handsOn02_linear-algebra

September 21, 2016

1 EECS 445: Introduction to Machine Learning

1.1 Hands-On Lecture 2: Linear Algebra

Wednesday, September 14, 2016

1.2 Outline

> This hands-on lecture corresponds to material from **Lecture 02: Linear Algebra & Optimization**.

Advice: Interpreting Matrix Operations (10-15mins) * Matrix-vector multiplication * **Problem 1:** Image of a matrix * Matrix-matrix multiplication
Hands-on Exercises (60mins) * **Problem 2:** Matrix Transpose * **Problem 3:** Infinity Norm * **Problem 4:** Matrix Inverse * **Problem 5:** Singular Values

1.3 Advice: Interpreting Matrix Operations

> Think of matrix operations in terms of row- and column-vectors, rather than in terms of individual entries.

This section borrows material heavily from the first chapter of Trefethen & Bau, "Numerical Linear Algebra"

1.3.1 Advice: Matrix-Vector Multiplication (Row View)

Recall the following definition of matrix multiplication:

> If $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$, then the ith entry of $b = Ax \in \mathbb{R}^m$ is:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, \ldots, m)$$

In other words, b is a dot product between x and the rows of A. If you think of A as data matrix, it is useful to keep in mind the following interpretation:
Each row of A is "scored" based on how well it aligns with the vector x.

1.3.2 Advice: Matrix-Vector Multiplication (Column View)

An alternative interpretation that comes handy when you want to study the subspaces of matrix is:

If $b = Ax$, then b is a linear combination of the columns of A, with coefficients from the vector x.

$$\left[\begin{array}{c|c|c|c} & & & \\ a_1 & a_2 & \cdots & a_n \\ & & & \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\begin{bmatrix} \\ a_1 \\ \\ \end{bmatrix} + x_2\begin{bmatrix} \\ a_2 \\ \\ \end{bmatrix} + \cdots + x_n\begin{bmatrix} \\ a_n \\ \\ \end{bmatrix}$$

Matrix-Vector Multiplication

1.3.3 Problem 1: Range and Nullspace

Recall that the **range** or **image** of $A \in \mathbb{R}^{m\times n}$ is the set of vectors $y \in \mathbb{R}^m$ that can be written as $y = Ax$ for some $x \in \mathbb{R}^n$,

$$\text{im A} = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, y = Ax\}$$

Problem 1: Argue that im A is the space spanned by the columns of A.

Hint: Use our "column view" of matrix-vector multiplication!

1.3.4 Solution 1: Range and Nullspace

Problem 1: Argue that im A is the space spanned by the columns of A.

Because any Ax is a linear combination of the columns of A, any vector $y \in$ im A can be written as a linear combination of the columns of A,

$$y = \sum_{j=1}^{n} x_j a_j$$

Forming a vector $x \in \mathbb{R}^n$ out of these coefficients x_j, we have $y = Ax$, and thus $y \in$ im A.

1.3.5 Advice: Matrix-Matrix Multiplication

The formula for matrix-matrix multiplication probably scares you a little:

If $A \in \mathbb{R}^{\ell\times m}$ and $C \in \mathbb{R}^{m\times n}$, then $B = AC \in \mathbb{R}^{\ell\times n}$ with entries

$$b_{ij} = \sum_{k=1}^{m} a_{ik}c_{kj}$$

1.3.6 Advice: Matrix-Matrix Multiplication

If $B = AC$, then *each column of B is a linear combination of the columns of A* with coefficients from the columns of C.

$$\left[\begin{array}{c|c|c|c} & & & \\ b_1 & b_2 & \cdots & b_n \\ & & & \end{array}\right] = \left[\begin{array}{c|c|c|c} & & & \\ a_1 & a_2 & \cdots & a_m \\ & & & \end{array}\right] \left[\begin{array}{c|c|c|c} c_1 & c_2 & \cdots & c_n \end{array}\right]$$

Matrix-Matrix Multiplication

$$\boxed{b_j = Ac_j}$$

1.3.7 Problem 2: Multiplication by Triangular Matrix

Problem 2: Consider $B = AR$, where R an upper-triangular matrix with entires all equal to one on and above the diagonal. Interpret the columns of B using our new interpretation of matrix multiplication.

$$\left[\begin{array}{c|c|c} & & \\ b_1 & \cdots & b_n \\ & & \end{array}\right] = \left[\begin{array}{c|c|c} & & \\ a_1 & \cdots & a_n \\ & & \end{array}\right] \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}.$$

Triangular matrix multiplication

1.3.8 Solution 2: Multiplication by Triangular Matrix

Problem 2: Consider $B = AR$, where R an upper-triangular matrix with entires all equal to one on and above the diagonal. Interpret the columns of B using our new interpretation of matrix multiplication.

Since $B = AR$, the columns of B are linear combinations of the columns of A, with coefficients taken from the columns of R. Because of the diagonal structure of R, the jth column of B is the sum of the first j columns of A:

$$b_j = Ar_j = \sum_{k=1}^{j} a_j$$

1.3.9 Advice: Conclusion

If you hadn't seen these interpretations before, they may seem a little strange. Keep at it! Thinking about linear algebra in this way will help a lot in the long run.

1.4 Hands-on Exercises

1.4.1 Review: Matrix Transposition

Recall that the **transpose** of $A \in \mathbb{R}^{m \times n}$ is $A^T \in \mathbb{R}^{n \times m}$ with indeces "flipped",

$$(A^T)_{ij} = A_{ji}$$

- Transposition flips entries across the diagonal
- Transposition turns rows into columns and vice-versa
- A matrix is **symmetric** if $A^T = A$.

1.4.2 Problem 3.1: Matrix Transpose

Problem 3.1: Let A and B be two matrices compatible with multiplication. Is it true that $AB = A^T B^T$? Either prove or give a counterexample.

1.4.3 Solution 3.1: Matrix Transpose

Problem 3.1: Let $A, B \in \mathbb{R}^{n \times n}$ Is it true that $AB = A^T B^T$? Either prove it or give a counterexample.

False. Let $A \in \mathbb{R}^{n \times n}$ be any square matrix and $B = I$ be the identity. Unless A is symmetric, then $AB = A \neq A^T = A^T B^T$. There are plenty of asymmetric matrices! For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1.4.4 Problem 3.2: Matrix Transpose

Problem 3.2: Is it true that $(AB)^T = B^T A^T$? Either prove it or give a counterexample.

1.4.5 Solution 3.2: Matrix Transpose

Problem 3.2: Is it true that $(AB)^T = B^T A^T$? Either prove it or give a counterexample.

True! Assume $A \in \mathbb{R}^{m \times p}, B \in \mathbb{R}^{p \times n}$. After verifying that the matrix dimensions match up, we can verify it the brute-force way,

$$
\begin{aligned}
{}_{ij} &= [AB]_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} \\
[B^T A^T]_{ij} &= \sum_{k=1}^{p} [B^T]_{ik} [A^T]_{kj} = \sum_{k=1}^{p} a_{jk} b_{ki}
\end{aligned}
$$

Try to interpret this result using what we've learned about matrix-vector products!

1.4.6 Problem 3.3: Matrix Transpose

Problem 3.3: Prove that $A + A^T$, $A^T A$, and AA^T are all symmetric.

1.4.7 Solution 3.3: Matrix Transpose

Problem 3.3: Prove that $A + A^T$, $A^T A$, and AA^T are all symmetric.
Verify the first one elementwise:

1. $[(A + A^T)^T]_{ij} = [A + A^T]_{ji} = [A]_{ij} + [A]_{ji} = [A + A^T]_{ij}$

Use Problem 3.2 to solve the other two:

2. $(A^T A)^T = A^T (A^T)^T = A^T A$
3. \$(A A$^{\text{T)}}$T = (A$^{\text{T)}}$T AˆT = A AˆT \$

1.4.8 Problem 4: Matrix Inverse

Problem 4: Let A and B be two $n \times n$ matrices. Prove, or find a counterexample, to the statement that

$$(AB)^{-1} = B^{-1} A^{-1}$$

1.4.9 Solution 4: Matrix Inverse

Problem 4: Let A and B be two $n \times n$ matrices. Prove, or find a counterexample, to the statement that

$$(AB)^{-1} = B^{-1} A^{-1}$$

Solution 4: Recall that, for any matrix M, the inverse M^{-1} is the unique matrix such that $MM^{-1} = I$, the identity matrix.

$\ldots$

1.4.10 Problem 4: Infinity Norm

Problem 4: Prove that the infinity norm $||x||_\infty = \max_k |x_k|$ is indeed a norm for $x \in \mathbb{R}^n$.

Recall that $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ is a norm if and only if for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, 1. $||x|| \geq 0$ 2. $||x|| = 0$ if and only if $x = 0$. 2. $||\alpha x|| = |\alpha| ||x||$ (Homogeneity) 4. $||x + y|| \leq ||x|| + ||y||$ (Triangle Inequality)