The force-free magnetosphere of a Kerr black hole: the role of boundary conditions

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(Dated: June 25, 2018)

The role of boundary conditions in the structure of a force-free black hole magnetosphere was rarely discussed, since previous studies have been focused on the the field lines entering the horizon which is causally disconnected and on which the boundary condition imposed usually makes no difference to the magnetosphere structure. However, recent high-resolution general relativistic (GR) force-free simulation shows that there are both field lines entering the horizon and field lines ending up on the equatorial current sheet within the ergosphere for asymptotic uniform field. For the latter, the equatorial boundary condition is well approximated being marginally force-free, i.e., $B^2-E^2=0$, where B and E are the magnetic and electric field strength, respectively. In this paper, we revisit this problem by solving the GR Grad-Shafranov equation which governs the structure of the force-free magnetosphere in steady state and self-consistently imposing the marginally force-free equatorial condition. We also discuss the applicability of this boundary condition and the numerical algorithm proposed in this paper for general magnetic field configurations.

I. INTRODUCTION

II. BASIC EQUATIONS

In this paper, we adopt the Kerr-Schild coordinate with the line element

$$ds^2 = -\left(1-\frac{2r}{\Sigma}\right)dt^2 + \left(\frac{4r}{\Sigma}\right)drdt + \left(1+\frac{2r}{\Sigma}\right)dr^2 + \Sigma d\theta^2 - \frac{4ar\sin^2\theta}{\Sigma}d\phi dt - 2a\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta d\phi dr + \frac{\beta}{\Sigma}\sin^2\theta d\phi^2$$

where $\Sigma = r^2+a^2\mu^2$, $\Delta = r^2-2r+a^2$, $\beta = \Delta\Sigma + 2r(r^2+a^2)$, and a is the dimensionless BH spin. In the force-free approximation, electromagnetic energy greatly exceeds that of matter. Consequently, the force-free magnetospheres is governed by energy conservation equation of electromagnetic field, or conventionally called as the GS equation. In the Kerr spacetime, the axisymmetric and steady GS equation can be written in a compact form [1]

$$\begin{aligned}&\left[A_{\phi,rr} + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu\mu}\right]\mathcal{K}(r,\theta;\Omega)\\ &+\left[A_{\phi,r}\partial_r^\Omega + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}\partial_\mu^\Omega\right]\mathcal{K}(r,\theta;\Omega)\\ &+\frac{1}{2}\left[A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right]\Omega'\partial_\Omega\mathcal{K}(r,\theta;\Omega)\\ &-\frac{\Sigma}{\Delta}II' = 0\ ,\end{aligned} \tag{1}$$

with $\mathcal{K}(r,\theta;\Omega) = \left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]$ being the LS function, $\mu\equiv\cos\theta$, and the primes designate derivatives with respect to A_ϕ. $\partial_i^\Omega(i=r,\mu)$ denotes the partial derivative with respect to coordinate i with Ω fixed, and ∂_Ω is the derivative with respect to Ω.

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III. UNIFORM FIELD SOLUTION

A. Boundary conditions

The boundary conditions

$$A_\phi(\mu=1)=0, \tag{2a}$$

$$A_{\phi,\mu}(\mu=0,r>2)=0, \tag{2b}$$

$$B^2-E^2(\mu=0,r_+\le r\le 2)=0. \tag{2c}$$

where r_+ is the radius of the event horizon.

In our computation, we use following numerical prescription

$$\int_{A_\phi^{\rm HE}}^{A_\phi^{\rm EE}}\frac{B^2-E^2}{B^2+E^2}dA_\phi \Bigg/ \left(A_\phi^{\rm EE}-A_\phi^{\rm HE}\right) < 10^{-3}. \tag{3}$$

as a proxy of the marginally force-free equatorial boundary condition (2c), where "HE" and "EE" are short for Horizon-Equator and Ergosphere-Equator, respectively; and we choose B^2+E^2 to be the energy density measured by the zero-angular-momentum-observers. Specifically,

$$\begin{aligned}B^2-E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[-\mathcal{K}\left(A_{\phi,r}^2+\frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right)+\frac{\Sigma}{\Delta}I^2\right]\\ B^2+E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[\left(\mathcal{K}+\frac{\Delta\Sigma}{\beta}\right)\left(A_{\phi,r}^2+\frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right)+\frac{\Sigma}{\Delta}I^2\right]\end{aligned} \tag{4}$$

B. Numerical method

We aim to find a pair of $\Omega(A_\phi)$ and $I(A_\phi)$ connected by the radiation condition

$$I = 2\Omega A_\phi, \tag{5}$$

and enabling field lines smoothly crossing the LS, and suitable normal derivative $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ on the equator guaranteeing the boundary condition (2c).

In our computation, we define a new radial coordinate $R = r/(1 + r)$, confine our computation domain $R \times \mu$ in the region $[R(r_+), 1] \times [0, 1]$, and implement a uniform 512×64 grid. The detailed algorithm is clarified in the following steps.

1. We choose an initial guess for the field configuration, functions $\{\Omega(A_\phi), I(A_\phi)\}$ and equatorial boundary condition as follows

$$A_\phi = \frac{B_0}{2} r^2 \sin^2\theta, \tag{6a}$$

$$\Omega = 0.5\Omega_{\rm H}\left(1 - A_\phi/A_\phi^{\rm HE}\right), \tag{6b}$$

$$I = \Omega_{\rm H} A_\phi \left(1 - A_\phi/A_\phi^{\rm HE}\right), \tag{6c}$$

$$A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2) = -(r/r_+)^3, \tag{6d}$$

where $\Omega_{\rm H} = a/2r_+$ is the angular velocity of the BH.

2. We evolve the GS equation (1) using the well-known relaxation method [2] and adjust $II'(A_\phi)$ until field lines smoothly cross the LS [see e.g. 3–6, for more details].

3. Usually the current I found in Step 2 neither satisfies the radiation condition (5) nor guarantees the boundary condition (2c). We adjust $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ as follows

$$A_{\phi,\mu}|_{\rm new} = A_{\phi,\mu}|_{\rm old} + \zeta_1 \times [2\Omega A_\phi(2\Omega A_\phi)' - II'], \tag{7}$$

where ζ_1 is an empirical step size. For each new $A_{\phi,\mu}$, we repeat Step 2 until $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ converges, i.e. the condition $2\Omega A_\phi(2\Omega A_\phi)' = II'$ is achieved for $A_\phi \in (A_\phi^{\rm HE}, A_\phi^{\rm EE})$.

4. The remaining task is to adjust $\Omega(0 < A_\phi < A_\phi^{\rm HE})$ enabling the radiation condition (5) for $A_\phi \in (0, A_\phi^{\rm HE})$ and to adjust $\Omega(A_\phi^{\rm HE} \le A_\phi \le A_\phi^{\rm EE})$ enabling the boundary condition (2c) for $A_\phi \in [A_\phi^{\rm HE}, A_\phi^{\rm EE}]$. The first part is straightforward, i.e.,

$$2A_\phi\Omega_{\rm new} = I|_{0<A_\phi<A_\phi^{\rm HE}}, \tag{8}$$

and the second part can be realized by iterative correction

$$2A_\phi(\Omega_{\rm new} - \Omega_{\rm old}) = -\zeta_2 \times \Delta(B^2 - E^2)|_{\mu=0, r_+\le r\le 2}, \tag{9}$$

where ζ_2 is again an empirical step size, and we have multiplied factor Δ in the correction term to avoid numerical difficulty in the vicinity of the event horizon. To eliminate unphysical discontinuity in the angular velocity at $A_\phi^{\rm HE}$, we fit $\Omega_{\rm new}(A_\phi)$ on the whole range $(0, A_\phi^{\rm EE})$ via a fifth-order polynomial.

5. For the new angular velocity $\Omega_{\rm new}(A_\phi)$ obtained in Step 4, we repeat Step 2 to Step 4, until the numerical prescription (3) for the boundary condition (2c) is satisfied.

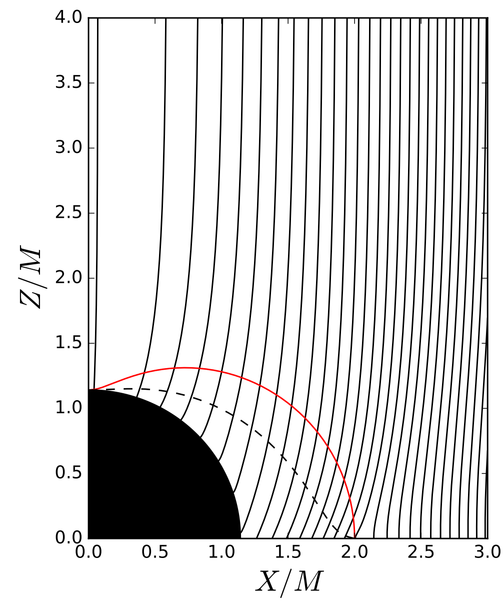

FIG. 1. The configuration of field lines for the magnetosphere of a Kerr BH with spin $a = 0.99$, where the solid/red line is the ergosphere and the dashed/black line is the LS, both of which intersect with the equator at $r = 2M$.

C. Numerical results

In Figure 1, we plot the magnetic field lines enclosing a BH with spin $a = 0.99$ as an example, and in the upper panel of Figure 2, we show the angular velocity functions $\Omega(A_\phi)$ for different BH spins. Combining the two figures, we see three generic features: (1) the LS runs from $r = r_+$ to $r = 2M$ as θ varies from 0 to $\pi/2$, (2) there is no current sheet within the magnetosphere except the equatorial current sheet extending from r_+ to $2M$, which gives rise to a cusp ($A_{\phi,\mu} \ne 0$) to the equatorial magnetic field lines, (3) magnetic field lines entering the ergosphere end up either on the horizon or on the equatorial current sheet, both of which carry electric current and therefore Poynting flux.

In the lower panel of Figure 2 and in Figure 3, we compare our numerical results with recent high-resolution simulations performed by East and Yang [7] (EY Sims). In Figure 3, the solid lines are curves fitted by polynomials

We find the angular velocity functions in two studies show a good agreement, while the value of our magnetic flux $A_\phi^{\rm EE}$ is lower by $\sim 5\%$, which gives to $\sim 10\%$ lower energy extraction rates.

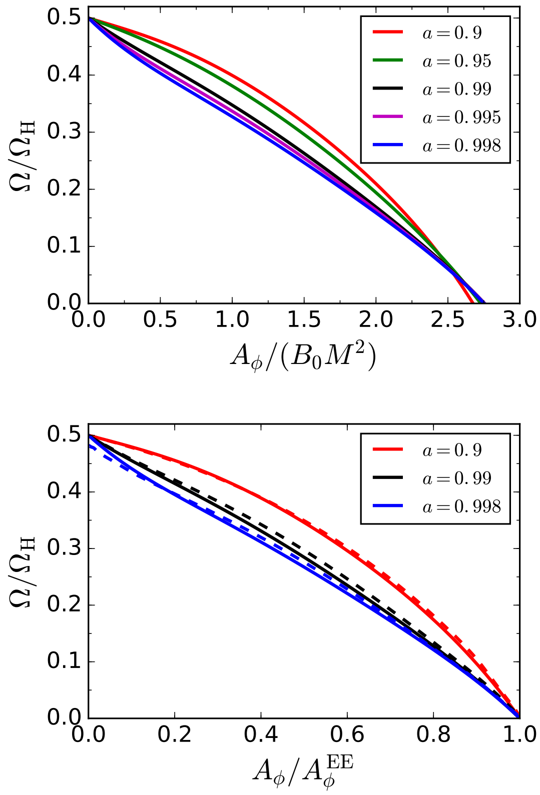

FIG. 2. Upper panel: the angular velocity functions $\Omega(A_\phi)$ for different BH spins. Lower Panel: comparison of our numerical results (solid lines) with the simulation results in [7] (dashed lines).

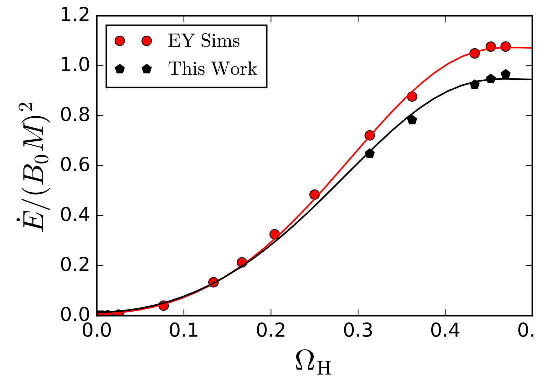

FIG. 3. Comparison of the energy extraction rate $\dot{E}(\Omega_{\rm H})$ derived from our numerical solutions and from the simulations [7], where the former is lower than the latter one by 10%.

IV. DISCUSSIONS

A. Solution uniqueness

B. Topology of near-horizon field lines

V. SUMMARY

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