The force-free magnetosphere of a Kerr black hole: the role of boundary conditions

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I. INTRODUCTION

II. BASIC EQUATIONS

In this paper, we adopt the Kerr-Schild coordinate with the line element

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2r}{\Sigma}\right)dt^2 + \left(\frac{4r}{\Sigma}\right)drdt + \left(1+\frac{2r}{\Sigma}\right)dr^2 \\
&+ \Sigma d\theta^2 - \frac{4ar\sin^2\theta}{\Sigma}d\phi dt - 2a\left(1+\frac{2r}{\Sigma}\right)\sin^2\theta d\phi dr \\
&+ \frac{\beta}{\Sigma}\sin^2\theta d\phi^2
\end{aligned}
$$

where $\Sigma = r^2 + a^2\mu^2$, $\Delta = r^2 - 2r + a^2$, $\beta = \Delta\Sigma + 2r(r^2 + a^2)$, and a is the dimensionless BH spin. In the force-free approximation, electromagnetic energy greatly exceeds that of matter. Consequently, the force-free magnetospheres is governed by energy conservation equation of electromagnetic field, or conventionally called as the GS equation. In the Kerr spacetime, the axisymmetric and steady GS equation can be written in a compact form [1]

$$
\begin{aligned}
&\left[A_{\phi,rr} + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu\mu}\right]\mathcal{K}(r,\theta;\Omega) \\
&+\left[A_{\phi,r}\partial_r^\Omega + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}\partial_\mu^\Omega\right]\mathcal{K}(r,\theta;\Omega) \\
&+\frac{1}{2}\left[A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right]\Omega'\partial_\Omega\mathcal{K}(r,\theta;\Omega) \\
&-\frac{\Sigma}{\Delta}II' = 0\ ,
\end{aligned}
\tag{1}
$$

with $\mathcal{K}(r,\theta;\Omega) = \left[\frac{\beta}{\Sigma}\Omega^2\sin^2\theta - \frac{4ra}{\Sigma}\Omega\sin^2\theta - \left(1-\frac{2r}{\Sigma}\right)\right]$ being the LS function, $\mu \equiv \cos\theta$, and the primes designate derivatives with respect to A_ϕ. $\partial_i^\Omega (i = r, \mu)$ denotes the partial derivative with respect to coordinate i with Ω fixed, and ∂_Ω is the derivative with respect to Ω.

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III. UNIFORM FIELD SOLUTION

A. Boundary conditions

$$A_\phi(\mu=1) = 0, \tag{2a}$$
$$A_{\phi,\mu}(\mu=0, r>2) = 0, \tag{2b}$$
$$B^2 - E^2(\mu=0, r_+ \le r \le 2) = 0. \tag{2c}$$

where

$$
\begin{aligned}
B^2 - E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[-\mathcal{K}\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2\right] \\
B^2 + E^2 &= \frac{1}{\Sigma\sin^2\theta}\left[\left(\mathcal{K} + \frac{\Delta\Sigma}{\beta}\right)\left(A_{\phi,r}^2 + \frac{\sin^2\theta}{\Delta}A_{\phi,\mu}^2\right) + \frac{\Sigma}{\Delta}I^2\right]
\end{aligned}
\tag{3}
$$

$$\int_{A_\phi^{\rm HE}}^{A_\phi^{\rm EE}} \frac{B^2-E^2}{B^2+E^2}dA_\phi \Big/ \left(A_\phi^{\rm EE} - A_\phi^{\rm HE}\right) < 10^{-3}. \tag{4}$$

where “HE” and “EE” are short for Horizon-Equator and Ergosphere-Equator, respectively.

B. Numerical method

We aim to find a pair of $\Omega(A_\phi)$ and $I(A_\phi)$ satisfying the radiation condition

$$I = 2\Omega A_\phi, \tag{5}$$

and suitable $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ enabling the boundary condition (2c).

1. We choose an initial guess for the field configuration, $\Omega(A_\phi)$ and equator boundary condition as follows

$$A_\phi = r^2\sin^2\theta, \tag{6a}$$
$$\Omega = 0.5\Omega_{\rm H}\left(1 - A_\phi/A_\phi^{\rm HE}\right), \tag{6b}$$
$$A_{\phi,\mu}(\mu=0, r_+ \le r \le 2) = -1\times(r/r_+)^3. \tag{6c}$$

2. We evolve the GS equation and find II' enabling smooth field lines across the LS [see 2–4, for details].

3. Usually the current I found in Step 2 neither satisfies the radiation condition (5) nor enables the boundary condition (2c). We find it is possible to adjust $A_{\phi,\mu}(\mu = 0, r_+ \le r \le 2)$ as follows

$$A_{\phi,\mu\rm new} = A_{\phi,\mu\rm old} + \zeta_1 \times [2\Omega A_\phi(2\Omega A_\phi)' - II'], \tag{7}$$

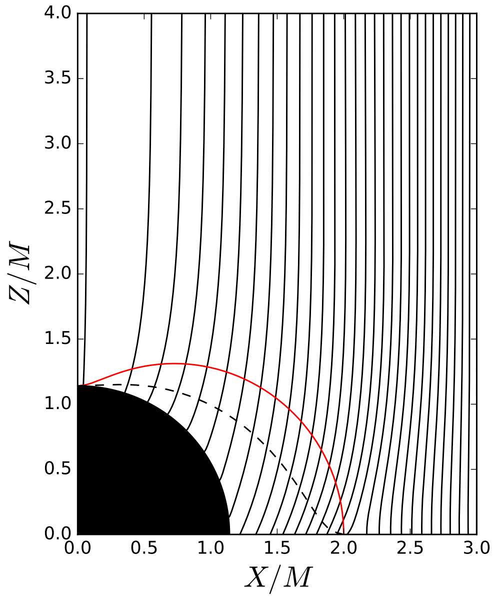

FIG. 1. The configuration of field lines for the magnetosphere of a Kerr BH with spin $a = 0.99$, where the solid/red line is the ergosphere and the dashed/black line is the LS, both of which intersect with the equator at $r = 2M$.

enabling the radiation condition (5) for $A_\phi \in [A_\phi^{\rm HE}, A_\phi^{\rm EE}]$, where ζ_1 is an empirical step size.

4. The final step is to adjust $\Omega(A_\phi < A_\phi^{\rm HE})$ enabling the radiation condition (5) for $A_\phi < A_\phi^{\rm HE}$ and to adjust $\Omega(A_\phi^{\rm HE} \le A_\phi \le A_\phi^{\rm EE})$ enabling the boundary condition (2c) for $A_\phi \in [A_\phi^{\rm HE}, A_\phi^{\rm EE}]$. The first part is straightforward, i.e.,

$$\Omega_{\rm new}(A_\phi < A_\phi^{\rm HE}) = \frac{I}{2A_\phi}, \tag{8}$$

and the second part can be realized by

$$\Omega_{\rm new} = \Omega_{\rm old} - \zeta_2 \times \frac{\Delta(B^2 - E^2)}{2A_\phi}\Big|_{\mu=0, r_+ \le r \le 2} \tag{9}$$

where ζ_2 is again an empirical step size, and we have multiplied factor Δ to avoid numerical difficulty in the vicinity of the event horizon. To avoid unphysical discontinuity in the angular velocity at $A_\phi^{\rm HE}$, we fit $\Omega_{\rm new}(A_\phi)$ via a fifth-order polynomial.

5. For the new angular velocity $\Omega_{\rm new}(A_\phi)$ obtained in Step 4, we repeat step 2 to step 4, until the numerical prescription (4) is satisfied.

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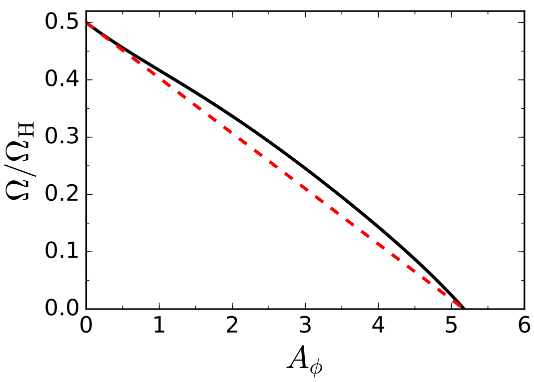

FIG. 2. $a = 0.99$