The force-free magnetosphere of a Kerr black hole: the role of boundary conditions

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The role of boundary conditions in the structure of a force-free black hole magnetosphere was rarely discussed, since previous studies have been focused on the the field lines entering the horizon which is causally disconnected and on which the boundary condition imposed usually makes no difference to the magnetosphere structure. However, recent high-resolution general relativistic (GR) force-free simulation shows that there are both field lines entering the horizon and field lines ending up on the equatorial current sheet within the ergosphere for asymptotic uniform field. For the latter, the equatorial boundary condition is well approximated being marginally force-free, i.e., $B^2 - E^2 = 0$, where B and E are the magnetic and electric field strength, respectively. In this paper, we revisit this problem by solving the GR Grad-Shafranov equation which governs the structure of the force-free magnetosphere in steady state and self-consistently imposing the marginally force-free equatorial condition. We also discuss the applicability of this boundary condition and the numerical algorithm proposed in this paper for general magnetic field configurations.

I. INTRODUCTION

II. BASIC EQUATIONS

In this paper, we adopt the Kerr-Schild coordinate with the line element

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2r}{\Sigma}\right)dt^2 + \left(\frac{4r}{\Sigma}\right)dr dt + \left(1+\frac{2r}{\Sigma}\right)dr^2 \\
&+ \Sigma d\theta^2 - \frac{4ar\sin^2\theta}{\Sigma}d\phi dt - 2a\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta d\phi dr \\
&+ \frac{\beta}{\Sigma}\sin^2\theta d\phi^2
\end{aligned}
$$

where $\Sigma = r^2 + a^2\mu^2$, $\Delta = r^2 - 2r + a^2$, $\beta = \Delta\Sigma + 2r(r^2 + a^2)$, and a is the dimensionless BH spin. In the force-free approximation, electromagnetic energy greatly exceeds that of matter. Consequently, the force-free magnetospheres is governed by energy conservation equation of electromagnetic field, or conventionally called as the GS equation. In the Kerr spacetime, the axisymmetric and steady GS equation can be written in a compact form [1]

$$
\begin{aligned}
&\left[A_{\phi,rr} + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu\mu}\right]\mathcal{K}(r,\theta;\Omega) \\
&+\left[A_{\phi,r}\partial_r^\Omega + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}\partial_\mu^\Omega\right]\mathcal{K}(r,\theta;\Omega) \\
&+\frac{1}{2}\left[A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right]\Omega'\partial_\Omega\mathcal{K}(r,\theta;\Omega) \\
&-\frac{\Sigma}{\Delta}II' = 0 \ , \qquad (1)
\end{aligned}
$$

with $\mathcal{K}(r,\theta;\Omega) = \left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]$ being the LS function, $\mu \equiv \cos\theta$, and the primes designate derivatives with respect to A_ϕ. $\partial_i^\Omega (i = r, \mu)$ denotes the partial derivative with respect to coordinate i with Ω fixed, and ∂_Ω is the derivative with respect to Ω.

III. UNIFORM FIELD SOLUTION

A. Boundary conditions

$$A_\phi(\mu = 1) = 0, \qquad (2a)$$

$$A_{\phi,\mu}(\mu = 0, r > 2) = 0, \qquad (2b)$$

$$B^2 - E^2(\mu = 0, r_+ \le r \le 2) = 0. \qquad (2c)$$

where

$$
\begin{aligned}
B^2 - E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[-\mathcal{K}\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2\right] \\
B^2 + E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[\left(\mathcal{K} + \frac{\Delta\Sigma}{\beta}\right)\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2\right]
\end{aligned} \qquad (3)
$$

$$\int_{A_\phi^{\rm HE}}^{A_\phi^{\rm EE}} \frac{B^2 - E^2}{B^2 + E^2} dA_\phi \Big/ \left(A_\phi^{\rm EE} - A_\phi^{\rm HE}\right) < 10^{-3}. \qquad (4)$$

where "HE" and "EE" are short for Horizon-Equator and Ergosphere-Equator, respectively.

B. Numerical method

We aim to find a pair of $\Omega(A_\phi)$ and $I(A_\phi)$ satisfying the radiation condition

$$I = 2\Omega A_\phi, \qquad (5)$$

enabling field lines smoothly crossing the LS, and suitable normal derivative $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ on the equator guaranteeing the boundary condition (2c).

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In our computation, we define a new radial coordinate $R = r/(1+r)$, confine our computation domain $R \times \mu$ in the region $[R(r_+), 1] \times [0, 1]$, and implement a uniform 512×64 grid. The detailed algorithm is listed as follows.

1. We choose an initial guess for the field configuration, $\Omega(A_\phi)$, $I(A_\phi)$ and equator boundary condition as follows

$$A_\phi = r^2 \sin^2\theta, \tag{6a}$$
$$\Omega = 0.5\Omega_{\rm H}\left(1 - A_\phi/A_\phi^{\rm HE}\right), \tag{6b}$$
$$I = \Omega_{\rm H} A_\phi \left(1 - A_\phi/A_\phi^{\rm HE}\right), \tag{6c}$$
$$A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2) = -(r/r_+)^3. \tag{6d}$$

2. We evolve the GS equation (1) and adjust $II'(A_\phi)$ enabling smooth field lines across the LS [see e.g. 2–5, for details].

3. Usually the current I found in Step 2 neither satisfies the radiation condition (5) nor guarantees the boundary condition (2c). We find it is possible to adjust $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ as follows

$$A_{\phi,\mu}|_{\rm new} = A_{\phi,\mu}|_{\rm old} + \zeta_1 \times [2\Omega A_\phi (2\Omega A_\phi)' - II'], \tag{7}$$

enabling the radiation condition (5) for $A_\phi \in [A_\phi^{\rm HE}, A_\phi^{\rm EE}]$, where ζ_1 is an empirical step size.

4. The remaining task is to adjust $\Omega(0 < A_\phi < A_\phi^{\rm HE})$ enabling the radiation condition (5) for $A_\phi \in (0, A_\phi^{\rm HE})$ and to adjust $\Omega(A_\phi^{\rm HE} \le A_\phi \le A_\phi^{\rm EE})$ enabling the boundary condition (2c) for $A_\phi \in [A_\phi^{\rm HE}, A_\phi^{\rm EE}]$. The first part is straightforward, i.e.,

$$\Omega_{\rm new} = \left.\frac{I}{2A_\phi}\right|_{0<A_\phi<A_\phi^{\rm HE}}, \tag{8}$$

and the second part can be realized by

$$\Omega_{\rm new} = \Omega_{\rm old} - \zeta_2 \times \left.\frac{\Delta(B^2 - E^2)}{2A_\phi}\right|_{\mu=0, r_+ \le r \le 2}, \tag{9}$$

where ζ_2 is again an empirical step size, and we have multiplied factor Δ in the correction term to avoid numerical difficulty in the vicinity of the event horizon. To eliminate unphysical discontinuity in the angular velocity at $A_\phi^{\rm HE}$, we fit $\Omega_{\rm new}(A_\phi)$ on the whole range $[0, A_\phi^{\rm EE}]$ via a fifth-order polynomial.

5. For the new angular velocity $\Omega_{\rm new}(A_\phi)$ obtained in Step 4, we repeat step 2 to step 4, until the numerical prescription (4) for the boundary condition (2c) is satisfied.

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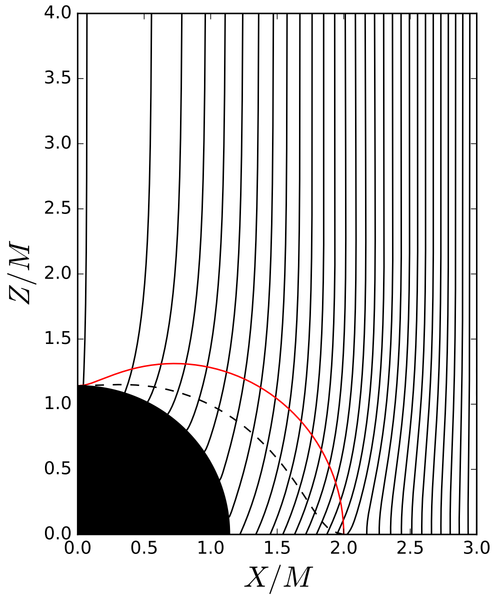

FIG. 1. The configuration of field lines for the magnetosphere of a Kerr BH with spin $a = 0.99$, where the solid/red line is the ergosphere and the dashed/black line is the LS, both of which intersect with the equator at $r = 2M$.

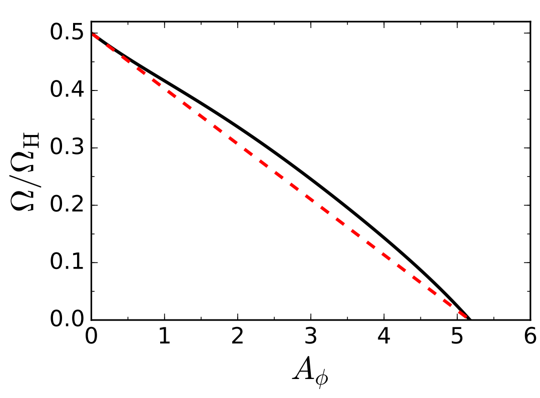

FIG. 2. $a = 0.99$