}\in\mathbb{R}^{(T+1)\times N}$, and the first T rows of $D_{\text{new}}^{\{D_T,(i,j)\}}$ is D_T, with the $(T+1)$-th row of $D_{\text{new}}^{\{D_T,(i,j)\}}$ being $D_{ij}^\top$. Imitating the derivation of SUN distribution in Eq. (16), we denote,

$$\tilde{\boldsymbol{\theta}}_{\text{new}}=D_{\text{new}}^{\{D_T,(i,j)\}}\boldsymbol{\theta}+\varepsilon_{\text{new}},\ \text{ where }\ \varepsilon_{\text{new}}\sim\mathcal{N}_{T+1}(\mathbf{0}_{T+1},I_{T+1})\text{ is independent of }\boldsymbol{\theta}.$$

Then we have

$$\Pr(\tilde{\boldsymbol{\theta}}_{\text{new}}>0|\boldsymbol{\theta})=\Pr(\varepsilon_{\text{new}}>-D_{\text{new}}^{\{D_T,(i,j)\}}\boldsymbol{\theta}|\boldsymbol{\theta})=\prod_{t=1}^{T+1}\Phi\left(D_{\text{new}}^{\{D_T,(i,j)\}}(t,:)\boldsymbol{\theta}\right)=(49)$$

where $D_{\text{new}}^{\{D_T,(i,j)\}}(t,:)$ is the t-th row of $D_{\text{new}}^{\{D_T,(i,j)\}}$. Equipped with the above equation we can also have

$$
\begin{aligned}
p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}}_{\text{new}}>0) \propto \Pr(\tilde{\boldsymbol{\theta}}_{\text{new}}>0|\boldsymbol{\theta})p(\boldsymbol{\theta}) &= \prod_{t=1}^{T+1} \Phi\left(D_{\text{new}}^{\{D_T,(i,j)\}}(t,:)\boldsymbol{\theta}\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N) \\
&= \Phi_{T+1}\left((D_{\text{new}}^{\{D_T,(i,j)\}})\boldsymbol{\theta}; I_{T+1}\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N) \\
&\overset{(1)}{=} \Phi_{T+1}\left(\begin{pmatrix}\Delta_T^\top\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix}; \begin{pmatrix} s_T^{-1}s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & 1\end{pmatrix}\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N) \\
&\overset{(2)}{=} \Phi_{T+1}\left(\begin{pmatrix}D_T\boldsymbol{\theta}\\ D_{ij}^\top\boldsymbol{\theta}\end{pmatrix}; \Omega_{\text{new}}\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N) \\
&\overset{(3)}{=} \Phi\left(M(\theta_i-\theta_j)\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N)\Phi_T\left(\Delta_T^\top\boldsymbol{\theta};\Gamma_T-\Delta_T^\top\Delta_T\right),
\end{aligned}
\tag{50}
$$

which is the kernel of the integration in Eq. (47). In Eq. (50), "(1)" is a consequence of Eq. (49) and Eq. (48), "(2)" arises from Eq. (48), and "(3)" is a result of Eq. (47). Recalling that $\tilde{\boldsymbol{\theta}}_{\text{new}} = D_{\text{new}}^{\{D_T,(i,j)\}}\boldsymbol{\theta} + \varepsilon_{\text{new}}$, we firstly have the $(N+T+1)$-dimensional multivariate normal distribution of $\begin{pmatrix}\boldsymbol{\theta}\\ \tilde{\boldsymbol{\theta}}_{\text{new}}\end{pmatrix}$ with mean and correlation matrix represented below:

$$
\begin{pmatrix}\boldsymbol{\theta}\\ \tilde{\boldsymbol{\theta}}_{\text{new}}\end{pmatrix} \sim \mathcal{N}_{N+T+1}\left(\begin{pmatrix}\mathbf{0}_N\\ \mathbf{0}_{T+1}\end{pmatrix}, \begin{pmatrix} I_N & \Delta_{\text{new}}^{\{\Delta_T,(i,j)\}}\\ (\Delta_{\text{new}}^{\{\Delta_T,(i,j)\}})^\top & \Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}}\end{pmatrix}\right),
\tag{51}
$$

parameters of which are defined as following. We first denote $s_{\text{new}}^{\{s_T,(i,j)\}} := \begin{pmatrix} s_T & \mathbf{0}_T\\ \mathbf{0}_T^\top & \sqrt{2M^2+1}\end{pmatrix}$, and then we have

$$
\begin{aligned}
\Delta_{\text{new}}^{\{\Delta_T,(i,j)\}} &= (D_{\text{new}}^{\{D_T,(i,j)\}})^\top (s_{\text{new}}^{\{s_T,(i,j)\}})^{-1} \\
&= (D_T^\top, D_{ij})\begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix}.
\end{aligned}
$$

Then $\Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}}$ can be represented as:

$$
\begin{aligned}
\Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}} &= \begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix}\left(D_{\text{new}}^{\{D_T,(i,j)\}}\left(D_{\text{new}}^{\{D_T,(i,j)\}}\right)^\top + I_{T+1}\right)\begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix} \\
&= \begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix}\left(\begin{pmatrix}D_T\\ D_{ij}^\top\end{pmatrix}(D_T^\top, D_{ij}) + I_{T+1}\right)\begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix} \\
&= \begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix}\begin{pmatrix} D_TD_T^\top + I_T & D_TD_{ij}\\ D_{ij}^\top D_T^\top & D_{ij}^\top D_{ij}+1\end{pmatrix}\begin{pmatrix} s_T^{-1} & \mathbf{0}_T\\ \mathbf{0}_T^\top & \frac{1}{\sqrt{2M^2+1}}\end{pmatrix} \\
&= \begin{pmatrix} s_T^{-1}(D_TD_T^\top + I_T)s_T^{-1} & \frac{s_T^{-1}D_TD_{ij}}{\sqrt{2M^2+1}}\\ \frac{D_{ij}^\top D_T^\top s_T^{-1}}{\sqrt{2M^2+1}} & \frac{D_{ij}^\top D_{ij}+1}{2M^2+1}\end{pmatrix} = \begin{pmatrix}\Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}}\\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1\end{pmatrix},
\end{aligned}
$$

recalling Γ_T defined below Eq. (15) and $\Delta_T(ij)$ defined in Eq. (23). From Lemma 2 and the derivation of SUN variable above Eq. (8), we have

$$
p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}}_{\text{new}}>0) \sim \text{SUN}_{N,T+1}\left(\mathbf{0}_N, I_N, \Delta_{\text{new}}^{\{\Delta_T,(i,j)\}}, \mathbf{0}_{T+1}, \Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}}\right).
$$

From Eq. (50), the result of the integration

$$
\int \Phi\left(M(\theta_i-\theta_j)\right)\phi_N(\boldsymbol{\theta};\mathbf{0}_N,I_N)\Phi_T\left(\Delta_T^\top\boldsymbol{\theta};\Gamma_T-\Delta_T^\top\Delta_T\right)d\boldsymbol{\theta}
$$

corresponds to the normalization constant of $p(\boldsymbol{\theta}|\tilde{\boldsymbol{\theta}}_{\text{new}} > 0)$, which is $\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}}\right)$. Then the integration result in Eq. (47) will be

$$(47) = \frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma_{\text{new}}^{\{\Gamma_T,(i,j)\}}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} = \frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}\Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}} \\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1\end{pmatrix}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}. \tag{52}$$

From Plackett [1954], we know that the multivariate normal CDF is continuous with respect to the correlation term in the variance-covariance matrix. Equipped with Lemma 3, we will have

$$(47) \to \frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}\Gamma_T & \frac{\Delta_T(ij)^\top}{\sqrt{2}} \\ \frac{\Delta_T(ij)}{\sqrt{2}} & 1\end{pmatrix}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} = \text{Pred}_T(i > j) \text{ as } M \to +\infty,$$

which ends our proof. □

A.3 Proof of Theorem 1 and Corollary 1 for the Absence of Loops

From the proof in Section A.2, $\Phi(M(\theta_i - \theta_j))$ can be regarded as a likelihood function in our SUN distribution, indicating that we observe a new sample with item i beating item j with a tuning parameter M (when $M = \frac{1}{\sqrt{2}}$, this likelihood function will become the TM model in Eq. (3)). Since we have the convergent equation in Eq. (45), we then denote the integration in Eq. (44) as:

$$\text{Pred}_T(i > j, M) \equiv \int \Phi\left(M(\theta_i - \theta_j)\right) p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N)\, d\boldsymbol{\theta}, \tag{53}$$

for notational consistency. This is in accordance with the definition of $\text{Pred}_T(i > j)$ in Eq. (21), since from Lemma 3 we have $\text{Pred}_T(i > j, +\infty) = \text{Pred}_T(i > j)$.

After $\text{Pred}_T(i > j, M)$ is defined in Eq. (53), we then propose a Lemma to state that $\text{Pred}_T(i > j, M)$ can also be derived in the form of an integration with respect to a univariate random variable.

Lemma 4. $\text{Pred}_T(i > j, M)$ *defined in Eq.* (53) *can be expressed as*

$$\text{Pred}_T(i > j, M) = \int \Phi\left(M\theta_{ij}\right) p(\theta_{ij}) d\theta_{ij}, \tag{54}$$

where θ_{ij} is a univariate SUN-distributed random variable introduced in Eq. (26).

Proof. of Lemma 4. Recall that $\boldsymbol{\theta} \sim \text{SUN}_{N,T}\left(\mathbf{0}_N, I_N, \Delta_T, \mathbf{0}_T, \Gamma_T\right)$ as shown in Eq. (16), according to Eq. (12) in Lemma 1, the distribution of θ_{ij} will be

$$\text{SUN}_{N,1}\left(\mathbf{0}_N + A_{ij}^\top \mathbf{0}_N, A_{ij}^\top I_N A_{ij}, \Delta_{A_{ij}}, \mathbf{0}_T, \Gamma_T\right). \tag{55}$$

$\Delta_{A_{ij}} = \text{diag}(A_{ij}^\top I_N A_{ij})^{-\frac{1}{2}} A_{ij}^\top I_N \Delta_T = \frac{\Delta_T(ij)}{\sqrt{2}}$, with $\Delta_T(ij)$ defined in Eq. (23). Simplify the expression in Eq. (55), we will have

$$\theta_{ij} \sim \text{SUN}_{N,1}\left(\mathbf{0}_N, 2, \frac{\Delta_T(ij)}{\sqrt{2}}, \mathbf{0}_T, \Gamma_T\right). \tag{56}$$

Then the integration in Eq. (54) can be expressed as

$$\int \Phi\left(M\theta_{ij}\right) \phi_1(\theta_{ij}; 0, 2) \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} d\theta_{ij}, \tag{57}$$

where $\phi_1(\theta_{ij};0,2)$ is a univariate normal distribution at θ_{ij}, with mean 0 and variance 2. To obtain the corresponding result in the above equation we first note that

$$(57) = \int \phi_1(\theta_{ij};0,2)\frac{\Phi_{T+1}\left(\left(\frac{\Delta_T(ij)}{2}, M\right)^\top \theta_{ij}; \begin{pmatrix} \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2} & \mathbf{0}_T \\ \mathbf{0}_T^\top & 1 \end{pmatrix}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} d\theta_{ij}. \tag{58}$$

Before going further, recall the term $\Delta_T(ij)$ which is a $1\times T$ matrix defined in Eq. (23) and the absolute maximum value of it can be achieved via its t-th element if item i beats item j in the t-th comparison. The corresponding maximum value can be $\frac{1}{2} - (-\frac{1}{2}) = 1$, and $\frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} = \frac{\Delta_T(ij)}{\sqrt{2+\frac{1}{M^2}}} < \frac{1}{\sqrt{2}}$ when $M > 0$. we then assume we have a $(T+2)$-dimensional multivariate normal distribution with mean and correlation matrix represented below:

$$\begin{pmatrix} \theta_{ij} \\ \tilde{\boldsymbol{\theta}}'_{\text{new}} \end{pmatrix} \sim \mathcal{N}_{T+2}\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0}_{T+1} \end{pmatrix}, \begin{pmatrix} 1 & \left(\frac{\Delta_T(ij)}{\sqrt{2}}, \frac{\sqrt{2}M}{\sqrt{2M^2+1}}\right) \\ \left(\frac{\Delta_T(ij)}{\sqrt{2}}, \frac{\sqrt{2}M}{\sqrt{2M^2+1}}\right)^\top & \begin{pmatrix} \Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}} \\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1 \end{pmatrix} \end{pmatrix}\right),$$

where $\frac{\sqrt{2}M}{\sqrt{2M^2+1}} = \frac{\sqrt{2}}{\sqrt{2+\frac{1}{M^2}}} < 1$ when $M \in (0, +\infty)$. From the derivation of SUN distribution below Eq. (7), we have

$$\sqrt{2}(\theta_{ij}|\tilde{\boldsymbol{\theta}}'_{\text{new}} > 0) \sim \text{SUN}_{1,T+1}(0, 2, \Delta', \mathbf{0}_{T+1}, \Gamma'),$$

where

$$\Delta' = \left(\frac{\Delta_T(ij)}{\sqrt{2}}, \frac{\sqrt{2}M}{\sqrt{2M^2+1}}\right), \text{ and } \Gamma' = \begin{pmatrix} \Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}} \\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1 \end{pmatrix}. \tag{59}$$

From Eq. (8), the probability density function of $\sqrt{2}(\theta_{ij}|\tilde{\boldsymbol{\theta}}'_{\text{new}} > 0)$ will be

$$\begin{aligned}
&\phi_1(\theta_{ij};0,2)\frac{\Phi_T\left(\frac{1}{\sqrt{2}}(\Delta')^\top\theta_{ij}; \Gamma' - (\Delta')^\top\Delta'\right)}{\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma'\right)} \\
&= \phi_1(\theta_{ij};0,2) \\
&\cdot \frac{\Phi_{T+1}\left((\frac{\Delta_T(ij)}{2}, \frac{M}{\sqrt{2M^2+1}})^\top\theta_{ij}; \begin{pmatrix} \Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}} \\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1 \end{pmatrix} - \begin{pmatrix} \frac{\Delta_T(ij)^\top}{\sqrt{2}} \\ \frac{\sqrt{2}M}{\sqrt{2M^2+1}} \end{pmatrix} \left(\frac{\Delta_T(ij)}{\sqrt{2}}, \frac{\sqrt{2}M}{\sqrt{2M^2+1}}\right)\right)}{\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma'\right)} \\
&= \phi_1(\theta_{ij};0,2)\frac{\Phi_{T+1}\left((\frac{\Delta_T(ij)}{2}, \frac{M}{\sqrt{2M^2+1}})^\top\theta_{ij}; \begin{pmatrix} \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2} & \mathbf{0}_T \\ \mathbf{0}_T^\top & \frac{1}{2M^2+1} \end{pmatrix}\right)}{\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma'\right)} \\
&= \phi_1(\theta_{ij};0,2)\frac{\Phi_{T+1}\left((\frac{\Delta_T(ij)}{2}, M)^\top\theta_{ij}; \begin{pmatrix} \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2} & \mathbf{0}_T \\ \mathbf{0}_T^\top & 1 \end{pmatrix}\right)}{\Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma'\right)},
\end{aligned}$$

the kernel of which is the same with the kernel in the integration in Eq. (58). Then we can have

$$\int \phi_1(\theta_{ij};0,2)\Phi_{T+1}\left(\left(\frac{\Delta_T(ij)}{2}, M\right)^\top \theta_{ij}; \begin{pmatrix} \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2} & \mathbf{0}_T \\ \mathbf{0}_T^\top & 1 \end{pmatrix}\right) d\theta_{ij} = \Phi_{T+1}\left(\mathbf{0}_{T+1};\Gamma'\right),$$

where $\Phi_{T+1}(\mathbf{0}_{T+1};\Gamma')$ is the normalization constant of $\mathrm{SUN}_{1,T+1}(0,2,\Delta',\mathbf{0}_{T+1},\Gamma')$. Then we can have

$$(58)=\frac{\Phi_{T+1}(\mathbf{0}_{T+1};\Gamma')}{\Phi_T(\mathbf{0}_T;\Gamma_T)}=\frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}\Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}}\\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1\end{pmatrix}\right)}{\Phi_T(\mathbf{0}_T;\Gamma_T)}=(57).$$

From Eq. (52), we know that

$$\mathrm{Pred}_T(i>j,M)=\frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}\Gamma_T & \frac{M\Delta_T(ij)^\top}{\sqrt{2M^2+1}}\\ \frac{M\Delta_T(ij)}{\sqrt{2M^2+1}} & 1\end{pmatrix}\right)}{\Phi_T(\mathbf{0}_T;\Gamma_T)}=(57),$$

which ends our proof for $M\in(0,+\infty)$.

For $M=\infty$, note that

$$\begin{aligned}\lim_{M\to+\infty}\mathrm{Pred}_T(i>j,M)&=\lim_{M\to+\infty}\int\Phi(M\theta_{ij})\,p(\theta_{ij})d\theta_{ij}\\&\overset{(1)}{=}\int\lim_{M\to+\infty}\Phi(M\theta_{ij})\,p(\theta_{ij})d\theta_{ij}\\&\overset{(2)}{=}\int_{\mathbf{1}_{\{\theta_{ij}>0\}}}p(\theta_{ij})d\theta_{ij}=\Pr(\theta_{ij}>0),\end{aligned}$$

where "(1)" is due to the dominated convergence theorem, and "(2)" is due to the converged function of $\Phi(M\theta_{ij})$ when $M\to\infty$ as depicted below Eq. (46). Given the distribution of θ_{ij} in Eq. (56) and the CDF function of SUN distribution in Eq. (9) in Lemma 1, we have that

$$\begin{aligned}\Pr(\theta_{ij}>0)=1-\Pr(\theta_{ij}\le 0)=1-&\frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}1 & \frac{-\Delta_T(ij)}{\sqrt{2}}\\ \frac{-\Delta_T(ij)^\top}{\sqrt{2}} & \Gamma_T\end{pmatrix}\right)}{\Phi_T(\mathbf{0}_T;\Gamma_T)}\\=&\frac{\Phi_{T+1}\left(\mathbf{0}_{T+1};\begin{pmatrix}1 & \frac{\Delta_T(ij)}{\sqrt{2}}\\ \frac{\Delta_T(ij)^\top}{\sqrt{2}} & \Gamma_T\end{pmatrix}\right)}{\Phi_T(\mathbf{0}_T;\Gamma_T)}=\mathrm{Pred}_T(i>j),\end{aligned}$$

which ends our proof.

□

Now, we establish Lemma 5, which forms the basis for the condition of loop absence, linking the posterior mean and $\mathrm{Pred}_T(i>j)$. Firstly, we state this lemma, and then we provide the proof of Theorem 1.

Lemma 5. *If $\boldsymbol{\theta}$ follows the posterior distribution as described in Eq. (17) and $\mathbb{E}[\theta_i-\theta_j]>0$, then there exists a specific value $M_0>0$ such that $\mathrm{Pred}_T(i>j,M_0)>0.5$, with $\mathrm{Pred}_T(i>j,M)$ defined in Eq. (53).*

Proof. of Lemma 5. From the definition of $\mathrm{Pred}_T(i>j,M)$ in Eq. (53) and Lemma 4, $\mathrm{Pred}_T(i>j,M)$ will take the form:

$$\begin{aligned}\mathrm{Pred}_T(i>j,M)&=\int\Phi(M(\theta_i-\theta_j))\,p(\boldsymbol{\theta}|\mathcal{S}_T,\mathcal{Y}_T(\mathcal{S}_T),\mathbf{0}_N,I_N)d\boldsymbol{\theta}\\&=\int\Phi(M\theta_{ij})\,p(\theta_{ij})d\theta_{ij}\\&=\int\Phi(M\theta_{ij})\,\phi_1(\theta_{ij};0,2)\frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{i-j};\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)}{\Phi_T(\mathbf{0}_T;\Gamma_T)}d\theta_{ij}.\end{aligned}\qquad(60)$$

Here, θ_{ij} is a univariate random variable the distribution of which is described in Eq. (26). Since M is a tuning parameter, to delve deeper, we take derivatives of $\text{Pred}_T(i > j, M)$ with respect to M:

$$\begin{aligned}\frac{d\text{Pred}_T(i > j, M)}{dM} = \int \theta_{ij}\phi_1\left(M\theta_{ij}; 0, 1\right)\phi_1(\theta_{ij}; 0, 2)&\\ \cdot \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)}d\theta_{ij}&\end{aligned} \tag{61}$$

Note that Eq. (61) is proportional to the expectation of a random variable, denoted as Y, whose probability density function takes the form:

$$\frac{1}{C_Y}\phi_1(MY; 0, 1)\phi_1(Y; 0, 2)\Phi_T\left(\frac{\Delta_\top(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right). \tag{62}$$

Here, C_Y is a normalization constant. The moment generating function of Y is:

$$\begin{aligned}&\int \frac{1}{C_Y}e^{tY}\phi_1(MY; 0, 1)\phi_1(Y; 0, 2)\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY\\ &= \int \frac{1}{C'_Y}e^{tY}e^{\frac{-M^2Y^2}{2}}e^{\frac{-Y^2}{4}}\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY\\ &= \int \frac{1}{C'_Y}e^{-\frac{(2M^2+1)Y^2-4tY}{4}}\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY\\ &= \int \frac{1}{C'_Y}e^{\frac{t^2}{2M^2+1}}e^{-\frac{(2M^2+1)\left(Y-\frac{2t}{2M^2+1}\right)^2}{4}}\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY\\ &= \frac{1}{C''_Y}e^{\frac{t^2}{2M^2+1}}\int \phi_1\left(Y; \frac{2t}{2M^2+1}, \frac{2}{2M^2+1}\right)\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY,\end{aligned} \tag{63}$$

where C'_Y and C''_Y are constants that do not affect the results. The integration part in Eq. (63), from Lemma 2.4.1 in Aziz [2011], will be:

$$\begin{aligned}&\int \phi_1\left(Y; \frac{2t}{2M^2+1}, \frac{2}{2M^2+1}\right)\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}Y; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)dY\\ &= \Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\cdot\frac{2t}{2M^2+1}; \Gamma_T - \frac{\Delta_T(ij)^\top\Delta_T(ij)}{2} + \frac{\Delta_T(ij)^\top}{2}\frac{2}{2M^2+1}\frac{\Delta_T(ij)}{2}\right)\\ &= \Phi_T\left(\frac{\Delta_T(ij)^\top}{2M^2+1}t; \Gamma_T - \frac{M^2}{2M^2+1}\Delta_T(ij)^\top\Delta_T(ij)\right).\end{aligned} \tag{64}$$

So, the moment generating function takes the form:

$$\frac{1}{C''_Y}e^{\frac{t^2}{2M^2+1}}\Phi_T\left(\frac{\Delta_T(ij)^\top}{2M^2+1}t; \Gamma_T - \frac{M^2}{2M^2+1}\Delta_T(ij)^\top\Delta_T(ij)\right). \tag{65}$$

Then the cumulant generating function of Y will be:

$$K_Y(t) = \log^{C''_Y} + \frac{t^2}{2M^2+1} + \log^{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2M^2+1}t; \Gamma_T - \frac{M^2}{2M^2+1}\Delta_T(ij)^\top\Delta_T(ij)\right)}. \tag{66}$$

Therefore, the expectation of Y derived from the cumulant generating function is shown in the next equation:

$$\begin{aligned}\left.\frac{dK_Y(t)}{dt}\right|_{t=0} = \mathbb{E}\left[Y\right] = &\frac{1}{\Phi_T\left(\mathbf{0}_T; \Gamma_T - \frac{M^2}{2M^2+1}\Delta_T(ij)^\top\Delta_T(ij)\right)}\\ &\cdot \left.\frac{d\Phi_T\left(\frac{\Delta_T(ij)^\top}{2M^2+1}t; \Gamma_T - \frac{M^2}{2M^2+1}\Delta_T(ij)^\top\Delta_T(ij)\right)}{dt}\right|_{t=0}.\end{aligned} \tag{67}$$

Note that when $M=0$:

$$\begin{aligned}\mathbb{E}\left[Y\right] &= \frac{1}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}\left.\frac{d\Phi_T\left(\Delta_T(ij)^\top t;\Gamma_T\right)}{dt}\right|_{t=0}\\ &= \frac{\Delta_T(ij)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}\nabla\Phi_T(\Gamma_T),\end{aligned}$$

which is actually the value of $\mathbb{E}\left[\theta_i-\theta_j\right]$ as depicted in Eq.(27). As we already know that $\mathbb{E}\left[\theta_i-\theta_j\right]$ is larger than 0, and this fact will lead to:

$$\left.\frac{d\mathrm{Pred}_T(i>j,M)}{dM}\right|_{M=0}>0 \tag{68}$$

Note that $\mathrm{Pred}_T(i>j,0)=0.5$, so there exists a certain $M_0>0$, which will ensure $\mathrm{Pred}_T(i>j,M_0)>0.5$. $\square$

Now, with Lemma 5 established, before going further, we will give Lemma 6, which comes from the non-symmetric condition in Eq. (25).

Lemma 6. *If the non-symmetric condition is satisfied for a pair (i,j), then, for $s\in(0,+\infty)$, we will have either*

$$\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)<\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right), \tag{69}$$

or

$$\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)>\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right). \tag{70}$$

Proof. of Lemma 6. Given $\Delta_T(ij)$, for notational simplicity, we denote

$$g(s)=\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right).$$

We can obtain the derivative of $g(s)$ with respect to s (for details of $\frac{dg(s)}{s}$, please refer to Section 2 in Arellano-Valle and Azzalini [2022]), and based on this fact $g(s)$ is continuous with respect to s. We also denote

$$\begin{aligned}f(s)&=\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)-\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}s;\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)\\ &=g(s)-g(-s).\end{aligned}$$

Note that $f(0)=0$, and $\lim_{s\to+\infty}f(s)=0$. Recall the non-symmetric condition in Eq. (25), we will have $f(s)\neq 0$ when $s\in(0,+\infty)$ since $f(s)$ is continuous. In other words we will have either $f(s)>0$ or $f(s)<0$ when $s\in(0,+\infty)$, which corresponds to Eq. (69) or Eq. (70). $\square$

In the following aspect of this section, we will give details of the proof of Theorem 1.

Proof. of Theorem 1. If Condition 1 is satisfied with $\mathbb{E}[\theta_i-\theta_i]\neq 0$, we can only consider the case when $\mathbb{E}[\theta_i-\theta_i]>0$ since when $\mathbb{E}[\theta_i-\theta_i]<0$ the proof is similar. From B.1 we know that the expectation $\mathbb{E}[\theta_i-\theta_i]>0$ has the same value with the expectation of θ_{ij} which is described in Eq. (26). With the known PDF of θ_{ij}, $\mathbb{E}[\theta_i-\theta_i]=\mathbb{E}[\theta_{ij}]$ takes the form of:

$$\begin{aligned}&\int_{\theta_{ij}>0}\phi_1(\theta_{ij};0,2)\frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}\theta_{ij}d\theta_{ij}\\ &+\int_{\theta_{ij}<0}\phi_1(\theta_{ij};0,2)\frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T-\frac{\Delta_T(ij)^\top\Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)}\theta_{ij}d\theta_{ij}>0.\end{aligned} \tag{71}$$

Then we will have:

$$
\begin{aligned}
\int_{\theta_{ij}>0} \phi_1(\theta_{ij};0,2)\theta_{ij} \Bigg(& \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} \\
& - \frac{\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} \Bigg) d\theta_{ij} > 0.
\end{aligned} \tag{72}
$$

From Lemma 6, and given the fact that $\phi_1(\theta_{ij};0,2) > 0$ when $\theta_{ij} \in (0,+\infty)$, we will have :

$$
\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right) > \Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right). \tag{73}
$$

Recall that in Lemma 5, since $\mathbb{E}[\theta_i - \theta_j] > 0$, we will have $\left.\frac{d\mathrm{Pred}_T(i>j,M)}{dM}\right|_{M=0} > 0$, and there exists a M_0 such that $\mathrm{Pred}_T(i > j, M_0) > 0.5$. When $M < M_0$, since $\mathrm{Pred}_T(i > j, 0) = 0.5$, from the properties of the derivative, for any $M < M_0$, we can have $\mathrm{Pred}_T(i > j, M) > \mathrm{Pred}_T(i > j, 0) = 0.5$. As for $M > M_0$, we can have:

$$
\begin{aligned}
& \mathrm{Pred}_T(i > j, M) - \mathrm{Pred}_T(i > j, M_0) \\
= & \int \Phi\left(M(\theta_i - \theta_j)\right) - \Phi\left(M_0(\theta_i - \theta_j)\right) p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) d\boldsymbol{\theta} \\
\overset{(1)}{=} & \int \left[\Phi\left(M\theta_{ij}\right) - \Phi\left(M_0\theta_{ij}\right)\right] p(\theta_{ij}) d\theta_{ij} \\
= & \int \left[\left(\Phi\left(M\theta_{ij}\right) - 0.5\right) - \left(\Phi\left(M_0\theta_{ij}\right) - 0.5\right)\right] p(\theta_{ij}) d\theta_{ij} \\
= & \int_{\theta_{ij}>0} \left[\left(\Phi\left(M\theta_{ij}\right) - 0.5\right) - \left(\Phi\left(M_0\theta_{ij}\right) - 0.5\right)\right] p(\theta_{ij}) d\theta_{ij} \\
& \quad + \int_{\theta_{ij}<0} \left[\left(\Phi\left(M\theta_{ij}\right) - 0.5\right) - \left(\Phi\left(M_0\theta_{ij}\right) - 0.5\right)\right] p(\theta_{ij}) d\theta_{ij} \\
\overset{(2)}{=} & \int_{\theta_{ij}>0} \left[\left(\Phi\left(M\theta_{ij}\right) - 0.5\right) - \left(\Phi\left(M_0\theta_{ij}\right) - 0.5\right)\right] p(\theta_{ij}) d\theta_{ij} \\
& \quad + \int_{\theta_{ij}>0} \left[\left(\Phi\left(-M\theta_{ij}\right) - 0.5\right) - \left(\Phi\left(-M_0\theta_{ij}\right) - 0.5\right)\right] p(-\theta_{ij}) d\theta_{ij},
\end{aligned} \tag{74}
$$

where "(1)" in Eq. (74) is due to Lemma 4 with the distribution of θ_{ij} also described in this Lemma, and "(2)" in Eq. (74) is due to that $\Phi\left(M\theta_{ij}\right) - 0.5$ is an odds function with respect to θ_{ij}. For notational simplicity, we denote $f_M(\theta_{ij}) = \Phi(M\theta_{ij})$, then we will have

$$
\begin{aligned}
(74) = & \int_{\theta_{ij}>0} f_M(\theta_{ij}) \left[p(\theta_{ij}) - p(-\theta_{ij})\right] d\theta_{ij} - \int_{\theta_{ij}>0} f_{M_0}(\theta_{ij}) \left[p(\theta_{ij}) - p(-\theta_{ij})\right] d\theta_{ij} \\
\overset{(3)}{=} & \int_{\theta_{ij}>0} \left[f_M(\theta_{ij}) - f_{M_0}(\theta_{ij})\right] \phi_1(\theta_{ij};0,2) \Bigg(\frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} \\
& - \frac{\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}\theta_{ij};\Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T;\Gamma_T\right)} \Bigg) d\theta_{ij} > 0,
\end{aligned} \tag{75}
$$

where ">" in "(3)" is based on Eq. (73) and $f_M(\theta_{ij}) - f_{M_0}(\theta_{ij}) = \Phi\left(M\theta_{ij}\right) - \Phi\left(M_0\theta_{ij}\right) > 0$, for $\theta_{ij} \in (0,+\infty)$, which makes elements in the integration in Eq. (75) is positive, resulting in

$\text{Pred}_T(i > j, M) - \text{Pred}_T(i > j, M_0) > 0$. From Lemma 3, when $M > M_0$, we have that

$$\text{Pred}_T(i > j) = \lim_{M \to +\infty} \text{Pred}_T(i > j, M) \geq \text{Pred}_T(i > j, M_0) > 0.5,$$

which ends our proof. $\square$

Finally, we will give the proof of the remaining corollary. This corollary ensures the absence of loops, which is always observed in our empirical study. Before going further, note that if $\Delta_T(ij) = \mathbf{0}_T$, from Eq. (24), we will definitely have $\text{Pred}_T(i > j) = \text{Pred}_T(j > i) = 0.5$. Once $\Delta_T(ij) = \Delta_T(i,:) - \Delta_T(j,:) = \mathbf{0}_T$, the comparison outcome of item i and item j is equal in the collected T samples. In this case, we can consider them to have the same ranking, as we will have $\text{Pred}_T(i > k) = \text{Pred}_T(j > k)$ for any other item k with $k \neq i, j$. So we can only consider pairs with $\Delta_T(ij) \neq \mathbf{0}_T$.

Proof. of Corollary 1. Note that

$$\mathbb{E}[\theta_i - \theta_j] \overset{(1)}{=} \mathbb{E}[\theta_{ij}] = \int_{\theta_{ij}>0} \phi_1(\theta_{ij}; 0, 2)\theta_{ij} \left(\frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} \right.$$
$$\left. - \frac{\Phi_T\left(\frac{-\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} \right) d\theta_{ij},$$

where θ_{ij} is described in Eq. (26), and "(1)" is given in B.1. With Condition 1 for a pair (i, j) with $i \neq j$ and Lemma 6, we observe either $\mathbb{E}[\theta_i - \theta_j] > 0$ or $\mathbb{E}[\theta_i - \theta_j] < 0$, as the integrand is always positive or negative over its interval. Thus, $\mathbb{E}[\theta_i - \theta_j] \neq 0$ for all (i, j) pairs.

By Theorem 1, if the posterior mean of item i is larger than that of item j, then $\text{Pred}_T(i > j) > 0.5$. Since $\text{Pred}_T(i > j) + \text{Pred}_T(j > i) = 1$, we conclude $\text{Pred}_T(i > j) > \text{Pred}_T(j > i)$. Recalling π_{pos} denotes the ranking sorted via the posterior mean, and $\pi_{pos}(i) > \pi_{pos}(j)$ implies $\text{Pred}_T(i > j) > 0.5$. Thus, $\pi_{pos} \in \arg\max_\pi C_T(\pi)$ based on Eq. (22), and this fact will end our proof. $\square$

A.4 Proof of Theorem 2 for the convergence of our SUN ranking algorithm

In this section we give the proof of lemma and theorem for the convergent property of our SUN Ranking framework, which shows the completeness of our algorithm. We firstly introduce the following lemma:

Lemma 7. *If (x, y) is a joint distribution of a bivariate normal distribution with parameters given by:*

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \right). \tag{76}$$

Then the integration in the following will become:

$$\int\int \Phi(M(x-y); b)P(x, y)dxdy = \Phi\left(\frac{M(\mu_x - \mu_y)}{\sqrt{b + M^2(\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy})}} \right) \tag{77}$$

Where M is a positive constant, $P(x, y)$ is the probability density function of (x, y), $\Phi(a; b)$ is the CDF of a Univariate Gaussian distribution evaluated at a with mean 0, and variance b.

we give the proof of the Lemma 7.

Proof. of Lemma 7. From Lemma 2.4.1 in Aziz [2011], (77) will become:

$$
\begin{aligned}
\Phi & \left(M(\mu_x - \mu_y); b + (M, -M) \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} M \\ -M \end{pmatrix} \right) \\
& = \Phi \left(M(\mu_x - \mu_y); b + M^2(\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}) \right) \\
& = \Phi \left(\frac{M(\mu_x - \mu_y)}{\sqrt{b + M^2(\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy})}} \right)
\end{aligned} \tag{78}
$$

□

With the above lemma is settled, then we will give the proof of the core theorem in the convergence of our framework.

Proof. of Theorem 2. In the proof, for disambiguation, we firstly denote $\boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_N^*)^\top$ as the ground-truth score, i.e., $\pi(i)^* > \pi(j)^*$ if and only if $\theta_i^* > \theta_j^*$. We then denote $\hat{\boldsymbol{\theta}}_T$ as the Maximum Likelihood Estimator of $\boldsymbol{\theta}^*$ after sampling T comparisons. From Bernstein-von Mises theorem Van der Vaart [2000], when T becomes extremely large, the posterior distribution of $\boldsymbol{\theta}$ will converges to a multivariate normal distribution as shown below:

$$
p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) \xrightarrow{d} \mathcal{N}_N \left(\hat{\boldsymbol{\theta}}_T, \frac{\mathcal{I}(\boldsymbol{\theta}^*)^{-1}}{T} \right), \tag{79}
$$

where $\mathcal{I}(\boldsymbol{\theta}^*)$ is the Fisher information matrix Ly et al. [2017] evaluated at the true population parameter value $\boldsymbol{\theta}^*$. On the other hand, as T goes to extremely large, as assumed in our theorem, $\hat{\boldsymbol{\theta}}_T$ will also converge to a multivariate Gaussian distribution:

$$
\hat{\boldsymbol{\theta}}_T \xrightarrow{d} \mathcal{N}_N (\boldsymbol{\theta}^*, \Sigma), \tag{80}
$$

where Σ is the variance-covariance matrix, which will not influence our results, of the corresponding multivariate normal distribution. When $M \in (0, +\infty)$, the expectation of the prediction probability in (53) will be carried out in normal distributions:

$$
\begin{aligned}
\mathrm{Pred}_T(i > j, M) &= \mathbb{E}\left[\mathbb{E}\left[\Phi\left(M(\theta_i - \theta_j)\right) | \hat{\boldsymbol{\theta}}_T \right] \right], \quad \boldsymbol{\theta} \sim p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N) \\
& \to \mathbb{E}_{\hat{\boldsymbol{\theta}}_T} \left[\Phi\left(\frac{M(\hat{\theta}_{T,i} - \hat{\theta}_{T,j})}{\sqrt{1 + M^2\left(\sigma_{T,ii}^2 + \sigma_{T,jj}^2 - 2\sigma_{T,ij}\right)}} \right) \right] \\
& \to \Phi\left(\frac{M(\theta_i^* - \theta_j^*)}{\sqrt{1 + M^2\left(\sigma_{T,ii}^2 + \sigma_{T,jj}^2 + \sigma_{ii}^2 + \sigma_{jj}^2 - 2\sigma_{T,ij} - 2\sigma_{ij}\right)}} \right),
\end{aligned} \tag{81}
$$

where $\sigma_{T,ij}^2$ is the i-th row and j-th column entry of the variance-covariance matrix $\frac{\mathcal{I}(\boldsymbol{\theta}^*)^{-1}}{T}$, σ_{ij} being the the i-th row and j-th column entry of the variance-covariance matrix Σ, and $\hat{\boldsymbol{\theta}}_{T,i}$ is the i-th element of $\hat{\boldsymbol{\theta}}_T$.

The first equation in Eq. (81) is due to the property of conditional expectation. The first $\to$ in Eq. (81) is due to that the posterior distribution $p(\boldsymbol{\theta}|\mathcal{S}_T, \mathcal{Y}_T(\mathcal{S}_T), \mathbf{0}_N, I_N)$ will converge to a multivariate normal distribution with parameters shown in Eq. (79), and since $\Phi(\cdot)$ is bounded continuous function, from theorem 1.9 in Shao [2003], the expectation of $\Phi(\cdot)$ over the ground-truth posterior will also converges to the expectation of $\Phi(\cdot)$ over the converged distribution. Since the converged distribution in Eq. (79) is multivariate normal, taking lemma 7 into consideration, will comes the result in the right hand side of the first $\rightarrow$.

The derivation of the second $\rightarrow$ is similar. Note that the distribution of the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_T$ will also converge to a multivariate distribution given in Eq. (80), since $\Phi(\cdot)$ is bounded continuous function from theorem 1.9 in Shao [2003] and lemma 7 we will also have the result in the right hand side of the second $\rightarrow$.

As for M approaches $+\infty$, from Lemma 3 we will have:

$$\begin{aligned}\mathrm{Pred}_T(i>j) &= \lim_{M\rightarrow+\infty}\mathrm{Pred}_T(i>j,M)\\ &\rightarrow \lim_{M\rightarrow+\infty}\Phi\left(\frac{M(\theta_i^*-\theta_j^*)}{\sqrt{1+M^2\left(\sigma_{T,ii}^2+\sigma_{T,jj}^2+\sigma_{ii}^2+\sigma_{jj}^2-2\sigma_{T,ij}-2\sigma_{ij}\right)}}\right)\\ &= \Phi\left(\frac{\theta_i^*-\theta_j^*}{\sqrt{\sigma_{T,ii}^2+\sigma_{T,jj}^2+\sigma_{ii}^2+\sigma_{jj}^2-2\sigma_{T,ij}-2\sigma_{ij}}}\right).\end{aligned}$$

Then, when $T\rightarrow+\infty$, $\mathrm{Pred}_T(i>j)>0.5$ if and only if $\theta_i^*>\theta_j^*$, and the proposed ranking from Algorithm 2 equals the sorting of the ground-truth score $\boldsymbol{\theta}^*$.

□

A.5 Proof of Theorem 3 for SUN Ranking Active Algorithm with MFVI

Proof. of Theorem 3. If all elements in $\bar{\boldsymbol{\theta}}_T^{(*)}$ are equal, then any ranking of $\{1,2,...,N\}$ will reach the upper bound of $\widehat{C_T(\pi)}$ and Eq. (38) is satisfied.

We then assume there exists at lest two elements in $\bar{\boldsymbol{\theta}}_T^{(*)}$ are not equal, and we first demonstrate that elements on the right-hand side of Eq. (38) belong to the left-hand side. Suppose a rank $\hat{\pi}$ maximizes $\widehat{C_T(\pi)}$, and assume it does not belong to the left-hand side of Eq. (38), i.e., there exist indices s and k such that $\bar{\boldsymbol{\theta}}_{T,s}^{(*)}>\bar{\boldsymbol{\theta}}_{T,k}^{(*)}$ and $\hat{\pi}(s)<\hat{\pi}(k)$. This assumption leads to a contradiction. Note that the estimated posterior distribution is a multivariate normal distribution denoted as $\phi_N\left(\boldsymbol{\theta};\bar{\boldsymbol{\theta}}_T^{(*)},V_T\right)$. For any pair (i,j) with $i\neq j$:

$$\widehat{\mathrm{Pred}_T(i>j)}=\int_{\theta_i>\theta_j}\phi_N\left(\boldsymbol{\theta};\bar{\boldsymbol{\theta}}_T^{(*)},\boldsymbol{V}_T\right)d\boldsymbol{\theta}=\Phi\left(\frac{\bar{\boldsymbol{\theta}}_{T,i}^{(*)}-\bar{\boldsymbol{\theta}}_{T,j}^{(*)}}{\sqrt{A_{ij}^\top V_T A_{ij}}}\right), \tag{82}$$

where A_{ij} is an N-dimensional vector with its i-th element set to 1, j-th element set to -1, and other elements set to 0. Then, from Eq. (82), we can conclude that

$$\widehat{\mathrm{Pred}_T(i>j)}>0.5 \text{ if and only if } \bar{\boldsymbol{\theta}}_{T,i}^{(*)}>\bar{\boldsymbol{\theta}}_{T,j}^{(*)}.$$

$$\widehat{\mathrm{Pred}_T(i>j)}=0.5 \text{ if and only if } \bar{\boldsymbol{\theta}}_{T,i}^{(*)}=\bar{\boldsymbol{\theta}}_{T,j}^{(*)}.$$

Since

$$\widehat{\mathrm{Pred}_T(i>j)}+\widehat{\mathrm{Pred}_T(j>i)}=1,$$

we will have

$$\widehat{\mathrm{Pred}_T(k>s)}<0.5<\widehat{\mathrm{Pred}_T(s>k)}$$

we denote $\hat{\pi}_T$ as the rank via the sorting of $\bar{\boldsymbol{\theta}}_T^{(*)}$:

$$\hat{\pi}_T(i)>\hat{\pi}_T(j) \text{ if } \bar{\boldsymbol{\theta}}_{T,i}^{(*)}>\bar{\boldsymbol{\theta}}_{T,j}^{(*)} \text{ or } \bar{\boldsymbol{\theta}}_{T,i}^{(*)}=\bar{\boldsymbol{\theta}}_{T,j}^{(*)} \text{ with } i<j.$$

Then,

$$\begin{aligned}
\widehat{C_T(\hat{\pi})} &= \sum_{\hat{\pi}(i)>\hat{\pi}(j)} \widehat{\mathrm{Pred}_T(i>j)} < \sum_{(i,j),i\neq j} \max(\widehat{\mathrm{Pred}_T(i>j)}, \widehat{\mathrm{Pred}_T(j>i)}) \\
&= \widehat{C_T(\hat{\pi}_T)} = \sum_{\hat{\pi}_T(i)>\hat{\pi}_T(j)} \widehat{\mathrm{Pred}_T(i>j)},
\end{aligned} \tag{83}$$

where the symbol $<$ in the above equation arises from the fact that $\widehat{\mathrm{Pred}_T(s>k)} > \widehat{\mathrm{Pred}_T(k>s)}$ while $\widehat{C_T(\hat{\pi})}$ adopts $\widehat{\mathrm{Pred}_T(k>s)}$. This contradiction arises because $\hat{\pi}$ will not maximize $\widehat{C_T(\pi)}$.

We then prove that elements on the left-hand side of Eq. (38) belong to the right-hand side. Note that a rank π on the left-hand side satisfies:

$$\pi(i) > \pi(j) \text{ only if } \bar{\boldsymbol{\theta}}^{(*)}_{T,i} \geq \bar{\boldsymbol{\theta}}^{(*)}_{T,j}. \tag{84}$$

The summation of the prediction probability becomes:

$$\begin{aligned}
\widehat{C_T(\pi)} &= \sum_{\pi(i)>\pi(j)} \widehat{\mathrm{Pred}_T(i>j)} \\
&= \sum_{\bar{\boldsymbol{\theta}}^{(*)}_{T,i} \geq \bar{\boldsymbol{\theta}}^{(*)}_{T,j}} \widehat{\mathrm{Pred}_T(i>j)} \\
&\overset{(1)}{=} \sum_{(i,j),i\neq j} \max(\widehat{\mathrm{Pred}_T(i>j)}, \widehat{\mathrm{Pred}_T(j>i)}) \\
&= \max_{\pi} \widehat{C_T(\pi)},
\end{aligned} \tag{85}$$

where "(1)" is due to Eq. (82), and the above equation ends our proof. □

Appendix B Some Computational Details

In this section, we provide details of several computations, including the moment-generating function of θ_{ij}, which is described in Lemma 4, and the transitional probability in Eq. (32).

B.1 Computation of the Moment Generating Function and the Mean of θ_{ij}

According to the definition of the moment generating function, the $M_T^{ij}(t)$ of θ_{ij} which is distributed as $\mathrm{SUN}_{1,T}\left(0, 2, \frac{\Delta_T(ij)}{\sqrt{2}}, \mathbf{0}_T, \Gamma_T\right)$ is defined as:

$$
\begin{aligned}
M_T^{ij}(t) &:= \int e^{t\theta_{ij}} \phi_1(\theta_{ij}; 0, 2) \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} d\theta_{ij} \\
&= \int \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{\theta_{ij}^2}{4}} e^{t\theta_{ij}} \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} d\theta_{ij} \\
&= e^{t^2} \int \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{\left(\theta_{ij}-2t\right)^2}{4}} \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} d\theta_{ij} \\
&= e^{t^2} \int \phi_1(\theta_{ij}; 2t, 2) \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2}\theta_{ij}; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} d\theta_{ij},
\end{aligned}
\tag{86}
$$

where $\phi_1(x; \mu, \sigma)$ denotes the PDF of normal distribution $N(\mu, \sigma)$ at x. From Lemma 2.4.1 in Aziz [2011], the above integration becomes:

$$
e^{t^2} \frac{\Phi_T\left(\frac{\Delta_T(ij)^\top}{2} \cdot 2t; \Gamma_T - \frac{\Delta_T(ij)^\top \Delta_T(ij)}{2} + \frac{\Delta_T(ij)^\top}{2} \cdot 2 \cdot \frac{\Delta_T(ij)}{2}\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)}. \tag{87}
$$

Simplify the equation, we will get the moment generating function:

$$
M_T^{ij}(t) = e^{t^2} \frac{\Phi_T\left(\Delta_T(ij)^\top t; \Gamma_T\right)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)}. \tag{88}
$$

From the moment generating function, its cumulant generating function $K_T^{ij}(t)$ will be

$$
K_T^{ij}(t) = t^2 + \log \Phi_T\left(\Delta_T(ij)^\top t; \Gamma_T\right) - \log \Phi_T\left(\mathbf{0}_T; \Gamma_T\right) \tag{89}
$$

Then the mean of θ_{ij} will be

$$
\mathbb{E}[\theta_{ij}] = \left.\frac{\mathrm{d}K_T^{ij}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{\Delta_T(ij)\nabla\Phi_T(\Gamma_T)}{\Phi_T\left(\mathbf{0}_T; \Gamma_T\right)} = \mathbb{E}[\theta_i - \theta_j] \tag{90}
$$

where $\nabla\Phi_T(\Gamma_T)$ is given in Eq. (27), and $\boldsymbol{\theta} \sim \mathrm{SUN}_{N,T}\left(\mathbf{0}_N, I_N, \Delta_T, \mathbf{0}_T, \Gamma_T\right)$ with its PDF given in Eq. (17).

B.2 Computation of the Transitional Probability in Eq. (32)

We only provide the computation details for $s = 1$, as the one for $s = -1$ is similar. As $\mathcal{S}_t$ contains collected pairs and $\mathcal{Y}_t(\mathcal{S}_t)$ records the comparison labels, we rewrite Eq. (32) when $s = 1$

$$
\begin{aligned}
\Pr(Y_{i_{t+1}j_{t+1}} = \pm 1 | \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), \mathbf{0}_N, I_N) &= \Pr(y_{t+1} = 1 | \mathcal{S}_t, \mathcal{Y}_t(\mathcal{S}_t), (i_{t+1}, j_{t+1}), \mathbf{0}_N, I_N) \\
&= \Pr(y_{t+1} = 1 | X_t, Y_t, x_{t+1}, \mathbf{0}_N, I_N),
\end{aligned}
\tag{91}
$$

where $Y_t = (y_1, ..., y_t)^\top$ denotes the label vector and $X_t \in \mathcal{R}^{t\times N}$ represents the pairwise comparison matrix for $\mathcal{S}_t$, such that for each sample (i_k, j_k), the k-th row of X_t has $X_t(k,l) = \frac{1}{\sqrt{2}}$ if $l = i_k$, $= -\frac{1}{\sqrt{2}}$ if $l = j_k$, and $= 0$ otherwise. x_{t+1} is a N-dimensional vector with its i_{t+1}-th element set to $\frac{1}{\sqrt{2}}$, j_{t+1}-th element set to $\frac{-1}{\sqrt{2}}$, and other elements set to 0. According to Eq. (3) and the fact that the prior distribution of $\boldsymbol{\theta}$ is $\mathcal{N}_N(\mathbf{0}, I_N)$, we can apply corollary 2 in Durante [2019] and obtain that

$$\Pr(y_{t+1} = 1 | X_t, Y_t, x_{t+1}, \mathbf{0}_N, I_N) = \frac{\Phi_{t+1}(\mathbf{0}_{t+1}; \Gamma_{t+1})}{\Phi_t(\mathbf{0}_t; \Gamma_t)},$$

where $\Gamma_t := s_t^{-1}(D_t I_N D_t^\top + I_t)s_t^{-1}$, with $s_t = [(x_1^\top x_1 + 1)^{1/2}, ..., (x_t^\top x_t + 1)^{1/2}]$ and $D_t = \mathrm{diag}(y_t)X_t$ for each t, which is the same with parameters of SUN distribution derived below Eq. (15). We thus obtain Eq. (32).

B.3 Specific Form of the ELBO in Our Framework

Assume we have finished the s-th iteration in Alg. 5, then the specific form of ELBO will be Fasano et al. [2022]:

$$\mathrm{ELBO}\left\{q_{\mathrm{MF}}(\boldsymbol{\theta}, \boldsymbol{z})\right\} = -\frac{1}{2}\left((\bar{\boldsymbol{z}}^{(s)})^\top X_T V_T V_T X_T^\top \bar{\boldsymbol{z}}^{(s)} + \sum_{l=1}^{T} \log \Phi\left(v_{\bar{\boldsymbol{z}}^{(s)},l}\right)\right), \tag{92}$$

where $v_{\bar{\boldsymbol{z}}^{(s)},l}$ is the l-the element of $v_{\bar{\boldsymbol{z}}^{(s)}} = \mathrm{diag}(\boldsymbol{y}_T)X_T V_T X_T^\top \bar{\boldsymbol{z}}^{(s)}$.

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