I. KERR-SCHILD COORDINATE

The Kerr-Schild coordinate

$$g_{\mu\nu} = \begin{pmatrix} -1+\frac{2r}{\Sigma} & \frac{2r}{\Sigma} & 0 & -\frac{2ar\sin^2\theta}{\Sigma} \\ \frac{2r}{\Sigma} & 1+\frac{2r}{\Sigma} & 0 & -a(1+\frac{2r}{\Sigma})\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ -\frac{2ar\sin^2\theta}{\Sigma} & -a(1+\frac{2r}{\Sigma})\sin^2\theta & 0 & \frac{\beta}{\Sigma}\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1-\frac{2r}{\Sigma} & \frac{2r}{\Sigma} & 0 & 0 \\ \frac{2r}{\Sigma} & \frac{\Delta}{\Sigma} & 0 & \frac{a}{\Sigma} \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ 0 & \frac{a}{\Sigma} & 0 & \frac{1}{\Sigma\sin^2\theta} \end{pmatrix}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2r + a^2$, $\sqrt{-g} = \Sigma\sin\theta$, and

$$\begin{aligned} \beta &= \Delta\Sigma + 2r(r^2+a^2) \\ &= (\Sigma + a^2\sin^2\theta)\Sigma + 2ra^2\sin^2\theta \\ &= (r^2+a^2)\Sigma + 2ra^2\sin^2\theta \\ &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta \end{aligned} \tag{1}$$

$$\begin{aligned} g^{tr}g^{rt} - g^{tt}g^{rr} &= \frac{4r^2 + \Delta(\Sigma+2r)}{\Sigma^2} = \frac{\beta}{\Sigma^2} \\ g^{\phi r}g^{r\phi} - g^{\phi\phi}g^{rr} &= \frac{a^2\sin^2\theta - \Delta}{\Sigma^2\sin^2\theta} = -\frac{\Sigma - 2r}{\Sigma^2\sin^2\theta} \end{aligned} \tag{2}$$

It is noticeable that the Kerr-Schild coordinate does not return to Schwarzschild coordinate when $a = 0$, i.e.

$$g_{\mu\nu}(a=0) = \begin{pmatrix} -1+2/r & 2/r & 0 & 0 \\ 2/r & 1+2/r & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and

$$g^{\mu\nu}(a=0) = \begin{pmatrix} -1-2/r & 2/r & 0 & 0 \\ 2/r & 1-2/r & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

II. CONSERVATION EQUATIONS IN KERR SPACE-TIME

$$F_{\theta\phi} = -F_{\phi\theta} = A_{\phi,\theta} \quad F_{r\phi} = -F_{\phi r} = A_{\phi,r} \tag{3}$$

$$F_{tr} = -F_{rt} = \Omega A_{\phi,r} \quad F_{t\theta} = -F_{\theta t} = \Omega A_{\phi,\theta} \tag{4}$$

$$F_{r\theta} = -F_{\theta r} = \sqrt{-g} B^{\phi} \tag{5}$$

$$\Omega \equiv \Omega(A_\phi) \quad \sqrt{-g} F^{\theta r} \equiv I(A_\phi). \tag{6}$$

Compare definitions of B^ϕ and I, it is clear that I is positive when electric current flows into BH.

$$-\Omega\left[(\sqrt{-g}F^{tr})_{,r} + (\sqrt{-g}F^{t\theta})_{,\theta}\right] + F_{r\theta} I'(A_\phi) + \left[(\sqrt{-g}F^{\phi r})_{,r} + (\sqrt{-g}F^{\phi\theta})_{,\theta}\right] = 0. \tag{7}$$

$$B^\phi = -\frac{I\Sigma + (2r\Omega - a)\sin\theta A_{\phi,\theta}}{\Delta\Sigma\sin^2\theta} = -\frac{I\Sigma - (2r\Omega - a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma\sin^2\theta} \tag{8}$$

$$\begin{aligned}
F_{tr} &= \Omega A_{\phi,r} \\
F^{tr} &= g^{tt}(g^{rr}F_{tr} + g^{r\phi}F_{t\phi}) + g^{tr}(g^{rt}F_{rt} + g^{r\phi}F_{r\phi}) \\
&= (g^{tt}g^{rr} - g^{tr}g^{rt})F_{tr} + g^{tr}g^{r\phi}F_{r\phi} \\
&= \left([g^{tt}g^{rr} - g^{tr}g^{rt}]\Omega + g^{tr}g^{r\phi}\right) A_{\phi,r}
\end{aligned} \tag{9}$$

$$F_{\phi r} = -A_{\phi,r} \tag{10}$$

$$F^{\phi r} = g^{\phi\alpha}g^{r\beta}F_{\alpha\beta} = g^{\phi r}(g^{rt}F_{rt} + g^{r\phi}F_{r\phi}) + g^{\phi\phi}g^{rr}F_{\phi r} \tag{11}$$

$$= (-\Omega g^{\phi r}g^{rt} + g^{\phi r}g^{r\phi} - g^{\phi\phi}g^{rr})A_{\phi,r} \tag{12}$$

$$F_{\phi r}F^{\phi r} + F_{tr}F^{tr} \tag{13}$$

$$\begin{aligned}
&= (\Omega g^{\phi r}g^{rt} - g^{\phi r}g^{r\phi} + g^{\phi\phi}g^{rr})A^2_{\phi,r} + \left([g^{tt}g^{rr} - g^{tr}g^{rt}]\Omega^2 + \Omega g^{tr}g^{r\phi}\right) A^2_{\phi,r} \\
&= \left([g^{tt}g^{rr} - g^{tr}g^{rt}]\Omega^2 + 2\Omega g^{tr}g^{r\phi} + g^{\phi\phi}g^{rr} - g^{\phi r}g^{r\phi}\right) A^2_{\phi,r} \\
&= \left(-\frac{\beta}{\Sigma^2}\Omega^2 + \frac{4ra}{\Sigma^2}\Omega + \frac{\Sigma - 2r}{\Sigma^2\sin^2\theta}\right) A^2_{\phi,r} \equiv -\frac{A^2_{\phi,r}}{\Sigma\sin^2\theta}\mathcal{K}(r,\theta;\Omega)
\end{aligned} \tag{14}$$

$$
\begin{aligned}
&-\Omega(\sqrt{-g}F^{tr})_{,r} + (\sqrt{-g}F^{\phi r})_{,r} \leftarrow X \\
&= -\Omega\left[\sqrt{-g}\left((g^{tt}g^{rr} - g^{tr}g^{rt})\Omega + g^{tr}g^{r\phi}\right)A_{\phi,r}\right]_{,r} + \left[\sqrt{-g}(-\Omega g^{\phi r}g^{rt} + g^{\phi r}g^{r\phi} - g^{\phi\phi}g^{rr})A_{\phi,r}\right]_{,r} \\
&= -\Omega\sqrt{-g}\left((g^{tt}g^{rr} - g^{tr}g^{rt})\Omega + g^{tr}g^{r\phi}\right)A_{\phi,rr} - \Omega\left[\sqrt{-g}\left((g^{tt}g^{rr} - g^{tr}g^{rt})\Omega + g^{tr}g^{r\phi}\right)\right]_{,r}A_{\phi,r} \\
&\quad + \sqrt{-g}(-\Omega g^{\phi r}g^{rt} + g^{\phi r}g^{r\phi} - g^{\phi\phi}g^{rr})A_{\phi,rr} + \left[\sqrt{-g}(-\Omega g^{\phi r}g^{rt} + g^{\phi r}g^{r\phi} - g^{\phi\phi}g^{rr})\right]_{,r}A_{\phi,r} \\
&= \sqrt{-g}\left[-(g^{tt}g^{rr} - g^{tr}g^{rt})\Omega^2 - 2g^{tr}g^{r\phi}\Omega - (g^{\phi\phi}g^{rr} - g^{\phi r}g^{r\phi})\right]A_{\phi,rr} \\
&\quad - \Omega\left[\Omega\left(\sqrt{-g}(g^{tt}g^{rr} - g^{tr}g^{rt})\right)_{,r} + \sqrt{-g}(g^{tt}g^{rr} - g^{tr}g^{rt})\Omega_{,r} + \left(\sqrt{-g}g^{tr}g^{r\phi}\right)_{,r}\right]A_{\phi,r} \\
&\quad - \left[\left(\sqrt{-g}g^{\phi r}g^{rt}\right)\Omega_{,r} + \left(\sqrt{-g}g^{\phi r}g^{rt}\right)_{,r}\Omega + \left(\sqrt{-g}(g^{\phi\phi}g^{rr} - g^{\phi r}g^{r\phi})\right)_{,r}\right]A_{\phi,r} \\
&= -\sqrt{-g}\left[(g^{tt}g^{rr} - g^{tr}g^{rt})\Omega^2 + 2g^{tr}g^{r\phi}\Omega + (g^{\phi\phi}g^{rr} - g^{\phi r}g^{r\phi})\right]A_{\phi,rr} \\
&\quad - \left[\Omega^2\left(\sqrt{-g}(g^{tt}g^{rr} - g^{tr}g^{rt})\right)_{,r} + 2\left(\sqrt{-g}g^{tr}g^{r\phi}\right)_{,r}\Omega + \left(\sqrt{-g}(g^{\phi\phi}g^{rr} - g^{\phi r}g^{r\phi})\right)_{,r}\right]A_{\phi,r} \\
&\quad - \left[\left(\sqrt{-g}(g^{tt}g^{rr} - g^{tr}g^{rt})\right)\Omega + \left(\sqrt{-g}g^{\phi r}g^{rt}\right)\right]\Omega' A_{\phi,r}^2 \\
&= \sin\theta\left[\frac{\beta}{\Sigma}\Omega^2 - \frac{4ra}{\Sigma}\Omega - \frac{\Sigma - 2r}{\Sigma\sin^2\theta}\right]A_{\phi,rr} \\
&\quad + \sin\theta\left[\Omega^2\left(\frac{\beta}{\Sigma}\right)_{,r} - \Omega\left(\frac{4ra}{\Sigma}\right)_{,r} - \left(\frac{\Sigma - 2r}{\Sigma\sin^2\theta}\right)_{,r}\right]A_{\phi,r} \\
&\quad + \sin\theta\left[\left(\frac{\beta}{\Sigma}\right)\Omega - \frac{2ra}{\Sigma}\right]\Omega' A_{\phi,r}^2 \\
&= \sin\theta\left[\left(\frac{\beta}{\Sigma}\right)\Omega^2 - \frac{4ra}{\Sigma}\Omega - \frac{1}{\sin^2\theta}\left(1 - \frac{2r}{\Sigma}\right)\right]A_{\phi,rr} \\
&\quad + \sin\theta\left[\Omega^2\left(\frac{\beta}{\Sigma}\right)_{,r} - \Omega\left(\frac{4ra}{\Sigma}\right)_{,r} + \frac{1}{\sin^2\theta}\left(\frac{2r}{\Sigma}\right)_{,r}\right]A_{\phi,r} \\
&\quad + \sin\theta\left[\left(\frac{\beta}{\Sigma}\right)\Omega - \frac{2ar}{\Sigma}\right]\Omega' A_{\phi,r}^2
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\frac{X}{\sin\theta} &= \left[\left(r^2 + a^2 + \frac{2ra^2\sin^2\theta}{\Sigma}\right)\Omega^2 - \frac{4ra}{\Sigma}\Omega - \frac{1}{\sin^2\theta}\left(1 - \frac{2r}{\Sigma}\right)\right]A_{\phi,rr} \\
&\quad + \left[\Omega^2\left(r^2 + a^2 + \frac{2ra^2\sin^2\theta}{\Sigma}\right)_{,r} - \Omega\left(\frac{4ra}{\Sigma}\right)_{,r} + \frac{1}{\sin^2\theta}\left(\frac{2r}{\Sigma}\right)_{,r}\right]A_{\phi,r} \\
&\quad + \left[\left(r^2 + a^2 + \frac{2ra^2\sin^2\theta}{\Sigma}\right)\Omega - \frac{2ar}{\Sigma}\right]\Omega' A_{\phi,r}^2
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
F_{t\theta} &= \Omega A_{\phi,\theta} = \Omega(-\sin\theta A_{\phi,\mu}) \\
F^{t\theta} &= g^{\theta\theta}(g^{tt}F_{t\theta} + g^{tr}F_{r\theta}) = g^{\theta\theta}(g^{tt}\Omega A_{\phi,\theta} + g^{tr}\sqrt{-g}B^{\phi})
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
F_{r\theta} &= \sqrt{-g}B^{\phi} \\
F_{r\theta}I' &= \Sigma\sin\theta\left(-\frac{I\Sigma - (2r\Omega - a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma\sin^2\theta}\right)I'
\end{aligned}
\tag{18}
$$

$$
F^{\phi\theta} = g^{\theta\theta}g^{\phi\alpha}F_{\alpha\theta} = g^{\theta\theta}(g^{\phi r}F_{r\theta} + g^{\phi\phi}F_{\phi\theta}) = g^{\theta\theta}(g^{\phi r}\sqrt{-g}B^{\phi} - g^{\phi\phi}A_{\phi,\theta}) \tag{19}
$$

$$
F_{t\theta}F^{t\theta} + F_{r\theta}F^{r\theta} + F_{\phi\theta}F^{\phi\theta} \tag{20}
$$

$$
\begin{aligned}
&= g^{\theta\theta}(g^{tt}\Omega A_{\phi,\theta} + g^{tr}\sqrt{-g}B^{\phi})\Omega A_{\phi,\theta} - IB^{\phi} - g^{\theta\theta}(g^{\phi r}\sqrt{-g}B^{\phi} - g^{\phi\phi}A_{\phi,\theta})A_{\phi,\theta} \\
&= g^{\theta\theta}(g^{tt}\Omega^2 + g^{\phi\phi})A^2_{\phi,\theta} + g^{\theta\theta}(g^{tr}\Omega - g^{\phi r})\sqrt{-g}B^{\phi}A_{\phi,\theta} - IB^{\phi} \\
&= g^{\theta\theta}(g^{tt}\Omega^2 + g^{\phi\phi})A^2_{\phi,\theta} + \frac{2r\Omega - a}{\Sigma^2}\sqrt{-g}B^{\phi}A_{\phi,\theta} - IB^{\phi} \\
&= g^{\theta\theta}(g^{tt}\Omega^2 + g^{\phi\phi})A^2_{\phi,\theta} - \left(\frac{2r\Omega - a}{\Sigma^2}\sqrt{-g}A_{\phi,\theta} - I\right)\left(\frac{I\Sigma + (2r\Omega - a)\sin\theta A_{\phi,\theta}}{\Delta\Sigma\sin^2\theta}\right) \\
&= \left[g^{\theta\theta}(g^{tt}\Omega^2 + g^{\phi\phi}) - \frac{(2r\Omega - a)^2}{\Delta\Sigma^2}\right]A^2_{\phi,\theta} + \frac{I^2}{\Delta\sin^2\theta} \\
&= \frac{1}{\Delta}\left[-\frac{\beta}{\Sigma^2}\Omega^2 + \frac{4ra}{\Sigma^2}\Omega + \frac{\Sigma - 2r}{\Sigma^2\sin^2\theta}\right]A^2_{\phi,\theta} + \frac{I^2}{\Delta\sin^2\theta} \\
&= -\frac{A^2_{\phi,\theta}}{\Delta\Sigma\sin^2\theta}\mathcal{K}(r,\theta;\Omega) + \frac{I^2}{\Delta\sin^2\theta}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\frac{1}{2}F\cdot F &= -\frac{A^2_{\phi,r}}{\Sigma\sin^2\theta}\mathcal{K}(r,\theta;\Omega) - \frac{A^2_{\phi,\theta}}{\Delta\Sigma\sin^2\theta}\mathcal{K}(r,\theta;\Omega) + \frac{I^2}{\Delta\sin^2\theta} \\
\frac{1}{2}(F\cdot F)(\Sigma\sin^2\theta) &= -\mathcal{K}(r,\theta;\Omega)\left(A^2_{\phi,r} + \frac{A^2_{\phi,\theta}}{\Delta}\right) + \frac{\Sigma}{\Delta}I^2
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&-\Omega(\sqrt{-g}F^{t\theta})_{,\theta} + F_{r\theta}I'(A_\phi) + (\sqrt{-g}F^{\phi\theta})_{,\theta} \leftarrow Y \\
=& -\Omega\left[\sqrt{-g}g^{\theta\theta}(g^{tt}\Omega A_{\phi,\theta} + g^{tr}\sqrt{-g}B^\phi)\right]_{,\theta} + \sqrt{-g}B^\phi I' + \left[\sqrt{-g}g^{\theta\theta}(g^{\phi r}\sqrt{-g}B^\phi - g^{\phi\phi}A_{\phi,\theta})\right]_{,\theta} \\
=& -\Omega\left[\sqrt{-g}g^{\theta\theta}g^{tt}\Omega A_{\phi,\theta}\right]_{,\theta} - \Omega\left[\sqrt{-g}g^{\theta\theta}g^{tr}\sqrt{-g}B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
&+ \left[\sqrt{-g}g^{\theta\theta}g^{\phi r}\sqrt{-g}B^\phi\right]_{,\theta} - \left[\sqrt{-g}g^{\theta\theta}g^{\phi\phi}A_{\phi,\theta}\right]_{,\theta} \\
=& -\Omega\left[\sqrt{-g}g^{\theta\theta}g^{tt}\Omega A_{\phi,\theta}\right]_{,\theta} - \left[\sqrt{-g}g^{\theta\theta}g^{\phi\phi}A_{\phi,\theta}\right]_{,\theta} \\
&-\Omega\left[\sqrt{-g}g^{\theta\theta}g^{tr}\sqrt{-g}B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' + \left[\sqrt{-g}g^{\theta\theta}g^{\phi r}\sqrt{-g}B^\phi\right]_{,\theta} \\
=& -\Omega\left[\sqrt{-g}g^{\theta\theta}g^{tt}\Omega A_{\phi,\theta}\right]_{,\theta} - \left[\sqrt{-g}g^{\theta\theta}g^{\phi\phi}A_{\phi,\theta}\right]_{,\theta} \\
&+(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
=& -\Omega\left[-(1+\frac{2r}{\Sigma})\Omega\sin\theta A_{\phi,\theta}\right]_{,\theta} - \left[\frac{1}{\Sigma\sin\theta}A_{\phi,\theta}\right]_{,\theta} \\
&+(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
=& -\Omega\left[(1+\frac{2r}{\Sigma})\Omega\sin^2\theta A_{\phi,\mu}\right]_{,\theta} + \left[\frac{1}{\Sigma}A_{\phi,\mu}\right]_{,\theta} \\
&+(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
=& \sin\theta\Omega\left[(1+\frac{2r}{\Sigma})\Omega\sin^2\theta A_{\phi,\mu}\right]_{,\mu} - \sin\theta\left[\frac{1}{\Sigma}A_{\phi,\mu}\right]_{,\mu} \\
&+(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
=& \sin\theta\Omega\left[(1+\frac{2r}{\Sigma})\Omega\sin^2\theta A_{\phi,\mu\mu} + \left((1+\frac{2r}{\Sigma})\Omega\sin^2\theta\right)_{,\mu}A_{\phi,\mu}\right] - \sin\theta\left[\left(\frac{1}{\Sigma}\right)A_{\phi,\mu\mu} + \left(\frac{1}{\Sigma}\right)_{,\mu}A_{\phi,\mu}\right] \\
&+(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\theta} + \sqrt{-g}B^\phi I' \\
=& \sin\theta\left[\Omega^2\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta - \left(\frac{1}{\Sigma}\right)\right]A_{\phi,\mu\mu} \\
&+\sin\theta\left[\Omega^2\left((1+\frac{2r}{\Sigma})\sin^2\theta\right)_{,\mu} - \left(\frac{1}{\Sigma}\right)_{,\mu}\right]A_{\phi,\mu} \\
&+\sin\theta\left[\left(1+\frac{2r}{\Sigma}\right)\Omega\Omega'\sin^2\theta A_{\phi,\mu}^2\right] - \sin\theta(a-2r\Omega)\left(\sin^2\theta B^\phi\right)_{,\mu} + \sin\theta\Sigma B^\phi I'
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\frac{Y}{\sin\theta} =& \left[\Omega^2\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta - \left(\frac{1}{\Sigma}\right)\right]A_{\phi,\mu\mu} \\
&+\left[\Omega^2\left((1+\frac{2r}{\Sigma})\sin^2\theta\right)_{,\mu} - \left(\frac{1}{\Sigma}\right)_{,\mu}\right]A_{\phi,\mu} + \left[\left(1+\frac{2r}{\Sigma}\right)\Omega\Omega'\sin^2\theta A_{\phi,\mu}^2\right] \\
&-(a-2r\Omega)\left(\sin^2\theta B^\phi\right)_{,\mu} + \Sigma B^\phi I'
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&-(a-2r\Omega)\left[\sin^2\theta B^\phi\right]_{,\mu}+\Sigma B^\phi I' \\
&=-(a-2r\Omega)\left[-\frac{I\Sigma-(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma}\right]_{,\mu}+\Sigma\left(-\frac{I\Sigma-(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma\sin^2\theta}\right)I' \\
&=-(a-2r\Omega)\left[-\frac{I}{\Delta}+\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma}\right]_{,\mu}+\Sigma\left(-\frac{I}{\Delta\sin^2\theta}+\frac{(2r\Omega-a)A_{\phi,\mu}}{\Delta\Sigma}\right)I' \\
&=+(2r\Omega-a)\left[-\frac{I'A_{\phi,\mu}}{\Delta}+\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma}\right)_{,\mu}\right]-\frac{\Sigma}{\Delta\sin^2\theta}II'+\frac{(2r\Omega-a)I'A_{\phi,\mu}}{\Delta} \\
&=+(2r\Omega-a)\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Delta\Sigma}\right)_{,\mu}-\frac{\Sigma}{\Delta\sin^2\theta}II' \\
&=\frac{1}{\Delta}\left\{(2r\Omega-a)\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Sigma}\right)_{,\mu}-\frac{\Sigma}{\sin^2\theta}II'\right\} \\
&=\frac{\Sigma}{\sin^2\theta\Delta}\left\{\frac{(2r\Omega-a)\sin^2\theta}{\Sigma}\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Sigma}\right)_{,\mu}-II'\right\} \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\frac{X+Y}{\sin\theta}=&\left[\left(\frac{\beta}{\Sigma}\right)\Omega^2-\frac{4ra}{\Sigma}\Omega-\frac{1}{\sin^2\theta}\left(1-\frac{2r}{\Sigma}\right)\right]A_{\phi,rr} \\
&+\left[\Omega^2\left(\frac{\beta}{\Sigma}\right)_{,r}-\Omega\left(\frac{4ra}{\Sigma}\right)_{,r}+\frac{1}{\sin^2\theta}\left(\frac{2r}{\Sigma}\right)_{,r}\right]A_{\phi,r} \\
&+\left[\left(\frac{\beta}{\Sigma}\right)\Omega-\frac{2ar}{\Sigma}\right]\Omega' A_{\phi,r}^2+\left(1+\frac{2r}{\Sigma}\right)\Omega\Omega'\sin^2\theta A_{\phi,\mu}^2 \\
&+\left[\Omega^2\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta-\left(\frac{1}{\Sigma}\right)\right]A_{\phi,\mu\mu} \\
&+\left[\Omega^2\left((1+\frac{2r}{\Sigma})\sin^2\theta\right)_{,\mu}-\left(\frac{1}{\Sigma}\right)_{,\mu}\right]A_{\phi,\mu} \\
&+\frac{\Sigma}{\sin^2\theta\Delta}\left\{\frac{(2r\Omega-a)\sin^2\theta}{\Sigma}\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Sigma}\right)_{,\mu}-II'\right\} \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
(X+Y)\sin\theta = & \left[\left(\frac{\beta}{\Sigma}\right)\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]A_{\phi,rr} \\
& + \left[\Omega^2\sin^2\theta\left(\frac{\beta}{\Sigma}\right)_{,r} - \Omega\sin^2\theta\left(\frac{4ra}{\Sigma}\right)_{,r} + \left(\frac{2r}{\Sigma}\right)_{,r}\right]A_{\phi,r} \\
& + \left[\left(\frac{\beta}{\Sigma}\right)\Omega - \frac{2ar}{\Sigma}\right]\Omega'\sin^2\theta A^2_{\phi,r} + \left(1+\frac{2r}{\Sigma}\right)\Omega\Omega'\sin^4\theta A^2_{\phi,\mu} \\
& + \left[\Omega^2\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta - \left(\frac{1}{\Sigma}\right)\right]\sin^2\theta A_{\phi,\mu\mu} \\
& + \left[\Omega^2\left(\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta\right)_{,\mu} - \left(\frac{1}{\Sigma}\right)_{,\mu}\right]\sin^2\theta A_{\phi,\mu} \\
& + \frac{(2r\Omega-a)\sin^2\theta}{\Delta}\left(\frac{(2r\Omega-a)\sin^2\theta A_{\phi,\mu}}{\Sigma}\right)_{,\mu} - \frac{\Sigma}{\Delta}II'
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
A_{\phi,\mu\mu} \leftarrow \quad & \sin^2\theta\left[\left(1+\frac{2r}{\Sigma}\right)\Omega^2\sin^2\theta - \frac{1}{\Sigma} + \frac{(2r\Omega-a)^2}{\Delta\Sigma}\sin^2\theta\right] \\
= & \sin^2\theta\left[\Omega^2\sin^2\theta\left(1+\frac{2r}{\Sigma}+\frac{4r^2}{\Delta\Sigma}\right) - \frac{4ar\Omega}{\Delta\Sigma}\sin^2\theta - \frac{1}{\Sigma} + \frac{a^2\sin^2\theta}{\Delta\Sigma}\right] \\
= & \sin^2\theta\left[\frac{\beta}{\Delta\Sigma}\Omega^2\sin^2\theta - \frac{4ar\Omega}{\Delta\Sigma}\sin^2\theta - \frac{\Sigma-2r}{\Delta\Sigma}\right] \\
= & \frac{\sin^2\theta}{\Delta}\left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ar\Omega}{\Sigma}\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
A_{\phi,\mu} \leftarrow \quad & \left[\Omega^2\left(\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta\right)_{,\mu} - \left(\frac{1}{\Sigma}\right)_{,\mu} + \frac{(2r\Omega-a)^2}{\Delta}\left(\frac{\sin^2\theta}{\Sigma}\right)_{,\mu}\right]\sin^2\theta \\
= & \left[\Omega^2\left(\frac{\beta}{\Delta\Sigma}\sin^2\theta\right)_{,\mu} - \Omega\left(\frac{4ar}{\Delta\Sigma}\sin^2\theta\right)_{,\mu} - \left(\frac{\Sigma-2r}{\Delta\Sigma}\right)_{,\mu}\right]\sin^2\theta \\
= & \left[\Omega^2\left(\frac{\beta}{\Sigma}\sin^2\theta\right)_{,\mu} - \Omega\left(\frac{4ar}{\Sigma}\sin^2\theta\right)_{,\mu} - \left(\frac{\Sigma-2r}{\Sigma}\right)_{,\mu}\right]\frac{\sin^2\theta}{\Delta}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
A^2_{\phi,\mu} \leftarrow \quad & \left(1+\frac{2r}{\Sigma}\right)\Omega\Omega'\sin^4\theta + \frac{(2r\Omega-a)}{\Delta}\frac{2r\Omega'}{\Sigma}\sin^4\theta \\
= & \Omega'\sin^4\theta\left[\Omega\left(1+\frac{2r}{\Sigma}+\frac{4r^2}{\Delta\Sigma}\right) - \frac{2ra}{\Delta\Sigma}\right] \\
= & \frac{\Omega'\sin^4\theta}{\Delta}\left[\frac{\beta}{\Sigma}\Omega - \frac{2ra}{\Sigma}\right]
\end{aligned} \tag{30}
$$

Sorted GS equation

$$
\begin{aligned}
&\left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]\left(A_{\phi,rr} + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu\mu}\right)\\
&+\left[\Omega^2\sin^2\theta\left(\frac{\beta}{\Sigma}\right)_{,r} - \Omega\sin^2\theta\left(\frac{4ra}{\Sigma}\right)_{,r} + \left(\frac{2r}{\Sigma}\right)_{,r}\right]A_{\phi,r}\\
&+\left[\Omega^2\left(\frac{\beta}{\Sigma}\sin^2\theta\right)_{,\mu} - \Omega\left(\frac{4ar}{\Sigma}\sin^2\theta\right)_{,\mu} + \left(\frac{2r}{\Sigma}\right)_{,\mu}\right]\frac{\sin^2\theta}{\Delta}A_{\phi,\mu}\\
&+\left[\left(\frac{\beta}{\Sigma}\right)\Omega - \frac{2ar}{\Sigma}\right]\Omega'\sin^2\theta\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) - \frac{\Sigma}{\Delta}II' = 0
\end{aligned}
\tag{31}
$$

or in a more compact form

$$
\begin{aligned}
&\left[A_{\phi,rr} + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu\mu}\right]\mathcal{K}(r,\theta;\Omega)\\
&+\left[A_{\phi,r}\partial_r^\Omega + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}\partial_\mu^\Omega\right]\mathcal{K}(r,\theta;\Omega)\\
&+\frac{1}{2}\left[A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right]\Omega'\partial_\Omega\mathcal{K}(r,\theta;\Omega)\\
&-\frac{\Sigma}{\Delta}II' = 0,
\end{aligned}
\tag{32}
$$

where

$$
\mathcal{K}(r,\theta;\Omega) = \left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]
\tag{33}
$$

$$
\begin{aligned}
E^2 &= \frac{\beta}{\Sigma^2}\left(\frac{2ar}{\beta} - \Omega\right)^2\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right)\\
B^2 &= \frac{\Delta}{\beta\sin^2\theta}\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2 + \frac{I^2\beta}{\Delta^2}\right)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
(B^2-E^2)\Sigma\sin^2\theta &= \frac{\Sigma}{\Delta}I^2 + \left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right)\left(\frac{\Delta\Sigma}{\beta} - \frac{\beta\sin^2\theta}{\Sigma}\left(\frac{2ar}{\beta} - \Omega\right)^2\right)\\
&= -\mathcal{K}\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
(B^2+E^2)\Sigma\sin^2\theta &= \frac{\Sigma}{\Delta}I^2 + \left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right)\left(\frac{\Delta\Sigma}{\beta} + \frac{\beta\sin^2\theta}{\Sigma}\left(\frac{2ar}{\beta} - \Omega\right)^2\right)\\
&= \left(\mathcal{K} + \frac{\Delta\Sigma}{\beta}\right)\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2
\end{aligned}
\tag{36}
$$

At the equator and with ILS, $A_{\phi,r} = 0, \sin\theta = 1$,

$$\frac{B^2 - E^2}{B^2 + E^2} = \frac{-\mathcal{K} A_{\phi,\mu}^2 + \Sigma I^2}{\left(\mathcal{K} + \Delta\Sigma/\beta\right) A_{\phi,\mu}^2 + \Sigma I^2} \tag{37}$$

At ILS, $A_{\phi,\mu} = 0$

$$\frac{B^2 - E^2}{B^2 + E^2} = 1 \tag{38}$$

At $r = r_+$,

$$\frac{B^2 - E^2}{B^2 + E^2} = 0 \tag{39}$$

$A_{\phi,\mu} = \Sigma I / 2r(\Omega - \Omega_{\rm H}) = -(r_+^2/a) I$,

$$-\mathcal{K} A_{\phi,\mu}^2 + \Sigma I^2 \tag{40}$$

$$= \left(1 - \frac{2}{r_+}\right) \frac{r_+^4}{(2r_+(\Omega - \Omega_{\rm H}))^2} I^2 + r_+^2 I^2 \tag{41}$$

$$= -\left(1 - \frac{2}{r_+}\right) \frac{r_+^4}{r_+^2 - 2r_+} \frac{I^2}{(\Omega/\Omega_{\rm H} - 1)^2} + r_+^2 I^2 \tag{42}$$

$$= r_+^2 I^2 \left[1 - \frac{1}{(1 - \Omega/\Omega_{\rm H})^2}\right] \tag{43}$$

A. Cross Check with Lei Huang's (10)

$A_{\phi,rr}\checkmark, \quad A_{\phi,\mu\mu}\checkmark, \quad A_{\phi,r}^2\checkmark, \quad A_{\phi,\mu}^2\checkmark$

$A_{\phi,r}$

$$\Omega^2 \to -\left(\frac{\beta_{,r}}{\beta} - \frac{\Sigma_{,r}}{\Sigma}\right) \frac{\beta \sin^2\theta}{\Sigma} = -\sin^2\theta \left(\frac{\beta_{,r}}{\Sigma} - \frac{\Sigma_{,r}}{\Sigma^2}\right) = -\sin^2\theta \left(\frac{\beta}{\Sigma}\right)_{,r} \checkmark \tag{44}$$

$$\Omega \to \frac{4ar\sin^2\theta}{\Sigma} \left(\frac{1}{r} - \frac{\Sigma_{,r}}{\Sigma}\right) = \sin^2\theta \left(\frac{4ar}{\Sigma}\right)_{,r} \checkmark \tag{45}$$

$$1 \to -\frac{2r}{\Sigma}\left(\frac{\beta_{,r}}{\beta} - \frac{\Sigma_{,r}}{\Sigma}\right) + \frac{2r}{\Sigma}\left(\frac{\beta_{,r}}{\beta} - \frac{1}{r}\right) = -\frac{2r}{\Sigma}\left(\frac{1}{r} - \frac{\Sigma_{,r}}{\Sigma}\right) = -\left(\frac{2r}{\Sigma}\right)_{,r} \checkmark \tag{46}$$

$$s^2 A_{\phi,\mu\mu} = A_{\phi,\theta\theta} - \frac{c}{s} A_{\phi,\theta} \tag{47}$$

$A_{\phi,\mu}$, considering $f_{,\theta} A_{\phi,\theta} = \sin^2\theta f_{,\mu} A_{\phi,\mu}$

$$\Omega^2 \to -\frac{\beta \sin^2\theta}{\Delta\Sigma}\left(2\frac{\cos\theta}{\sin\theta} + \frac{\beta_{,\theta}}{\beta} - \frac{\Sigma_{,\theta}}{\Sigma}\right)$$
$$= -\frac{1}{\Delta}\left(\frac{\beta(\sin^2\theta)_{,\theta}}{\Sigma} + \sin^2\theta(\frac{\beta_{,\theta}}{\Sigma} - \frac{\beta\Sigma_{,\theta}}{\Sigma^2})\right)$$
$$= -\frac{1}{\Delta}\left(\frac{\beta(\sin^2\theta)_{,\theta}}{\Sigma} + \sin^2\theta(\frac{\beta}{\Sigma})_{,\theta}\right)$$
$$= -\frac{1}{\Delta}\left(\frac{\beta\sin^2\theta}{\Sigma}\right)_{,\theta} \checkmark \tag{48}$$

$$\Omega \to \frac{4ar\sin^2\theta}{\Delta\Sigma}\left(-\frac{\beta_{,\theta}}{\beta} + 2\frac{\cos\theta}{\sin\theta} + \frac{\beta_{,\theta}}{\beta} - \frac{\Sigma_{,\theta}}{\Sigma}\right)$$
$$= \frac{4ar\sin^2\theta}{\Delta\Sigma}\left(2\frac{\cos\theta}{\sin\theta} - \frac{\Sigma_{,\theta}}{\Sigma}\right)$$
$$= \frac{4ar}{\Delta}\left(\frac{\sin^2\theta}{\Sigma}\right)_{,\theta} \checkmark \tag{49}$$

$$1 \to \frac{1}{\Delta}\frac{2r\Sigma_{,\theta}}{\Sigma^2} = -\frac{1}{\Delta}\left(\frac{2r}{\Sigma}\right)_{,\theta} \checkmark \tag{50}$$

B. Znajek Horizon Condition

$$I = \frac{2r(\Omega - \Omega_{\rm H})\sin^2\theta}{\Sigma} A_{\phi,\mu}\Big|_{r=r_+} \tag{51}$$

Def $\mathcal{I} = I/A_\phi^{\rm H}, \mathcal{A} = A_\phi/A_\phi^{\rm H}$, where $\mathcal{A}$ falls in $[0,1]$

$$\mathcal{I} = \frac{2r(\Omega - \Omega_{\rm H})\sin^2\theta}{\Sigma} \mathcal{A}_{,\mu}\Big|_{r=r_+} \tag{52}$$

Usually, $\Omega|_{\mu=1} = \Omega_{\rm H}/2$, and $\mathcal{I}|_{\mu=1} = 0$. Given any $\mathcal{I}(\mathcal{A})$ and $\Omega(\mathcal{A})$, there exists a solution $\mathcal{A}(\mu)$ to above equation with boundary condition $\mathcal{A}|_{\mu=0} = 1, \mathcal{A}|_{\mu=1} = 0$? The answer is no.

If we write $\mathcal{I} = 2(\Omega_{\rm H} - \Omega)f(\mathcal{A})$, then

$$f(\mathcal{A}) = -\frac{r_+\sin^2\theta}{r_+^2 + a^2\mu^2}\mathcal{A}_{,\mu} \tag{53}$$

i.e.

$$\frac{d\mathcal{A}}{f(\mathcal{A})} = -\frac{r_+^2 + a^2\mu^2}{r_+(1-\mu^2)}d\mu = -\left(\frac{2}{1-\mu^2} - \frac{a^2}{r_+}\right)d\mu \tag{54}$$

we may formally write the solution as

$$\int_1^{\mathcal{A}(\mu)} \frac{d\mathcal{A}}{f(\mathcal{A})} = -\ln\frac{1+\mu}{1-\mu} + \frac{a^2}{r_+}\mu\Big|_0^\mu = \ln\frac{1-\mu}{1+\mu} + \frac{a^2}{r_+}\mu \tag{55}$$

When $\mu = 0$, both sides are equal to zero, if $1/f(\mathcal{A})$ does not diverge or diverge slowly there. For example, finite $f(\mathcal{A}=1)$ or $f(\mathcal{A}=1^-) \sim \sqrt{1-\mathcal{A}}$ are allowed. When $\mu = 1$, RHS is divergent. To deal with the divergence, we write the equation as

$$e^{\int_1^{\mathcal{A}(\mu)} \frac{d\mathcal{A}}{f(\mathcal{A})}} = e^{\ln\frac{1-\mu}{1+\mu} + \frac{a^2}{r_+}\mu} = \frac{1-\mu}{1+\mu} \times e^{\frac{a^2}{r_+}\mu} \tag{56}$$

as a result,

$$e^{\int_1^0 \frac{d\mathcal{A}}{f(\mathcal{A})}} = e^{-\int_0^1 \frac{d\mathcal{A}}{f(\mathcal{A})}} = 0 \tag{57}$$

which implies $f(\mathcal{A})|_{\mu=1} = f(\mathcal{A}=0) = 0$. ✓

Solution can be obtained from eq.(56):

$$F(\mathcal{A}) \equiv e^{\int_1^{\mathcal{A}} \frac{d\mathcal{A}}{f(\mathcal{A})}} \quad G(\mu) \equiv \frac{1-\mu}{1+\mu} \times e^{\frac{a^2}{r_+}\mu} \tag{58}$$

Both functions $F(\mathcal{A})$ and $G(\mu)$ can be computed, and obviously $\mathcal{A}(\mu)$ is obtained by enabling $F(\mathcal{A}) = G(\mu)$.

C. Variable Transform and Finite Difference

$$R = 1 - \frac{1}{1+r} \qquad 1+r = \frac{1}{1-R} \qquad R_{,r} = \frac{1}{(1+r)^2} = (1-R)^2 \tag{59}$$

$$\begin{aligned} A_{\phi,r} &= A_{\phi,R}(1-R)^2 \\ A_{\phi,rr} &= [A_{\phi,R}(1-R)^2]_{,R}(1-R)^2 = A_{\phi,RR}(1-R)^4 - 2A_{\phi,R}(1-R)^3 \end{aligned} \tag{60}$$

$$C_{RR}A_{\phi,RR} + C_{\mu\mu}A_{\phi,\mu\mu} + C_R A_{\phi,R} + C_\mu A_{\phi,\mu} = S \tag{61}$$

where

$$\begin{aligned} C_{RR} &= (1-R)^4 C_{rr} = (1-R)^4\mathcal{K} \\ C_R &= (1-R)^2 C_r - 2(1-R)^3 C_{rr} = (1-R)^2 C_r - 2(1-R)^3\mathcal{K} \end{aligned} \tag{62}$$

Finite Difference, $j \to \mu$**,** $l \to R$

$$\begin{aligned}
&C_{RR}\frac{A_{j,l+1}-2A_{j,l}+A_{j,l-1}}{\delta R^2}+C_{\mu\mu}\frac{A_{j+1,l}-2A_{j,l}+A_{j-1,l}}{\delta\mu^2}\\
&+C_{R\mu}\frac{(A_{j+1,l+1}-A_{j+1,l-1})-(A_{j-1,l+1}-A_{j-1,l-1})}{4\delta R\delta\mu}\\
&+C_R\frac{A_{j,l+1}-A_{j,l-1}}{2\delta R}+C_\mu\frac{A_{j+1,l}-A_{j-1,l}}{2\delta\mu}=S_{j,l}
\end{aligned}\tag{63}$$

$$\begin{aligned}
&(A_{j,l+1}-2A_{j,l}+A_{j,l-1})C_{RR}\frac{\delta^2}{\delta R^2}+(A_{j+1,l}-2A_{j,l}+A_{j-1,l})C_{\mu\mu}\frac{\delta^2}{\delta\mu^2}\\
&+(A_{j+1,l+1}+A_{j-1,l-1}-A_{j+1,l-1}-A_{j-1,l+1})C_{R\mu}\frac{\delta^2}{4\delta R\delta\mu}\\
&+(A_{j,l+1}-A_{j,l-1})C_R\frac{\delta^2}{2\delta R}+(A_{j+1,l}-A_{j-1,l})C_\mu\frac{\delta^2}{2\delta\mu}=S_{j,l}\delta^2
\end{aligned}\tag{64}$$

$$\begin{aligned}
\mathcal{L}A=\;&a_{j,l}A_{j+1,l}+b_{j,l}A_{j-1,l}+c_{j,l}A_{j,l+1}+d_{j,l}A_{j,l-1}\\
&+w_{j,l}(A_{j+1,l+1}-A_{j+1,l-1}-A_{j-1,l+1}+A_{j-1,l-1})+e_{j,l}A_{j,l}-f_{j,l}=0
\end{aligned}\tag{65}$$

$$\begin{aligned}
a&=\frac{\delta^2}{\delta\mu^2}C_{\mu\mu}+\frac{\delta^2}{2\delta\mu}C_\mu\\
b&=\frac{\delta^2}{\delta\mu^2}C_{\mu\mu}-\frac{\delta^2}{2\delta\mu}C_\mu\\
c&=\frac{\delta^2}{\delta R^2}C_{RR}+\frac{\delta^2}{2\delta R}C_R\\
d&=\frac{\delta^2}{\delta R^2}C_{RR}-\frac{\delta^2}{2\delta R}C_R\\
w&=\frac{\delta^2}{4\delta R\delta\mu}C_{R\mu}\\
e&=-2\left(\frac{\delta^2}{\delta R^2}C_{RR}+\frac{\delta^2}{\delta\mu^2}C_{\mu\mu}\right)\\
f&=S_{j,l}\delta^2=\frac{\Sigma}{\Delta}II'\delta^2
\end{aligned}\tag{66}$$

$$A_{j,l}^{\mathrm{new}}=A_{j,l}-\frac{\mathcal{L}A_{j,l}}{\partial(\mathcal{L}A_{j,l})/\partial A_{j,l}}=A_{j,l}-\frac{\mathcal{L}A_{j,l}}{e_{j,l}+\partial(-f_{j,l})/\partial A_{j,l}}\tag{67}$$

D. Returning Current

$$I^2/2 = \int_0^{A_0} II' dA \tag{68}$$

Returning Current, assuming $II' = \alpha \times (A_0 - A)(A_0 + \delta A_0 - A)$

$$-I^2/2 = \int_{A_0}^{A_0+\delta A_0} II' dA = -\frac{\alpha}{6}\delta A_0^3 \Rightarrow \alpha = (6/\delta A_0^3) \times I^2/2 \tag{69}$$

OR $II' = \alpha = I^2/2/\delta A_0$.

E. LS Boundary

$$(1-R)^2 C_r A_{\phi,R} + C_\mu A_{\phi,\mu} = S|_{\mathcal{K}=0} \tag{70}$$

$$dA_\phi = A_{\phi,R} dR + A_{\phi,\mu} d\mu = x \times S, \tag{71}$$

if

$$\frac{dR}{(1-R)^2 C_r} = \frac{d\mu}{C_\mu} = x \tag{72}$$

With

$$\begin{aligned} \frac{dR}{\epsilon \delta R} &= \frac{(1-R)^2 C_r/\delta R}{\sqrt{[(1-R)^2 C_r/\delta R]^2 + [C_\mu/\delta\mu]^2}} \\ \frac{d\mu}{\epsilon \delta\mu} &= \frac{C_\mu/\delta\mu}{\sqrt{[(1-R)^2 C_r/\delta R]^2 + [C_\mu/\delta\mu]^2}} \end{aligned} \tag{73}$$

we have

$$x = \frac{\epsilon}{\sqrt{[(1-R)^2 C_r/\delta R]^2 + [C_\mu/\delta\mu]^2}} \tag{74}$$

III. NHEK COORDINATE

$$g_{\mu\nu} = 2\Gamma \begin{pmatrix} -r^2(1-\Lambda^2) & 0 & 0 & r\Lambda^2 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r\Lambda^2 & 0 & 0 & \Lambda^2 \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{2\Gamma} \begin{pmatrix} -\frac{1}{r^2} & 0 & 0 & \frac{1}{r} \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{r} & 0 & 0 & \frac{1}{\Lambda^2}-1 \end{pmatrix}$$

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta} = \frac{\sin\theta}{\Gamma}$$

$$\sqrt{-g} = 4\Gamma^2\Lambda = 4\sin\theta\Gamma$$

$$\sqrt{-g}g^{rr} = 2r^2\sin\theta \qquad \sqrt{-g}g^{\theta\theta} = 2\sin\theta$$

$$F_{r\phi} = -F_{\phi r} = \psi_{,r}\ , F_{\theta\phi} = -F_{\phi\theta} = \psi_{,\theta}\ , \tag{75}$$

$$F_{tr} = -F_{rt} = \Omega\psi_{,r}\ , F_{t\theta} = -F_{\theta t} = \Omega\psi_{,\theta}\ , \tag{76}$$

$$F_{r\theta} = -F_{\theta r} = -\frac{I}{r^2\Lambda}\ . \tag{77}$$

$$-\Omega\left[(\sqrt{-g}F^{tr})_{,r} + (\sqrt{-g}F^{t\theta})_{,\theta}\right] + F_{r\theta}I'(A_\phi) + \left[(\sqrt{-g}F^{\phi r})_{,r} + (\sqrt{-g}F^{\phi\theta})_{,\theta}\right] = 0. \tag{78}$$

$$\begin{aligned} F_{tr} &= \Omega A_{\phi,r} \\ F^{tr} &= g^{rr}(g^{tt}F_{tr} + g^{t\phi}F_{\phi r}) \\ &= \left(g^{tt}\Omega - g^{t\phi}\right)g^{rr}A_{\phi,r} \end{aligned} \tag{79}$$

$$\begin{aligned} F_{\phi r} &= -A_{\phi,r} \\ F^{\phi r} &= g^{\phi\alpha}g^{rr}F_{\alpha r} = g^{rr}(g^{\phi t}F_{tr} + g^{\phi\phi}F_{\phi r}) \\ &= (g^{\phi t}\Omega - g^{\phi\phi})g^{rr}A_{\phi,r} \end{aligned} \tag{80}$$

$$\begin{aligned} F_{t\theta} &= \Omega A_{\phi,\theta} = \Omega(-\sin\theta A_{\phi,\mu}) \\ F^{t\theta} &= g^{\theta\theta}(g^{tt}F_{t\theta} + g^{t\phi}F_{\phi\theta}) = g^{\theta\theta}(g^{tt}\Omega - g^{t\phi})A_{\phi,\theta} \end{aligned} \tag{81}$$

$$F_{r\theta} = -\frac{I}{r^2\Lambda}, \qquad F_{r\theta}I' = -\frac{II'}{r^2\Lambda} \tag{82}$$

$$\begin{aligned}
&F_{\phi\theta} = -A_{\phi,\theta} \\
&F^{\phi\theta} = g^{\theta\theta}(g^{\phi t}F_{t\theta} + g^{\phi\phi}F_{\phi\theta}) = g^{\theta\theta}(g^{t\phi}\Omega - g^{\phi\phi})A_{\phi,\theta}
\end{aligned} \tag{83}$$

$$\begin{aligned}
&-\Omega(\sqrt{-g}F^{tr})_{,r} + (\sqrt{-g}F^{\phi r})_{,r} \leftarrow X \\
&= -\Omega\left[\sqrt{-g}g^{rr}(g^{tt}\Omega - g^{t\phi})A_{\phi,r}\right]_{,r} + \left[\sqrt{-g}g^{rr}(g^{\phi t}\Omega - g^{\phi\phi})A_{\phi,r}\right]_{,r} \\
&= \Omega\left[\Lambda(\Omega+r)A_{\phi,r}\right]_{,r} + \left[\left(\Lambda r(\Omega+r) - \frac{r^2}{\Lambda}\right)A_{\phi,r}\right]_{,r} \\
&= \left[\Lambda(\Omega+r)^2 - \frac{r^2}{\Lambda}\right]A_{\phi,rr} + \Lambda(\Omega+r)\Omega' A_{\phi,r}^2 + 2\left[\Lambda(\Omega+r) - \frac{r}{\Lambda}\right]A_{\phi,r}
\end{aligned} \tag{84}$$

$$\begin{aligned}
&-\Omega(\sqrt{-g}F^{t\theta})_{,\theta} + F_{r\theta}I'(A_\phi) + (\sqrt{-g}F^{\phi\theta})_{,\theta} \leftarrow Y \\
&= -\Omega\left[\sqrt{-g}g^{\theta\theta}(g^{tt}\Omega - g^{t\phi})A_{\phi,\theta}\right]_{,\theta} - \frac{II'}{r^2\Lambda} + \left[\sqrt{-g}g^{\theta\theta}(g^{t\phi}\Omega - g^{\phi\phi})A_{\phi,\theta}\right]_{,\theta} \\
&= \Omega\left[\frac{\sin\theta}{\Gamma}(\frac{\Omega}{r^2} + \frac{1}{r})A_{\phi,\theta}\right]_{,\theta} - \frac{II'}{r^2\Lambda} + \left[\frac{\sin\theta}{\Gamma}\left(\frac{\Omega}{r} - (\frac{1}{\Lambda^2} - 1)\right)A_{\phi,\theta}\right]_{,\theta} \\
&= \Omega\left[\frac{\sin\theta}{\Gamma}\frac{\Omega+r}{r^2}A_{\phi,\theta}\right]_{,\theta} - \frac{II'}{r^2\Lambda} + \left[\frac{\sin\theta}{\Gamma}\left(\frac{\Omega+r}{r} - \frac{1}{\Lambda^2}\right)A_{\phi,\theta}\right]_{,\theta} \\
&= \Omega\sin\theta\left[\frac{\sin^2\theta}{\Gamma}\frac{\Omega+r}{r^2}A_{\phi,\mu}\right]_{,\mu} - \frac{II'}{r^2\Lambda} + \sin\theta\left[\frac{\sin^2\theta}{\Gamma}\left(\frac{\Omega+r}{r} - \frac{1}{\Lambda^2}\right)A_{\phi,\mu}\right]_{,\mu} \\
&= \frac{\sin\theta}{r^2}\left[\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2 - \Gamma r^2\right]A_{\phi,\mu\mu} - \frac{II'}{r^2\Lambda} \\
&+ \frac{\sin\theta}{r^2}\partial_\mu^\Omega\left[\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2 - \Gamma r^2\right]A_{\phi,\mu} + \frac{\sin\theta}{r^2}\frac{\sin^2\theta}{\Gamma}(\Omega+r)\Omega' A_{\phi,\mu}^2
\end{aligned} \tag{85}$$

$$\begin{aligned}
0 &= (X+Y)\sin\theta \\
&= \left[\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2 - \Gamma r^2\right]A_{\phi,rr} + \frac{\sin^2\theta}{\Gamma}(\Omega+r)\Omega' A_{\phi,r}^2 + 2\left[\frac{\sin^2\theta}{\Gamma}(\Omega+r) - \Gamma r\right]A_{\phi,r} \\
&+ \frac{\sin^2\theta}{r^2}\left[\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2 - \Gamma r^2\right]A_{\phi,\mu\mu} - \frac{\Gamma}{r^2}II' \\
&+ \frac{\sin^2\theta}{r^2}\partial_\mu^\Omega\left[\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2 - \Gamma r^2\right]A_{\phi,\mu} + \frac{\sin^2\theta}{r^2}\frac{\sin^2\theta}{\Gamma}(\Omega+r)\Omega' A_{\phi,\mu}^2
\end{aligned}$$

$$
\begin{aligned}
&\left[A_{\phi,rr}+\frac{\sin^2\theta}{r^2}A_{\phi,\mu\mu}\right]\mathcal{K}(r,\theta;\Omega)\\
&+\left[A_{\phi,r}\partial_r^\Omega+\frac{\sin^2\theta}{r^2}A_{\phi,\mu}\partial_\mu^\Omega\right]\mathcal{K}(r,\theta;\Omega)\\
&+\frac{1}{2}\left[A_{\phi,r}^2+\frac{\sin^2\theta}{r^2}A_{\phi,\mu}^2\right]\Omega'\partial_\Omega\mathcal{K}(r,\theta;\Omega)-\frac{\Gamma}{r^2}II'=0, \qquad (86)
\end{aligned}
$$

with

$$\mathcal{K}=\frac{\sin^2\theta}{\Gamma}(\Omega+r)^2-r^2\Gamma \qquad (87)$$

$$Y=\frac{1}{2r^2\Lambda A_{\phi,\theta}}\left\{[\Lambda(\Omega+r)A_{\phi,\theta}]^2-I^2\right\}_{,\theta}-\left(\frac{A_{\phi,\theta}}{\Lambda}\right)_{,\theta} \qquad (88)$$

Regularity condition at $r=0$, requires

$$\left\{[\Lambda(\Omega+r)A_{\phi,\theta}]^2-I^2\right\}_{r=0}=0, \quad \partial_r\left\{[\Lambda(\Omega+r)A_{\phi,\theta}]^2-I^2\right\}_{r=0}=0$$

which imply

$$[\Lambda(\Omega+r)A_{\phi,\theta}]^2-I^2=\sum_{n=2}^{\infty}r^nf_n(\theta) \qquad (89)$$