HW4_Python

April 9, 2024

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```
# imports
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
from scipy.stats import norm

# set up the environment
%matplotlib inline
plt.rcParams["figure.figsize"] = (12, 9)  # set default figure size
sns.set_style("whitegrid")  # set default style
rng = np.random.default_rng()  # random number generator
```

0.1 Quanto Review

Let X denotes the exchange rate measured in domestic currency for a unit of foreign currency:

$$\frac{dX}{X} = \mu_x dt + \sigma_x dB_x$$

Let S denotes the foreign asset measured in foreign currency:

$$\frac{dS}{S} = \mu_s dt + \sigma_s dB_s$$

where

- $(dB_x)(dB_s) = \rho dt$
- q - constant dividend yield of the foreign asset
- r - risk-free interest rate in domestic currency
- r_f - risk-free interest rate in foreign currency

A "**quanto**" is a derivative written on foreign asset and converted to domestic currency at a fixed exchange rate $\bar{X}$, i.e. $V(T) = \bar{X}S(T)$. The price of the quanto at time $t \leq T$ is given by Eq. (6.7):

$$V(t) = e^{(r_f - r - q - \rho\sigma_x\sigma_s)(T-t)}\bar{X}S(t)$$

Following Eq. (6.14), a replication strategy for the quanto is:

- Long (invest) $V(t)$ units of domestic currency in the foreign asset
- Short (borrow) $V(t)$ units of domestic currency at the foreign risk-free rate r_f;
- Long (invest) $V(t)$ units of domestic currency in the domestic risk-free asset.

0.2 Question 6.5 Discretely-Rebalanced Quanto Hedge

Following the similar logic from "**Discretely-Rebalanced Delta Hedges**" (Sect. 3.10 in the textbook), the money-market hedging strategy for the quanto can be implemented as follows:

1. At time $t_0 = 0$:
   - long $V(0)$ units of domestic currency in the foreign asset, which means $\delta(0) = \frac{V(t_0)}{X(t_0)S(t_0)}$ shares;
   - short $V(0)$ units of domestic currency at the foreign risk-free rate r_f;
   - long $V(0)$ units of domestic currency in the domestic risk-free asset, i.e., cash account $C(0) = V(0)$.
2. At each rebalancing time t_i, $i = 1, 2, \dots, N-1$:
   - long $\delta(t_i) = \frac{V(t_i)}{X(t_i)S(t_i)}$ shares of the foreign asset;
   - short $V(t_i)$ units of domestic currency at the foreign risk-free rate r_f;
   - the new cash account $C(t_i)$ consists of:
     - adjusting shares of the foreign asset: $-(\delta(t_i) - \delta(t_{i-1}))S(t_i)X(t_i)$;
     - adjusting short position at the foreign risk-free asset: $V(t_i) - V(t_{i-1})e^{r_f \Delta t}X(t_i)/X(t_{i-1})$;
     - continuous dividend payment from the foreign asset: $\delta(t_{i-1})S(t_{i-1})(e^{q\Delta t} - 1)X(t_i)$;
     - accumulation/payment of interest on the cash position of domestic currency: $C(t_{i-1})e^{r\Delta t}$.
3. At date $t_N = T$, the final value of the portfolio consists of:
   - foreign asset: $\delta(T-1)S(T)X(T)$;
   - dividend payment: $\delta(T-1)S(T-1)(e^{q\Delta t} - 1)X(T)$;
   - foreign risk-free asset: $-V(T-1)e^{r_f \Delta t}X(T)/X(T-1)$;
   - cash account: $C(T-1)e^{r\Delta t}$.
   - promised payment: $-\bar{X}S(T)$.

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def Quanto_Hedge_Profit(
    X0, mu_x, sigma_x, S0, mu_s, sigma_s, q, r, r_f, rho, X_bar, T, N
):
    dt = T / N  # time step
    t = np.linspace(0, T, N + 1)  # time grid
    # two sets of standard normal random numbers
    z1 = rng.standard_normal(size=N)
    z2 = rng.standard_normal(size=N)
    # two Brownian motions with correlation rho
    B1, B2 = np.zeros(N + 1), np.zeros(N + 1)
    B1[1:] = np.cumsum(np.sqrt(dt) * z1)
    B2[1:] = np.cumsum(np.sqrt(dt) * (rho * z1 + np.sqrt(1 - rho**2) * z2))
```

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    # simulate the exchange rate
    X = X0 * np.exp((mu_x - 0.5 * sigma_x**2) * t + sigma_x * B1)
    # simulate the foreign asset prices
    S = S0 * np.exp((mu_s - 0.5 * sigma_s**2 - q) * t + sigma_s * B2)
    # quanto values
    V = np.exp((r - r_f - q - rho * sigma_s * sigma_x * t) * (T - t)) * S *␣
↪X_bar
    # number of shares invested in foreign asset
      = V / (S * X)
    # cash position in domestic currency
    cash = np.zeros(N + 1)
    cash[0] = V[0]
    # rebalancing the portfolio from time t(1) to t(N-1)
    for i in range(1, N):
        cash[i] = (
            (cash[i - 1] * np.exp(r * dt))  # cash accumulation/payment
            + ( [i - 1] * S[i - 1] * (np.exp(q * dt) - 1) * X[i])  # dividend
            - (( [i] -  [i - 1]) * S[i] * X[i])  # foreign asset purchase/sale
            + (
                V[i] - V[i - 1] * np.exp(r_f * dt) * X[i] / X[i - 1]
            )  # foreign risk-free asset purchase/sale
        )
    # portfolio value at maturity time t(N) = T
    cash[N] = (
        (cash[N - 1] * np.exp(r * dt))  # cash accumulation/payment
        + ( [N - 1] * S[N - 1] * (np.exp(q * dt) - 1) * X[N])  # dividend
        + ( [N - 1] * S[N] * X[N])  # foreign asset sale
        - (
            V[N - 1] * np.exp(r_f * dt) * X[N] / X[N - 1]
        )  # payment of foreign risk-free asset
    )
    profit = cash[N] - X_bar * S[N]  # profit at maturity
    return profit


def Simulate_Quanto_Hedge_Profit(P, M):
    profits = np.zeros(M)
    for i in range(M):
        profits[i] = Quanto_Hedge_Profit(*P)
    return profits
```

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```
# Parameters
X0 = 0.5  # initial exchange rate
mu_x = 0.05  # drift of exchange rate
sigma_x = 0.2  # volatility of exchange rate
S0 = 100  # initial foreign asset
mu_s = 0.1  # drift of foreign asset
```

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sigma_s = 0.2  # volatility of foreign asset
q = 0.01  # dividend yield of foreign asset
r = 0.02  # risk-free rate in domestic currency
r_f = 0.02  # risk-free rate in foreign currency
rho = 0  # correlation between two Brownian motions
X_bar = 0.5  # fixed exchange rate of the quanto
T = 1  # maturity
N = 252  # number of periods
M = 10000  # number of simulations

# Simulation
quato_hedge_profits = Simulate_Quanto_Hedge_Profit([X0, mu_x, sigma_x, S0,␣
 ↪mu_s, sigma_s, q, r, r_f, rho, X_bar, T, N], M)
```

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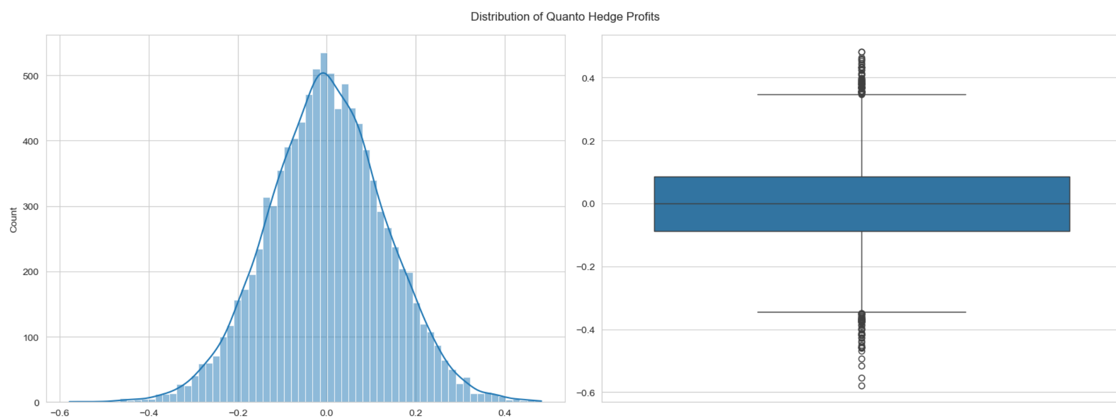
```
# Percentiles
percentiles = np.percentile(quato_hedge_profits, [1, 5, 25, 50, 75, 95, 99])
print("Percentiles of profits:")
for i, p in enumerate([1, 5, 25, 50, 75, 95, 99]):
    print(f"{p}th percentile: {percentiles[i]:.2f}")

# Distribution of profits in histogram and boxplot
fig, ax = plt.subplots(1, 2, figsize=(16, 6))
sns.histplot(quato_hedge_profits, kde=True, ax=ax[0])
sns.boxplot(quato_hedge_profits, ax=ax[1])
fig.suptitle("Distribution of Quanto Hedge Profits")
fig.tight_layout()
plt.show()
```

```
Percentiles of profits:
1th percentile: -0.31
5th percentile: -0.22
25th percentile: -0.09
50th percentile: -0.00
75th percentile: 0.09
95th percentile: 0.22
99th percentile: 0.31
```



0.3 Question 7.7 Discretely-Rebalanced Delta Hedges for Foward Call Option

As stated in Sect. 7.6, the delta-hedge portfolio for a written call option (option maturing at T, forward maturing at T', and $T < T'$) consists of:

- Long $F(0)N(d_1) - KN(d_2)$ units of the discounted bond maturing at T'
- Long $N(d_1)$ forward contracts.

This is a zero-cost portfolio when we include the proceeds from selling the call. Since forward contracts were worth zero at initiation, we don't need to adjust the bond position because there is no other cash flow before the maturity.

Each time the forward contracts were rebalanced, leaving a cash flow of $x(t_{i-1})[F(t_i) - F(t_{i-1})]$ dollars to be received at forward maturity T' (see details in Sect. 7.10), where $x(t)$ is the hedge position of forward contracts. So, the profit of the hedging strategy at option maturity T is:

$$\text{Profit} = P(T,T')\,[F(0)N(d_1) - KN(d_2)] + P(T,T')\sum_{i=1}^{N} x(t_{i-1})\,[F(t_i) - F(t_{i-1})] - P(T,T')\max(F(T) - K, 0)$$

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```python
def Black_Call_Delta(F, K, sigma, T):
    d1 = (np.log(F / K) + 0.5 * sigma**2 * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
     _F = norm.cdf(d1)
     _P = F * norm.cdf(d1) - K * norm.cdf(d2)
    return  _F,  _P


def Black_Call_Hedge_Profit(F0, K, mu, sigma, y, T, T_p, N):
    dt = T / N  # time step
    t = np.linspace(0, T, N + 1)  # time grid
    # Brownian motion B(t)
    B = np.zeros(N + 1)
    B[1:] = np.cumsum(np.sqrt(dt) * rng.standard_normal(size=N))
```

```python
    # Forward price F(t)
    F = F0 * np.exp((mu - 0.5 * sigma**2) * t + sigma * B)
    # Discount bond price P(t, T')
    P = np.exp(-y * (T_p - t))
    # Positions of forward contract and discounted bond to hedge the call option
    _F, _P = Black_Call_Delta(F[:-1], K, sigma, T - t[:-1])
    # Cumulative cash flow at maturity T' from the change value of the forward␣
↪contracts
    dF = np.zeros(N + 1)
    for i in range(1, N + 1):
        dF[i] = dF[i - 1] + _F[i - 1] * (F[i] - F[i - 1])
    # portfolio value at option maturity t(N) = T
    profit = _P[0] * P[N] + dF[N] * P[N] - max(F[N] - K, 0) * P[N]
    return profit


def Simulate_Black_Call_Hedge_Profit(P, M):
    profits = np.zeros(M)
    for i in range(M):
        profits[i] = Black_Call_Hedge_Profit(*P)
    return profits
```

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```python
# Parameters
F0 = 100  # initial forward price
K = 100  # strike price
mu = 0.05  # drift of the forward price
sigma = 0.2  # volatility of the forward price
y = 0.05  # constant yields of the discount bond
T = 0.5  # maturity of the call option
T_p = 1  # maturity of the forward contract
N = 252  # number of periods
M = 10000  # number of simulations

# Simulation
profits = Simulate_Black_Call_Hedge_Profit([F0, K, mu, sigma, y, T, T_p, N], M)
```

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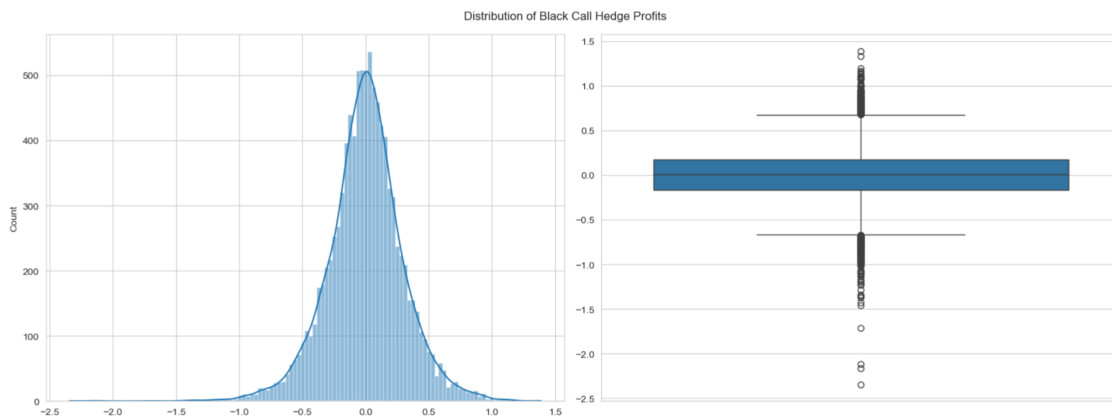
```python
# Percentiles
percentiles = np.percentile(profits, [1, 5, 25, 50, 75, 95, 99])
print("Percentiles of profits")
for i, p in enumerate([1, 5, 25, 50, 75, 95, 99]):
    print(f"{p}th percentile: {percentiles[i]:.2f}")

# Distribution of profits in histogram and boxplot
fig, ax = plt.subplots(1, 2, figsize=(16, 6))
sns.histplot(profits, kde=True, ax=ax[0])
sns.boxplot(profits, ax=ax[1])
fig.suptitle("Distribution of Black Call Hedge Profits")
```

```
fig.tight_layout()
plt.show()
```

```
Percentiles of profits
1th percentile: -0.81
5th percentile: -0.49
25th percentile: -0.17
50th percentile: 0.00
75th percentile: 0.17
95th percentile: 0.48
99th percentile: 0.76
```

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