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The Vasicek Model Fitted By The Kalman Filter Applied To The Long-Run Term Structure

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The Vasicek Model Fitted By The Kalman Filter Applied To The Long-Run Term Structure

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Abstract

This thesis discusses methods to construct a term structure of interest rates up to maturities of 50 years and gives new insights about the dynamics of the long end of the yield curve. The thesis focuses on the Vasicek model, in which the parameters are estimated by the Kalman filter. The theoretical frameworks of the Vasicek model as well as the Kalman filter are discussed, as well as the results of the two, three, four and five factor versions of the Vasicek model. It is concluded that the model works significantly better for every extra factor added, up to the fourth factor. Especially considering the very long end of maturities, the four factor model gives graphically significantly better results than the two and three factor models. The five factor model performs better on paper, however the Likelihood Ratio Test leaves room for doubt in proving that the five factor model is significantly better than the four factor model, since the cut off value is close to the value of the test. Overall, the four and five factor model fit the data well for the short term and the long term interest rates. However, in times of financial crises the model seems to perform worse.

Contents

1 Introduction

There are many uncertainties in the world. One of the most important for insurance and pension funds is the interest rate. An important agent regarding interest rates in Europe is the ECB. The ECB sets the overnight interest rate for banks, called the marginal lending facility, which is used to offer banks credit from the ECB itself (ECB, 2019b). The overnight inter-bank interest rate has an impact on the interest rates in the market. However, a large part of the term structure of interest rates is determined by market dynamics. This results in interest rates being uncertain and unpredictable. While the interest rate in the short run is fairly easy to predict, the interest rate over larger periods of time is difficult to predict. When looking at Figure 1, it can be seen that the yield curve (retrieved from the ECB at the time of writing) seems to converge in the long term to a number somewhat around 0,45 percent. There is no certainty that the interest rates that the ECB retrieves from information in the market are the true interest rates. This uncertainty is caused by market irregularities, such as the illiquidity of long term bonds. Additionally, it is unknown what will happen in the world in 30 years time and some events can influence the interest rate greatly. In Europe the European Insurance and Occupational Pensions Authority (EIOPA) contributes to creating guidelines and regulations for pension funds and insurance companies, trying to eliminate some of the interest rate risk that these companies are exposed to (EIOPA, 2019a). However, even if companies stick to these regulations, the uncertainty of interest rates cannot be fully eliminated. The Dutch Central Bank (DNB) publishes the interest rate term structure used by pension funds. Here the interest rates up to 100 years are shown, see Figure 2. As can be seen, the yield curve has a similar shape as the curve from the ECB.

The long term interest rates are crucial to actuaries, since the payments that need to be considered in for example a pension cover a time span of 50 years or more. For the medium term, 10 to 20 years, one can still rely on market rates of government bonds because of replication arguments. This is often done by the no arbitrage assumption, which results in the fact that the market rates fit the term structure perfectly. An example is bootstrapping (BundesBank, 2006), (Leombroni et al., 2018). For very long term maturities, market rates are often considered not representative due to liquidity problems that arise because of the fact that the long term bonds are not traded in the market frequently, or because the demand is too large in comparison to the supply which can cause miss-pricing. Additionally, bootstrapping might not be possible due to a lack of available data. This causes central banks and pension funds to rely on other methods to come up with a long term interest rate. As can be seen from recent developments in the Netherlands, a change in the long term interest rate (in Dutch: "de rekenrente") (Haegens, 2019) (Pelgrim, 2019) can cause problems for pension funds. For example, if the long term interest rate is lower than expected, pension funds can get into trouble because the expected return on their money is lower. Therefore they might not have enough money available to pay out the pensions that they promised to pay out. This occurrence has been in the Dutch news recently because the Dijsselbloem committee, a committee appointed to give advise to the Dutch government about the long term interest rate, suggested a new way to calculate the long term interest rate based on scientific research in the field. The new calculations suggest a lower long term interest rate than the rate that is being used now (Dijsselbloem et al., 2019) (Valck, 2019) (RTL-Z, 2019).

Unconditionally on the method that is used to construct (and/or forecast) the yield curve, data is needed to do so. When focusing on swap rates instead of bond rates more "correct" data is available. Swaps are more liquid than bonds and swaps with long maturities are traded in the market are reasonably liquid. That is why swap data is used in this thesis instead of bond price data. For the maturities to 20 years, there are many extensive methods that one can use to create a term structure. As discussed above, bootstrapping is a method that is often used when constructing a yield curve. When bootstrapping does not work, methods like the Nelson Siegel method can be useful. The method of Nelson Siegel fits a curve

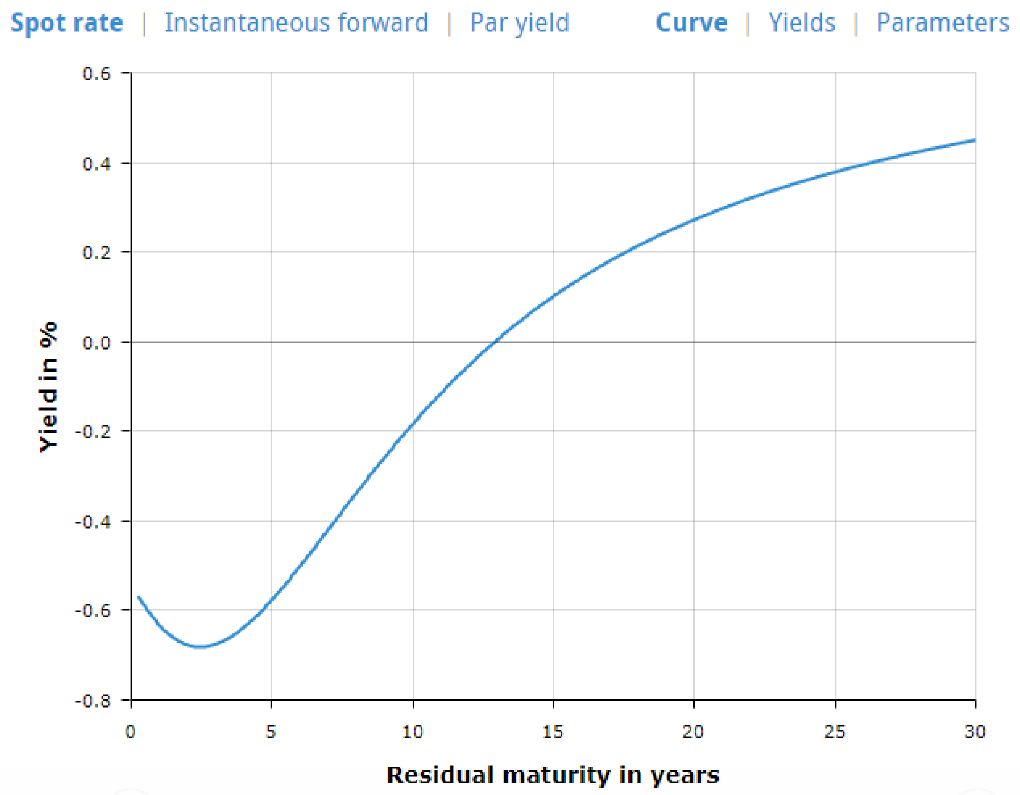


Figure 1: Spot rates according to the ECB at the time of writing. Retrieved from ECB (2019a)

to the interest rates. The Nelson Siegel (Nelson and Siegel, 1987) method is often used because of the parsimonious property of the model. The curve that the method fits is

$$y(m) = a + b(1 - \exp(-m/\tau)) + c(\exp(-m/\tau)),$$

where y is the yield corresponding to maturity m and a, b, c and τ are parameters. As can be seen, the model only needs to fit 4 parameters, but has nevertheless appeared to have a good fit in practise. The original method is not arbitrage free since it does not fit the market rates exactly but instead fits a smooth curve. Another downside of the model is that it lacks a theoretical background. Nevertheless, it has proven to perform well in practise (De Pooter, 2007) and is therefore used by multiple central banks, as well as by the Bank of International Settlements (BIS, 2005). Other versions of the Nelson Siegel model exist in which the model is expended. Diebold and Li (2006) researched a extended version of the Nelson Siegel model in which the parameters are time dependent, thus dynamic. This is useful for forecasting purposes and works well in practise when compared to the actual interest rates. Another curve fitting model is the Smith-Wilson model (Smith and Wilson, 2001). The model solves a linear system of equations and is therefore not dependant on minimizing a least squares etc. An additional advantage of the model is that it gives a perfect fit to the liquid data points. After the liquid points, the curve is dependant on the convergence rate α, and the Ultimate Forward Rate (UFR), the rate the yield curve converges to ultimately. One of the disadvantages of the model is that the α is not determined in the model, but externally as well as the UFR. Therefore, an expert is needed to assign a value to α and the UFR. Additionally, there is no constraint that makes the discount function a decreasing function

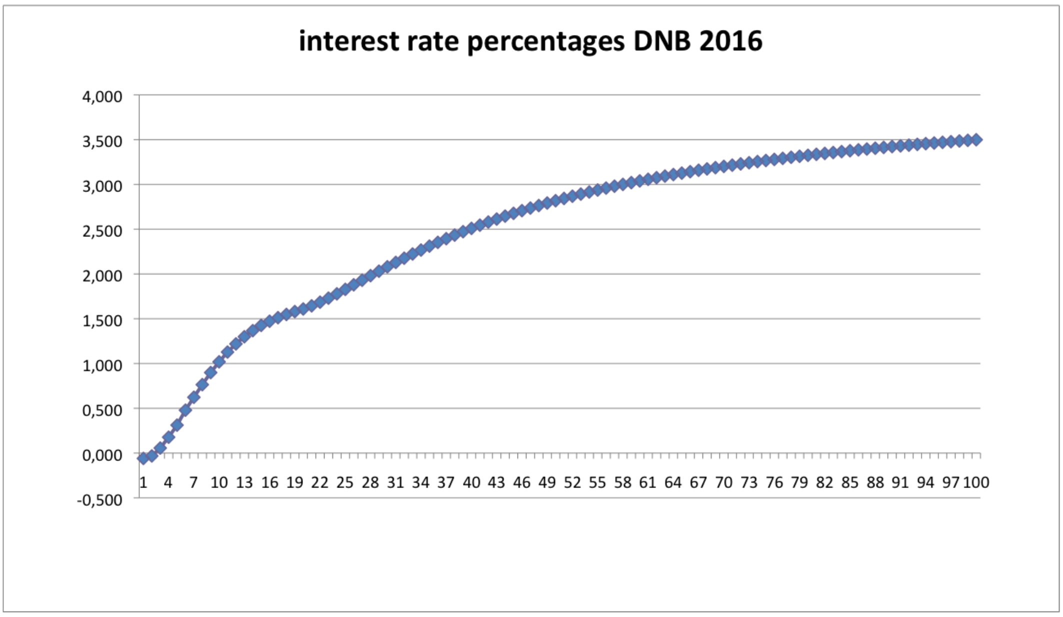


Figure 2: Interest rate term structure as published by DNB in 2016 DNB (2016), y-axis the interest rate in percentages, x-axis the time to maturity in years.

(EIOPA, 2019b).

Other classes of interest rate models are the equilibrium models and the arbitrage free models. Both have a extensive theoretical background, in contrast to the Nelson Siegel model. One of the most well known models of the equilibrium models is the Vasicek model (Vasicek, 1977). The model describes the interest rate as a rate that converges to the equilibrium rate and is affected by random processes that represent economics news that causes the interest rate to diverge from the long term interest rate. Because the model takes into account that mispricing can occur, the market prices are almost never exactly equal to the prices that result from the term structure retrieved from the Vasicek model, or other equilibrium models. However, the equilibrium models are extensively researched and there is many theoretical background for these type of models (CFI, 2019). The Vasicek model will be explained in more detail in Section 2. Because the Vasicek model assigned positive probability to negative interest rate, new types of models have been researched to solve the negative interest rate problem such as the Black and Karasinski model (Black and Karasinski, 1991), which models the interest rate as being log-normal, and the Cox Ingersoll and Ross model (Cox et al., 2005), which does not assume that the interest rates are Gaussian. Besides equilibrium models and the Nelson Siegel model, there also exist arbitrage free models to model the term structure of interest rates. The arbitrage-free models take the market rates as input to calculate the interest rate, under the assumption that the market does not allow for arbitrage. In contrast to the equilibrium models, no assumptions are made about the generating process of the interest rates. An example of an arbitrage free model is the Ho-Lee (Ho and Lee, 1986) model, which was one of the first arbitrage free models. They state that the approach has several benefits, such as using the total of information available (the whole term structure) and that the prices that result from the model are certainly the correct theoretical price according to the model. The model of Ho and Lee is modeled with constraints such that the model is still consistent with an equilibrium framework. There additionally exists many versions of interest rate models that are combinations of one of these models. For example, Christensen et al. (2011) wrote a paper about the affine arbitrage-free class of Nelson Siegel term structure models, which would be combining all three of the above stated models. It is hard to state the goodness of the performance of these kind of models, since not much research is done for the specific models that are used and since the theoretical substantiated models are combined with the models that are not, it is hard to theoretically proof that the models are correct. That is why in this thesis we stay with one type of model. Since it is unknown if the market rates that we use in this thesis reflect the true term structure of interest rate, we will focus on the equilibrium models.

What we try to research in this thesis is if term structure models can be applied to the maturities between 20 and 50 years. We focus on the Vasicek model, following the example of Babbs and Nowman (1999) and De Jong (2000), a Gaussian model often used to create term structures up to 20 or 30 years. In the work of Babbs and Nowman, they show that the two factor version of the Vasicek model often performs as well as the three factor model for maturities up to 10 years. In other works of Babbs and Nowman, the Vasicek model is applied to interest rate data and it is concluded that there is a need for a multi factor version of the Vasicek model (Babbs and Nowman, 1998a), (Babbs and Nowman, 1998b). Therefore we focus on the multi-factor model in this thesis. The Vasicek model can be calibrated by numerous methods. However, we have chosen to do so by the Kalman filter, because of the properties that the Kalman filter has that allows for measurement errors. In the work of Duan and Simonato (1999), it is shown with a Monte Carlo simulation that the Kalman filter approach works sufficiently good for the calibration of the Vasicek model and the non-Gaussian version of the Vasicek model, the Cox Ingersol and Ross model. For this reason the method has been used in several other literature. For example, a two factor Vasicek model with a Kalman filter has been used in an analysis on the effect of the European Monetary Union (Lund, 1999).

The rest of this thesis is organized as follows: Firstly, the Vasicek model is derived and discussed in Section 2. In Section 3, the Kalman filter and its state equations as well as the derivations are discussed. In Section 4, the data and the algorithm are discussed. In Section 5 the results are displayed from different factor versions of the Vasicek model and a comparison between the models is made. Lastly, a conclusion is given.

2 The Vasicek Model

2.1 Introduction

Oldrich Vasicek (Vasicek, 1977) published his work in 1977, in which he derived a general structure of the term structure of interest rates. The model assumes that the instantaneous interest rate follows a diffusing process, that the price of a discount bond depends only on the spot rate over its term and that the market is efficient. Since the Vasicek model assumes normally distributed interest rates, there is a positive probability of negative interest rates. However, given the current state of the Euro-Zone and the given data set, this occurrence is not impacting the model as negatively as is described in the literature. Hull and White (1993) researched in their paper if and how versions of the Vasicek model, together with other one-factor models, could be implemented to use for price derivations of financial securities. It was found out that the Vasicek model was a good model for doing so and that the model gave good estimation results. The same authors (Hull and White, 1990) additionally compared the Vasicek model with other interest rate models and found out that the Vasicek model performs well both on practical implementation as on results.

2.2 Notation and derivation

As said, in this thesis the Vasicek model is used to model the term structure of interest rates. First the price of a zero coupon bond is calculated under the risk neutral probability measure $\mathbb{Q}$. Thereafter we apply Girsanov's theory to obtain the dynamics of price change under the real world probability $\mathbb{P}$, where a risk premium ψ is introduced. The instantaneous interest rate is a theoretical concept that can not be observed in real life. However, as we will explain later, it can be used to model the term structure of interest rates when applying the Kalman filter. We calculate both prices, under $\mathbb{Q}$ and $\mathbb{P}$, because both are needed to apply the Kalman filter, as can be seen in Section 3.

The variables of the Vasicek model can be interpreted as κ being the mean reversion of the model, θ being the long term mean of the model, and α being the volatility of the Vasicek model. The model can be used as a one factor model, or as a more factor model. This aspect of the model will be discussed in Section 2.3. Lastly, W_t is a stochastic Wiener process.

2.2.1 Risk neutral probability measure

The Vasicek model assumes that the instantaneous interest rate under $\mathbb{Q}$ is defined by the stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \alpha dW_t^{\mathbb{Q}}. \tag{1}$$

$\mathbb{Q}$ is the probability measure under which we obtain a Brownian motion, which results in the process being a martingale. We obtain $\mathbb{Q}$ by first choosing a numeric to measure the interest rates with. We choose the bond price

$$B_t = e^{\int_0^t r_u du}.$$

It is known that there exists a probability measure (Delbaen and Schachermayer, 1994), $\mathbb{Q}$, under which all prices of zero coupon bonds, defined as $D(s, \tau, r_t)$ divided by B_s are martingales. It results in

$$\frac{D(0,\tau,r_t)}{B_s} = \mathbb{E}^{\mathbb{Q}}\left[\frac{D(0,\tau,r_t)}{B_s}|\mathcal{F}_s\right] = D(s,\tau,r_t) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_s^\tau r_t dt\right)\right] \tag{2}$$

where $\mathcal{F}_s$ is the sigma algebra of the diffusion process.

Let $0 \leq s < t \leq T$, where T is the latest time point of the sample. The stochastic differential equation of Equation (2) has the solution

$$\begin{aligned} r_t = r_s \cdot e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \alpha \int_s^t e^{-\kappa(t-s)} dW_t^{\mathbb{Q}} \\ = \theta + (r_s - \theta)e^{-\kappa(t-s)} + \alpha \int_s^t e^{-\kappa(t-s)} dW_t^{\mathbb{Q}}, \end{aligned} \tag{3}$$

which is verifiable by Ito's Lemma (Mamon, 2014) (Uhlenbeck and Ornstein, 1930) (De Jong, 2000).

Since the third term of the model is identified as an Ito Integral, which is a martingale (Mikosch, 1998) with an expectation of 0, the expectation of the interest rate equals

$$\mathbb{E}^{\mathbb{Q}}\left[r_t \mid \mathcal{F}_s\right] = \theta + (r_s - \theta)e^{-\kappa(t-s)}$$

Additionally the variance can be computed from Equation (3) by making use of Ito Isometry, by solving

$$\begin{aligned} \mathrm{Var}^{\mathbb{Q}}[r_t \mid \mathcal{F}_s] = \mathrm{Var}^{\mathbb{Q}}\left[\alpha \int_s^t e^{-\kappa(t-s)} dW_t^{\mathbb{Q}}\right] \\ = \alpha^2 \int_s^t e^{-2\kappa(t-s)} \text{ dt By Ito's isometry} \\ = \frac{\alpha^2}{2\kappa}\left(1 - e^{-2\kappa(t-s)}\right). \end{aligned} \tag{4}$$

To calculate the solution to Equation 2, we need the value of the variance and the value of $\mathbb{E}(-\int_s^\tau r_t dt)$. The expectation can be obtained by substituting Equation (3). This results in

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[-\int_s^\tau r_t dt \mid \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{Q}}\left[-\int_s^\tau \left(\theta + (r_s - \theta)e^{-\kappa(t-s)} + \alpha \int_s^t e^{-\kappa(t-s)} dW_t^{\mathbb{Q}}\right) dt \mid \mathcal{F}_s\right] \\ = \mathbb{E}^{\mathbb{Q}}\left[-\int_s^\tau \left(\theta + (r_s - \theta)e^{-\kappa(t-s)}\right) dt \mid \mathcal{F}_s\right] \\ = \frac{r_s - \theta}{\kappa} \cdot (1 - e^{-\kappa(\tau - s)}) - \theta(\tau - s) \end{aligned} \tag{5}$$

The variance can be calculated as

$$\begin{aligned}
\mathrm{Var}^{\mathbb{Q}}\left[-\int_s^\tau r_t dt \mid \mathcal{F}_s\right] &= \mathrm{Var}^{\mathbb{Q}}\left[-\int_s^\tau \left(\theta + (r_s-\theta)e^{-\kappa(t-s)} + \alpha\int_s^u e^{-\kappa(u-s)} dW_t^{\mathbb{Q}}\right) du \mid \mathcal{F}_s\right] \\
&= \mathrm{Var}^{\mathbb{Q}}\Big[-\alpha\int_s^\tau \int_s^u e^{-\kappa(u-s)} dW_t^{\mathbb{Q}} du \mid \mathcal{F}_s\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\Big(-\alpha\int_s^\tau \int_s^u e^{-\kappa(u-s)} dW_t^{\mathbb{Q}} du\Big)^2 \mid \mathcal{F}_s\right] \text{ Because of the martingale property} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\Big(-\alpha\int_s^\tau \int_t^\tau e^{-\kappa(u-s)} du \; dW_t^{\mathbb{Q}}\Big)^2 \mid \mathcal{F}_s\right] \text{ By the stochastic Fubini theorem (Billingsley, 2008)} \\
&= \alpha^2 \int_s^\tau \left[\int_t^\tau e^{-\kappa(u-s)} du\right]^2 dt \text{ By Isometry} \\
&= \alpha^2 \int_s^\tau \left[\frac{1-e^{-\kappa(\tau-t)}}{\kappa}\right]^2 dt \\
&= \frac{\alpha^2}{\kappa^2}\int_s^\tau \left[1 - 2e^{-\kappa(\tau-t)} + e^{-2\kappa(\tau-t)}\right] dt \\
&= \frac{\alpha^2}{\kappa^2}\left(\tau - s - 2\frac{1-e^{-\kappa(\tau-s)}}{\kappa} + \frac{1-e^{-2\kappa(\tau-s)}}{2\kappa}\right)
\end{aligned} \tag{6}$$

We can now solve for $D(s,\tau,r_t)$, using the Ito Formula.

$$D(s,\tau,r_t) = \exp\left(\mathbb{E}^{\mathbb{Q}}\left[-\int_s^\tau r_t dt \mid \mathcal{F}_s\right] + \frac{1}{2}\mathrm{Var}^{\mathbb{Q}}\left[-\int_s^\tau r_t dt \mid \mathcal{F}_s\right]\right)$$

$$= \exp\left[\frac{r_s-\theta}{\kappa}(1-e^{-\kappa(\tau-s)}) - \theta(\tau-s) + \frac{\alpha^2}{\kappa^2}\Big(t - s - 2\frac{1-e^{-\kappa(t-s)}}{\kappa} + \frac{1-e^{-2\kappa(t-s)}}{2\kappa}\Big)\right]$$

$$= \exp\left[\frac{r_s-\theta}{\kappa}(1-e^{-\kappa(\tau-s)}) - \theta(\tau-s) + \frac{\alpha^2}{4\kappa^3}(1 - 2e^{-\kappa(t-s)} + e^{-2\kappa(t-s)} + 2(1-e^{-\kappa(t-s)}) + (t-s)\kappa)\right]$$

We note τ as $t-s$, which represents the time to maturity. We make the notation

$$B^{\mathbb{Q}}(\tau) = \frac{1-e^{-\kappa(t-s)}}{\kappa} = \frac{1-e^{-\kappa\tau}}{\kappa} \tag{7}$$

Which results in

$$\exp\left[(r_t-\theta)B^{\mathbb{Q}}(\tau) - \theta\tau + \frac{\alpha^2}{4\kappa}B(\tau)^{\mathbb{Q}2} + \frac{\alpha^2}{2\kappa^2}(B^{\mathbb{Q}}(\tau) - \tau)\right]$$

The final result under risk neutral probability is

$$D(s,\tau,r_t) = \exp\Big[-B^{\mathbb{Q}}(\tau)r_t + A^{\mathbb{Q}}(\tau)\Big] \tag{8}$$

where

$$A^{\mathbb{Q}}(\tau) = (\theta - \frac{\alpha^2}{2\kappa^2})(B^{\mathbb{Q}}(\tau) - \tau) - \frac{\alpha^2 B^{\mathbb{Q}}(\tau)^2}{4\kappa} \tag{9}$$

and

$$B^{\mathbb{Q}}(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}$$

2.2.2 Real world probability measure

In the model we use, the price dimension of the model is measured in the $\mathbb{Q}$ probability measure. The dynamics in the time direction however, are measured in the $\mathbb{P}$ probability measure. To help understand the notation in Section 3, which explains the time dynamics in more detail, we derive the price equation under the $\mathbb{Q}$ probability measure to make clear what the differences are. It is to note that the specific price equations are not used in the model itself, only to give an understanding about the time dynamics of the model and its parameters. To do so, we apply the theory of Girsanov, (Girsanov, 2012), to measure our interest rate model with the real world probability risk measure. We recall Equation (1), the interest rates modeled with a risk neutral probability measure $\mathbb{Q}$:

$$dr_t = \kappa(\theta - r_t)dt + \alpha dW_t^{\mathbb{Q}}.$$

When changing the probability measure we get

$$\alpha dW_t^{\mathbb{Q}} = \alpha(dW_t^{\mathbb{P}} + \psi dt), \tag{10}$$

where ψ is our risk premium, defined as the drift term of our process.

Applying Equation (10) to Equation (1) results in

$$\begin{aligned} dr_t^P &= \kappa(\theta - r_t)dt + \alpha(dW_t^{\mathbb{P}} + \psi dt) \\ &= \kappa\big[\big(\theta + \frac{\alpha\psi}{\kappa}\big) - r_t\big]dt + \alpha dW_t^{\mathbb{P}} \\ &= \kappa\big[\theta^* - r_t\big]dt + \alpha dW_t^{\mathbb{P}} \end{aligned} \tag{11}$$

where

$$\theta^* = \theta + \frac{\alpha\psi}{\kappa}$$

In the Multi-factor model the problem arises that the parameters of the model can not be fully identified. To solve this problem it is implicated that

$$\theta^* = 0.$$

This results in

$$\theta = -\frac{\alpha\psi}{\kappa}.$$

The Multi-factor model will be explained in more detail in the next subsection.

2.3 Multi-factor model

A multi factor model can describe multiple shocks of 'economic news' (Babbs and Nowman, 1999) (Duan and Simonato, 1999) (Bergstrom and Nowman, 1999) by describing the instantaneous interest rate r as

$$r_t = c(t) + \sum_{i=1}^{n} x_{it},$$

where x is a process defined by

$$\mathrm{d}x = \kappa x \mathrm{dt} + \alpha \mathrm{d}W_t^{\mathbb{P}}$$

and $c(t)$ is a function of time. The x multi factor processes can be summarized in the notation

$$\mathrm{d}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \Lambda x \mathrm{dt} + \alpha \mathrm{d}\begin{pmatrix} W_{t,1}^{\mathbb{P}} \\ W_{t,2}^{\mathbb{P}} \\ \vdots \\ W_{t,n}^{\mathbb{P}} \end{pmatrix}$$

$$= \Lambda x \mathrm{dt} + \alpha \left(\mathrm{d}\begin{pmatrix} W_{t,1}^{\mathbb{Q}} \\ W_{t,2}^{\mathbb{Q}} \\ \vdots \\ W_{t,n}^{\mathbb{Q}} \end{pmatrix} + \psi \mathrm{dt} \right),$$

where Λ is a diagonal matrix with entries $\kappa_1, \kappa_2 ... \kappa_n$, ψ is a vector and α is a square matrix. Similarly, in $\mathbb{Q}$ probability measure

$$r_t = c(t) + \iota' x_t \tag{12}$$

and in $\mathbb{P}$ probability measure it will result in

$$r_t = c(t) + \iota' x_t + \iota' \alpha \psi$$

Obviously, the more factors there are, the more variables are needed to describe the model. In the n factor model model

$$y_t = A + B x_t,$$

A is defined as

$$-\frac{A^{\mathbb{Q}}(\tau)_1}{\tau} - \frac{A^{\mathbb{Q}}(\tau)_2}{\tau} - ... - \frac{A^{\mathbb{Q}}(\tau)_n}{\tau}.$$

Because, as seen from Equation (12), the constants sum up in the equation. $A^{\mathbb{Q}}(\tau)_i$ is defined as

$$\left(\theta_i - \sum_{j=1}^{j} \left(\frac{\alpha_{i,j}^2}{2\kappa_i^2} \right) \left(B_i^{\mathbb{Q}}(\tau) - \tau \right) \right) - \sum_{j=0}^{j} \left(\frac{\alpha_{i,j}^2 B^{\mathbb{Q}}(\tau)_i^2}{4\kappa_i} \right).$$

B becomes a transposed matrix that is multiplied with x_t. Bx_t is defined as

$$\frac{B^{\mathbb{Q}}(\tau)_1}{\tau} x_{1,t} + \frac{B^{\mathbb{Q}}(\tau)_2}{\tau} x_{2,t} + ... + \frac{B^{\mathbb{Q}}(\tau)_n}{\tau} x_{n,t},$$

where $B^{\mathbb{Q}}(\tau)_i$ is defined as

$$\frac{1 - e^{-\kappa_i \tau}}{\kappa_i}.$$

This will additionally result in ψ being a vector of length n and θ being a vector of length n.

3 The Kalman filter

3.1 Literature

The Kalman filter assumes that there is a non-observable process that influences an observable process. It takes into account that measurement errors occur when using the observable process to measure the non-observable process. The filter uses the previous data to make a prediction of the non-observable process. Then it updates this prediction by using the observable information for that time period and uses the information for the next prediction. It operates in a way such that the observations are weighted by the variance of their measurement error. That way the observations with the largest measurement errors are taken into account the least. The Kalman filter is often used in the fields of physics, because in physics a lot of processes are physically impossible to measure, such as the location of a rocket in space or the temperature of extremely hot substances. An illustration of the rocket example is shown in Figure 3. It shows illustratively what the principles of the Kalman filter are. It can be seen that the on-board measurements are not describing the exact place where the rocket is. However, the Kalman filter is still able to find the real path of the rocket by filtering the measurement errors. It starts by making a rough estimate of the place of the rocket, but gradually it can be seen that the Kalman filter describes the place of the rocket almost exactly, even though the information given by the measurements on board is still flawed.

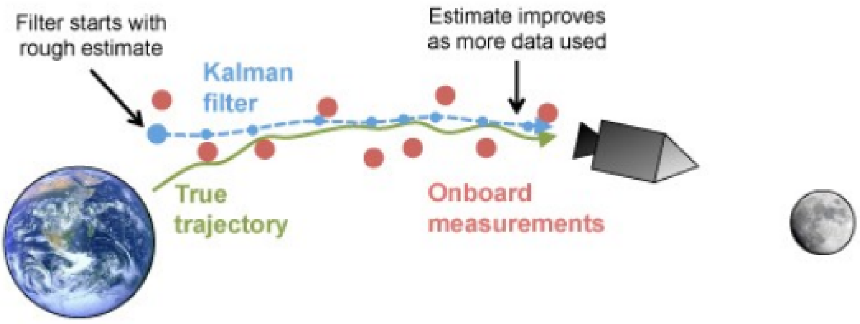


Figure 3: Kalman filter example. Retrieved from Chertkov (2019)

The Kalman filter can be used for the interest rates modeled by the Vasicek model in the same manner as it is used for rockets in Figure 3. The Vasicek model models the instantaneous interest rate r_t, by estimating the values of x_t, that influence the values of r_t. Both processes are not observed in the market, since x_t is unknown and r_t cannot be found anywhere in the market. y_t is observed in the market, so measurement noise will still occur. Therefore, the not observable process is r_t, and the observable process is y_t, the yield rates for zero coupon bonds. y_t in the case of the Vasicek model would be similar to the on-board measurements. They do say something about the un-observable rate r_t and therefore x_t(or in the case of Figure 3, the place of the rocket) but they are not entirely correct.

The output of the Kalman filter is the series of filtered state variables x_t, that influence the instantaneous interest rate r_t. The Kalman filter takes the parameters of the Vasicek model as an input. These are in our case however unknown. Fortunately it is possible to estimate the variables of the Vasicek model by making use of the Kalman filter. We use the Kalman filter to calculate the log likelihood value for every time step. Since the logs are taken, the total log-likelihood can be calculated by summing the values for every time step. This can be done because it is known (Hamilton, 1994) that if the Kalman Filter is applied to Gaussian ARMA processes, it calculates the exact log-likelihood value. If the log-likelihood function is maximized by repeatedly running the Kalman filter with different parameters, which is best done by a numerical optimizer, we have found the best parameters of the Vasicek model by making use of a combination of an MLE method and the Kalman filter. Other methods are also possible for estimating the parameters. For example, the Least Square Method (LSM) and pure Maximum Likelihood Estimation (MLE) methods. The LSM minimizes the squared difference with the data and the model, MLE uses the likelihood function of the distribution of, in our case, y_t, takes the log and minimizes this function with respect to the parameters that we are interested in. However, they both do not have the property that they take into account the measurement error that occurs when measuring an unobservable process, they both assume that the data does not contain any flaws, which is unlikely with real world market data. Therefore, these methods result in worse estimates for the Vasicek parameters than the Kalman filter estimates (Amin, 2012).

3.2 State Equations

It is important to note that the state space formulas of the Kalman filter operate in different probability measures. The equation that describes the zero coupon discount rate, moves in the maturity direction of the data. This is why the equation is observed under the $\mathbb{Q}$ measure. The x_t process however, moves in the direction of time. This means that it is observed under the $\mathbb{P}$ probability measure.

3.2.1 Kalman Filter Equation 1, Observation Equation

To calculate the observed interest rate with probability measure $\mathbb{Q}$, we use the definition of $D(s, \tau, r_t)$ to retrieve (Brown and Dybvig, 1986)

$$y_t = \frac{-\ln\,(D(s,\tau,r_t))}{\tau}$$

With a multi-factor model, we get

$$-A^{\mathbb{Q}}(\tau)/\tau + \left(B^{\mathbb{Q}}(\tau)/\tau\right)' x_t.$$

Since y_t is the observable process in the Kalman Filter, this equation is often called the Observation equation and can be written down as

$$y_t = A + Bx_t + \eta_t \qquad \text{where } \eta_t \sim N(0, H),$$

Since our maturities go up to 50 years, A is equal to

$$\begin{pmatrix} -A(1)^{\mathbb{Q}}/1 \\ -A(2)^{\mathbb{Q}}/2 \\ \vdots \\ -A(50)^{\mathbb{Q}}/50 \end{pmatrix}$$

and B is the matrix

$$\begin{pmatrix} B(1,1),\ldots,B(1,f) \\ B(2,1),\ldots,B(2,f) \\ \vdots \\ B(50,1),\ldots,B(50,f) \end{pmatrix}$$

and H is a matrix that in our model we have restricted to a diagonal matrix, because we assume that the measurement errors are not dependant on each other. H can be defined as a full matrix, however, this will impact the running time of our algorithm significantly. Additionally, it is probable that the covariances are close to zero. This results in

$$H = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_{50} \end{pmatrix}$$

3.2.2 Kalman Filter Equation 2, State Equation

The second Kalman filter equation describes in which way x_t is dependant on x_{t-1}. Since this is the non-observable process, this equation is often called the State equation. As stated before, this equation moves in the time step direction, and therefore operates under $\mathbb{P}$ probability measure.

We start with the multi factor version of Equation (12), the real world probability measured version of the interest rate model:

$$dx_t = \Lambda x_t dt + \alpha d \begin{pmatrix} W^{\mathbb{P}}_{t,1} \\ W^{\mathbb{P}}_{t,2} \\ \vdots \\ W^{\mathbb{P}}_{t,n} \end{pmatrix}.$$

Where Λ is a diagonal matrix, with on the diagonal $\{\kappa_1, \kappa_2, ...\kappa_n\}$. The solution to this equation is

$$de^{-\Lambda t}(x_t - \theta^*) = e^{-\Lambda t}\alpha d \begin{pmatrix} W^{\mathbb{P}}_{t,1} \\ W^{\mathbb{P}}_{t,2} \\ \vdots \\ W^{\mathbb{P}}_{t,n} \end{pmatrix}, \tag{13}$$

verifiable by Ito's formula. This has the implication that

$$x_{t+h} = e^{\Lambda h} x_t + \int_0^h e^{\Lambda(h-s)}\alpha d \begin{pmatrix} W^{\mathbb{P}}_{t,1} \\ W^{\mathbb{P}}_{t,2} \\ \vdots \\ W^{\mathbb{P}}_{t,n} \end{pmatrix}. \tag{14}$$

As the integral is a martingale, it gives us

$$\mathbb{E}\left[x_{t+h}|x_t\right] = e^{\Lambda h} x_t. \tag{15}$$

(Pang and Hodges, 1995)

In the n-factor model where $n \leq 1$, θ^* is set equal to zero to make the model identifiable.

The state equation in which this results is

$$x_{t+1} = \phi x_t + \epsilon_t \qquad \text{where } \epsilon_t \sim N(0, Q).$$

Q is the covariance matrix of the error terms and ϕ is defined as a diagonal matrix where the entries are equal to $e^{\Lambda h}$.

When applying Ito Isometry to Equation 14, the form of Q is retrieved. The form of Q is determined by the parameters of the Vasicek model

$$Q = \int_0^{\Delta t} \exp\left(-\Lambda s\right) \alpha\alpha' \exp\left(-\Lambda' s\right) \text{ ds}. \tag{16}$$

Q is a matrix of $f \times f$.

3.2.3 Summary

The Observation Equation is:

$$y_t = A + Bx_t + \eta_t \qquad \text{where } \eta_t \sim N(0, H)$$

and the State Equation is:

$$x_{t+1} = \phi x_t + \epsilon_t \qquad \text{where } \epsilon_t \sim N(0, Q).$$

Here, y is our observed rate, and is an $i \times j$ matrix, where i is the number of observations per maturity, and j is the number of maturities. Therefore, y_t is an $1 \times j$ matrix. A is defined as $\frac{-A(\tau)}{\tau} - B\theta^*$ and B by $\frac{B(\tau)}{\tau}$. A is an $1 \times j$ matrix, where f is the amount of factors that are included in the model. B is an $j \times f$ matrix, R_t is an $f \times 1$ and ϕ is the conditional mean of R_t as derived above ($\psi = e^{\Lambda h}$) and an $f \times f$ matrix. H is the error term matrix for the first state space equation and is an $j \times j$ diagonal matrix in our model.

3.3 Derivation of the Kalman filter

We use the Kalman filter equations derived in the previous sections and start with a one factor model. The error terms in both of the equations are assumed to be Gaussian, as derived in Section 3.3. When combining the Kalman Filter equations, we can make an update for $\widehat{x}_t$, by taking the estimate of $x_{t|t-h}$ and adding the "Kalman Gain" multiplied by the measurement difference u_t, defined as

$$u_t = y_t - A - B\widehat{x}_t. \tag{17}$$

This results in

$$\widehat{x}_t = x_{t|t-h} + K_t u_t. \tag{18}$$

First, we obtain the projection values of x_t and P_t, defined as $x_{t|t-h}$ and $P_{t|t-h}$. From the State equation it is easily seen that

$$x_{t|t-h} = \phi \widehat{x}_{t-h},$$

where $\widehat{x}_{t-h}$ is our estimate of x_{t-h}, estimated by the Kalman Filter.

$$\begin{aligned} P_{t|t-h} &= \mathbb{E}\left[(x_{t|t-h} - \widehat{x}_{t|t-h})(x_{t|t-h} - \widehat{x}_{t|t-h})'\right] \\ &= \phi\, \mathbb{E}[(x_{t-h} - \widehat{x}_{t-h})(x_{t-h} - \widehat{x}_{t-h})']\phi' + Q_t \\ &= \phi \widehat{P}_{t-h}\phi' + Q_t \end{aligned} \tag{19}$$

The initial values can be initialized by

$$\widehat{R}_0 = \mathbb{E}[x_\infty]$$

and

$$\widehat{P}_0 = \mathrm{Var}[x_\infty].$$

So we get

$$\widehat{x}_0 = \theta$$

and

$$\widehat{P}_0 = \frac{1}{2}\alpha^2\kappa^{-1}$$

Equation (18) is the updating formula for the estimate of x_t. To calculate the Kalman Gain K_t, we need the error covariance matrix.

We define P_t as our error covariance matrix at time t. Substituting Equation (18) in the definition of P_t, which is $\mathbb{E}\left[(x_t - \widehat{x}_t)(R_t - \widehat{x}_t)'\right]$, we get

$$\widehat{P}_t = P_{t|t-h} - K_t B P_{t|t-h} - P_{t|t-h} B' K_t' + K_t (B P_{t|t-h} B' + H) K_t'. \tag{20}$$

We need to find a K_t, the Kalman Gain, that minimizes the error covariance matrix. Therefore, we take the derivative of Equation (20) with respect to K_t and setting it equal to zero. The full derivation of the derivative can be found in Lacey (1998). However, the intuition seems correct that if Equation (20) is considered as a quadratic function in K_t, the minimum of the function will have approximately the form $\frac{||b||}{2a}$. Substituting the values of Equation (20) yields

$$K_t = P_{t|t-h} B' (B P_{t|t-h} B' + H)^{-1}, \tag{21}$$

which is the correct result for the Kalman Gain. We define the measurement prediction error as

$$V_t = B P_{t|t-h} B' + H. \tag{22}$$

Therefore Equation (21) can we written as

$$K_t = P_{t|t-h} B' V_t^{-1}. \tag{23}$$

This concludes the Kalman Gain and the update of $\widehat{R}_t$. What is left is the update of $\widehat{P}_t$. We first substitute Equation (21) into Equation (20). When written out, it results in

$$\begin{aligned} \widehat{P}_t &= P_{t|t-h} - P_{t|t-h} B' V_t^{-1} B P_{t|t-h} \\ &= P_{t|t-h} - K_t B P_{t|t-h} \\ &= (I - K_t B) P_{t|t-h}. \end{aligned} \tag{24}$$

3.4 Likelihood

To use the Kalman filter as a method of estimating parameters of a model, it is needed to calculate the maximum likelihood function in the process of the Kalman filter. The Kalman filter assumes normality in order to calculate the exact likelihood function. r_t is normally distributed, because of the fact that r_t is a Ornstein Uhlenback Process. A process is an Orstein Uhlenbeck process if "it is stationary, Gaussian, Markovian, and continuous in probability", cited from Finch (2004). Since y_t is a linear combination of r_t, it is normally distributed as well. Since the process r_t is Gaussian and the error terms are Gaussian, the conditional distribution of y_t given r_t is Gaussian (Hamilton, 1994). A Gaussian likelihood function with mean $y_{t|t-1}$ and variance $\text{Var}_{t|t-1}$, where the estimated value according to the Kalman Filter is denoted as V_t, for all t, is given by the formula (Harvey, 1990).

$$\prod_{t=1}^{i} \exp\left[-u_t' V_t^{-1} u_t\right] (2\pi|V_t)^{-\frac{1}{2}}, \tag{25}$$

where i is our number of time points as stated above. Taking the natural logarithm results in

$$-\frac{i}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{i}\ln(|V_t|) - \frac{1}{2}\sum_{t=1}^{i} u_t' 2V_t^{-1} u_t. \tag{26}$$

We can get rid of the constants since constants in the log likelihood function do not matter for maximization or minimization purposes. It results in

$$-\frac{1}{2}\sum_{t=1}^{i}\ln(|V_t|) - \frac{1}{2}\sum_{t=1}^{i} u_t' V_t^{-1} u_t. \tag{27}$$

Equation (21) can be rewritten as

$$-\frac{1}{2}\sum_{t=1}^{i}\ln(|V_t|) + u_t' V_t^{-1} u_t. \tag{28}$$

4 The Data and The Algorithm

The data used in this research is the zero coupon swap curve of the Bundesbank. The data provides monthly observations from January 2002 to November 2010. This time period included the financial crisis and therefore is interesting to do research on. It provides a challenging but rich data set to fit a term structure on.

Mean Total Data	0.0307		
Mean 1 Year Mat	0.0179	SD 1 Year Mat	0.0160
Mean 2 Year Mat	0.0180	SD 2 Year Mat	0.0160
Mean 3 Year Mat	0.0195	SD 3 Year Mat	0.0161
Mean 4 Year Mat	0.0212	SD 4 Year Mat	0.0160
Mean 5 Year Mat	0.0241	SD 5 Year Mat	0.0160
Mean 6 Year Mat	0.0242	SD 6 Year Mat	0.0158
Mean 7 Year Mat	0.0255	SD 7 Year Mat	0.0157
Mean 8 Year Mat	0.0267	SD 8 Year Mat	0.0155
Mean 9 Year Mat	0.0278	SD 9 Year Mat	0.0154
Mean 10 Year Mat	0.0297	SD 10 Year Mat	0.0151
Mean 15 Year Mat	0.0322	SD 15 Year Mat	0.0147
Mean 20 Year Mat	0.0335	SD 20 Year Mat	0.0146
Mean 25 Year Mat	0.0336	SD 25 Year Mat	0.0145
Mean 30 Year Mat	0.0331	SD 30 Year Mat	0.0143
Mean 35 Year Mat	0.0328	SD 35 Year Mat	0.0142
Mean 40 Year Mat	0.0324	SD 40 Year Mat	0.0140
Mean 45 Year Mat	0.0322	SD 45 Year Mat	0.0139
Mean 50 Year Mat	0.0320	SD 50 Year Mat	0.0137

Table 1: Descriptives of the Bundesbank Data

We have monthly data on the term structure, which means we have 203 time points. The term structure of January 2015 for example, can be found in our data and is depicted by Figure 4. It can be seen that there seems to be a steeper increase in the first 25 years. Afterwards, the interest rates seem to slowly decrease from 40 years onward, which is hard to be modeled by many of the models named in the introduction that make use of an UFR and a conversion rate.

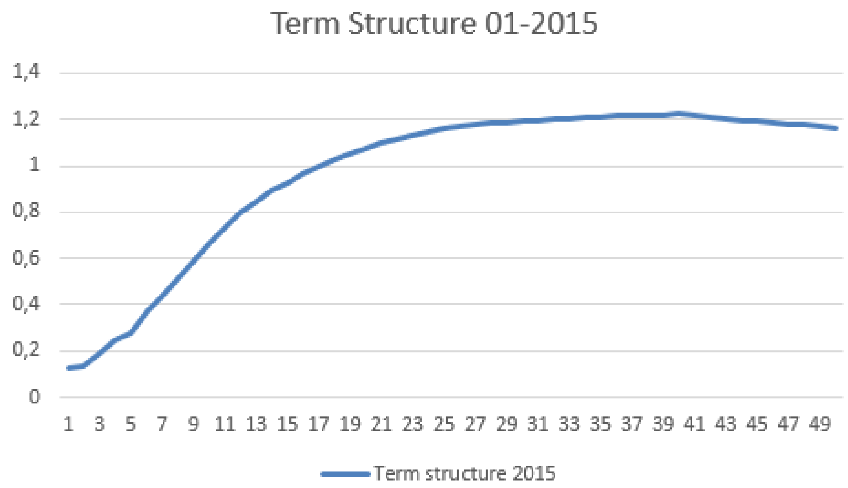


Figure 4: Term-structure of Interest rate according to the swap rates of the Bundesbank, January 2015.

4.1 Algorithm

In the previous sections, the Vasicek model and the Kalman filter are separately discussed. In this section, the complete algorithm is presented in pseudo-code that is used to do the calculations in Matlab. The pseudo-code is additionally suitable for similar languages such as R. The code works by first initializing values for the variables κ, α, ψ and H. The code that is used for this thesis took the data for the maturities 1 up to 10 year, 15, 20, 25, 30, 35, 40, 45 and 50 year. This is done to reduce the computations. If one wants to, all the data can be used in the model. The code consists of a numerical solver, *fminunc*, which is a numerical optimizer in Matlab. The numerical optimizer is used when calculating the maximum likelihood of the Kalman filter, applied to the Vasicek model. The Kalman filter is implemented as described in Section 3, except that the function in the code returns the negative likelihood in order to maximize the likelihood value.

4.2 Pseudo-code

In the pseudo-code the notation Ω is used to describe all the parameters of the Vasicek model, $\{\kappa, \psi, \alpha, H\}$. Formulas used for A, B and Q can be found in Section 2.

The running time of the algorithm depends on the input. When taking f as the number of factors, m as the number of maturities and t as the number of observations per maturities, the number of parameters is equal to $f + f^2 + m + f$. This is denoted in a regular running time analysis notation (Dasgupta et al., 2008) as an input of $O(f^2)$. The run time of the Kalman filter is equal to $O(t \cdot (2f + f^2 + m)^3)$, as the most complex operations are calculating the determinant and the inverse of a matrix. These operations can both be done in $O(n^3)$. The running time can be rewritten as $O(t \cdot f^6)$ when simplifying, since the running time is denoted in big O notation. The algorithm is therefore polynomial itself as well as polynomial in the input size. However, the numerical optimization process has an unknown big O running time. This means that the running time of the whole of the algorithm is undefined. It can still be concluded that if the parameters of the Vasicek model are known from prior research, the algorithm is efficient and runs relatively fast.

Algorithm 1 Kalman filter applied to Vasicek model - Algorithm

procedure KALMAN FILTER - VASICEK MODEL
    $\Omega \leftarrow$ Initialization values, size: $f + f^2 + m + f$
    ans $\leftarrow$ Optimizer(Function, Ω)
Function:
    $A(\Omega) \leftarrow -A(\tau)/\tau$
    $B(\Omega) \leftarrow B(\tau)'/\tau$
    $R_0(\Omega) \leftarrow \theta$
    $P_0(\Omega) \leftarrow \alpha^2/2\kappa$
    $Q \leftarrow Q(\Omega)$
Kalman Filter Loop:
    for all t $\in$ Observations **do**
        $R_{t|t-h} \leftarrow \phi R_{t-h}$
        $P_{t|t-h} \leftarrow \phi P_{t-h} \phi'$
        $u \leftarrow y - A - BR_{t|t-h}$
        $v \leftarrow BP_{t|t-h}B' + \mathrm{H}$
        Loglike $\leftarrow -\frac{1}{2}(\ln(det(v)) + u'v^{-1}u)$
        $K \leftarrow P_{t|t-h}B'v^{-1}$
        $L \leftarrow I - KB$
        $R_{t-h} \leftarrow R_{t|t-h} + Ku$
        $P_{t-h} \leftarrow LP_{t|t-h}$
        goto *loop*
    ans $\leftarrow$ -sum(Loglike)

4.3 Standard errors

When using the algorithm of Section 4.2, the standard errors can be calculated with the use of the Hessian matrix (Yuan and Hayashi, 2006). This is because the kalman filter function can be used as an MLE function. When calculating a MLE value, the standard errors can be calculated by the fact that the variance of the MLE parameters, now denoted by Ω, are defined by

$$\mathrm{Var}[\Omega] = (-\,\mathbb{E}[\mathrm{Hes}(\Omega)])^{-1}\,.$$

The Hes matrix is the Hessian which can be calculated by taking the second derivative of the MLE function, which can be done by a computer software or by hand (Henningsen and Toomet, 2011). We can calculate the variance like this because if we calculate the derivative of the likelihood function, we get

$$\frac{\partial \mathcal{L}}{\partial \Omega}|\widehat{\Omega} = 0$$

$$= \frac{\partial \mathcal{L}}{\partial \Omega}|\Omega + \frac{\partial^2 \mathcal{L}}{\partial \Omega \partial \widehat{\Omega}}(\widehat{\Omega} - \Omega)$$

which implies

$$\widehat{\Omega} - \Omega = -\left[\frac{\partial^2 \mathcal{L}}{\partial \Omega \partial \widehat{\Omega}}\right]^{-1} \frac{\partial \mathcal{L}}{\partial \Omega}$$

$$\mathrm{Var}[\widehat{\Omega}] = \mathbb{E}(\widehat{\Omega} - \Omega).$$

Substituting above equations yield

$$\mathrm{Var}[\widehat{\Omega}] = \left(-\,\mathbb{E}\left[\frac{\partial^2 \mathcal{L}}{\partial \Omega \partial \widehat{\Omega}}\right]\right)^{-1}.$$

hereby we can see that asymptotically, we can calculate the variance of Ω. In the case of our described algorithm, the function returns the negative log likelihood. This will result in

$$\mathrm{Var}[\Omega] = \left(\mathbb{E}[\mathrm{Hes}^*(\Omega)]\right)^{-1},$$

where Hes^* is the hessian associated with the described algorithm. The standard deviation is calculated by taking the square root of the diagonal of the variance matrix.

$$\mathrm{sd} \; = \sqrt{\mathrm{diag}\left(\mathbb{E}[\mathrm{Hes}^*(\Omega)]\right)^{-1}}.$$

5 Results

In this section the results of different multi-factor models of the Vasicek model are presented. The section starts with the Two-factor model and builds up to a Five-factor model. The different values of the parameters are presented together with the standard deviation of the different measurement errors. Additionally the function value is given. The function value in our case is the likelihood measure.

5.1 Two-factor Vasicek model

Parameter	Value	stddev
κ_1	0.0178	0.0005
κ_2	10.1206	1.1727
$\alpha_{1,1}$	0.0136	0.0012
$\alpha_{1,2}$	-0.0100	0.0001
$\alpha_{2,1}$	-0.0100	0.0001
$\alpha_{2,2}$	0.3186	0.0024
ψ_1	2.4353	2.1210
ψ_2	-5.5462	5.4822
Loglikelihood	$2.0157 \cdot 10^4$	-

Table 2: Standard deviations per maturity, 2 factor model

These standard deviations seem reasonable when compared with the values of other standard deviations from similar research (De Jong, 2000). The numerical value of the likelihood measure does not provide information on its own. However, it can be used to compare with the other factor models, as is described in Subsection 5.5.

std. ME	Value	stddev
$\sqrt{h_1}$	0.0075	0.0001
$\sqrt{h_2}$	0.0055	0.0001
$\sqrt{h_3}$	0.0038	0.0001
$\sqrt{h_4}$	0.0027	< 0.0001
$\sqrt{h_5}$	0.0019	0.0001
$\sqrt{h_6}$	0.0011	< 0.0001
$\sqrt{h_7}$	0.0005	< 0.0001
$\sqrt{h_8}$	0.0000	< 0.0001
$\sqrt{h_9}$	0.0006	< 0.0001
$\sqrt{h_{10}}$	0.0017	< 0.0001
$\sqrt{h_{15}}$	0.0010	< 0.0001
$\sqrt{h_{20}}$	0.0015	0.0001
$\sqrt{h_{25}}$	0.0027	0.0001
$\sqrt{h_{30}}$	0.0038	< 0.0001
$\sqrt{h_{35}}$	0.0037	0.0001
$\sqrt{h_{40}}$	0.0040	0.0001
$\sqrt{h_{45}}$	0.0053	0.0002
$\sqrt{h_{50}}$	0.0077	0.0005

Table 3: Measurement errors per maturity

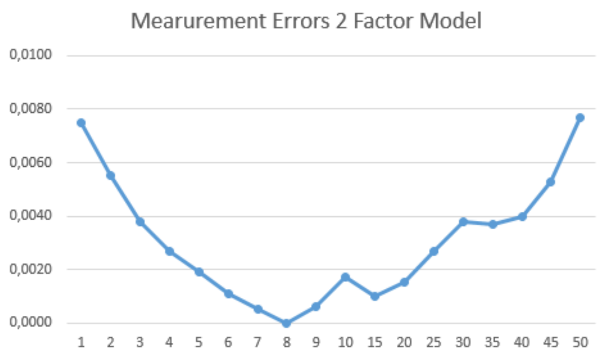


Figure 5: Graph of measurement errors plotted against maturity, 2 factor model

5.2 Three-factor Vasicek model

Parameter	Value	stddev
κ_1	5.4429	0.0151
κ_2	0.0144	0.0028
κ_3	1.2126	0.0109
$\alpha_{1,1}$	0.2800	0.0031
$\alpha_{1,2}$	-0.0098	0.0016
$\alpha_{1,3}$	-0.0587	0.0018
$\alpha_{2,1}$	-0.0098	0.0016
$\alpha_{2,2}$	0.0147	0.0012
$\alpha_{2,3}$	-0.0026	0.0001
$\alpha_{3,1}$	-0.0587	0.0018
$\alpha_{3,2}$	-0.0026	0.0001
$\alpha_{3,3}$	0.0640	0.0003
ψ_1	-2.5867	3.2932
ψ_2	6.7782	5.2836
ψ_3	9.3597	28.2955
Loglikelihood	$2.2825 \cdot 10^4$	-

Table 4: Parameter results 3 factor model

std. ME	Value	stddev
$\sqrt{h_1}$	0.0071	0.0001
$\sqrt{h_2}$	0.0049	0.0001
$\sqrt{h_3}$	0.0031	< 0.0001
$\sqrt{h_4}$	0.0019	0.0001
$\sqrt{h_5}$	0.0006	< 0.0001
$\sqrt{h_6}$	0.0004	< 0.0001
$\sqrt{h_7}$	0.0000	< 0.0001
$\sqrt{h_8}$	0.0003	< 0.0001
$\sqrt{h_9}$	0.0011	< 0.0001
$\sqrt{h_{10}}$	0.0014	< 0.0001
$\sqrt{h_{15}}$	0.0007	< 0.0001
$\sqrt{h_{20}}$	0.0011	0.0001
$\sqrt{h_{25}}$	0.0027	< 0.0001
$\sqrt{h_{30}}$	0.0030	0.0001
$\sqrt{h_{35}}$	0.0036	0.0001
$\sqrt{h_{40}}$	0.0039	< 0.0001
$\sqrt{h_{45}}$	0.0045	0.0001
$\sqrt{h_{50}}$	0.0075	0.0001

Table 5: Measurement errors per maturity

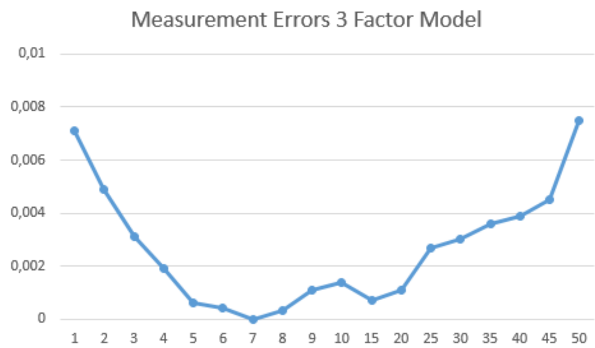


Figure 6: Graph of measurement errors plotted against maturity, 3 factor model

5.3 Four-factor Vasicek model

Parameter	Value	stddev
κ_1	4.1219	0.0062
κ_2	-0.0001	< 0.0001
κ_3	0.0849	0.0092
κ_4	0.7562	0.0149
$\alpha_{1,1}$	0.0292	0.0036
$\alpha_{1,2}$	0.0016	0.0008
$\alpha_{1,3}$	-0.0149	0.0011
$\alpha_{1,4}$	-0.0318	0.0006
$\alpha_{2,1}$	0.0016	0.0008
$\alpha_{2,2}$	0.0031	0.0010
$\alpha_{2,3}$	0.0001	< 0.0001
$\alpha_{2,4}$	0.0001	< 0.0001
$\alpha_{3,1}$	-0.0149	0.0011
$\alpha_{3,2}$	< 0.0001	< 0.0001
$\alpha_{3,3}$	0.0232	0.0018
$\alpha_{3,4}$	0.0163	0.0044
$\alpha_{4,1}$	-0.0318	0.0006
$\alpha_{4,2}$	< 0.0001	< 0.0001
$\alpha_{4,3}$	0.0163	0.0044
$\alpha_{4,4}$	0.0219	0.0031
ψ_1	-1.4879	6.1649
ψ_2	0.0001	1.9477
ψ_3	0.4683	5.2232
ψ_4	1.7163	2.5830
Loglikelihood	$2.7087 \cdot 10^4$	-

Table 6: Parameter results 4 factor model

std. ME	Value	stddev
$\sqrt{h_1}$	0.0072	< 0.0001
$\sqrt{h_2}$	0.0039	0.0001
$\sqrt{h_3}$	0.0030	0.0001
$\sqrt{h_4}$	0.0017	< 0.0001
$\sqrt{h_5}$	0.0003	< 0.0001
$\sqrt{h_6}$	0.0002	< 0.0001
$\sqrt{h_7}$	0.0000	< 0.0001
$\sqrt{h_8}$	0.0003	< 0.0001
$\sqrt{h_9}$	0.0006	< 0.0001
$\sqrt{h_{10}}$	0.0002	< 0.0001
$\sqrt{h_{15}}$	0.0012	< 0.0001
$\sqrt{h_{20}}$	0.0014	0.0001
$\sqrt{h_{25}}$	0.0009	< 0.0001
$\sqrt{h_{30}}$	0.0020	< 0.0001
$\sqrt{h_{35}}$	0.0019	< 0.0001
$\sqrt{h_{40}}$	0.0023	< 0.0001
$\sqrt{h_{45}}$	0.0024	< 0.0001
$\sqrt{h_{50}}$	0.0039	< 0.0001

Table 7: Measurement errors per maturity

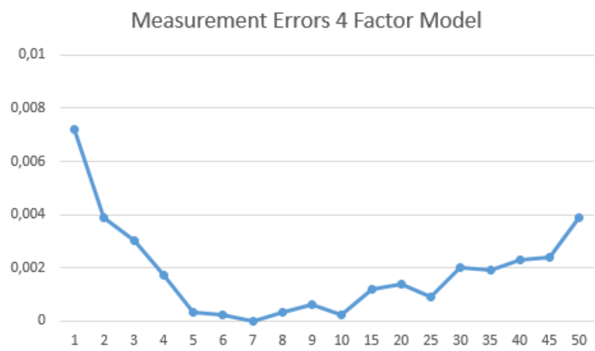


Figure 7: Graph of measurement errors plotted against maturity, 4 factor model

5.4 Five-factor model

Parameter	Value	stddev
κ_1	4.9432	0.3028
κ_2	0.7267	0.0250
κ_3	0.6482	0.0193
κ_4	0.1051	0.0094
κ_5	-0.0002	< 0.0001
$\alpha_{1,1}$	0.1857	0.0522
$\alpha_{1,2}$	-0.0002	< 0.0001
$\alpha_{1,3}$	-0.0125	0.0063
$\alpha_{1,4}$	-0.0287	0.0002
$\alpha_{1,5}$	-0.0025	0.0005
$\alpha_{2,1}$	-0.0002	< 0.0001
$\alpha_{2,2}$	0.0010	< 0.0001
$\alpha_{2,3}$	-0.0002	< 0.0001
$\alpha_{2,4}$	0.0032	0.0009
$\alpha_{2,5}$	0.0017	< 0.0001
$\alpha_{3,1}$	-0.0125	0.0063
$\alpha_{3,2}$	-0.0002	< 0.0001
$\alpha_{3,3}$	0.0026	0.0013
$\alpha_{3,4}$	-0.0034	0.0018
$\alpha_{3,5}$	-0.0019	0.0009
$\alpha_{4,1}$	-0.0287	0.0002
$\alpha_{4,2}$	0.0032	0.0009
$\alpha_{4,3}$	-0.0034	0.0018
$\alpha_{4,4}$	0.0484	0.0130
$\alpha_{4,5}$	-0.0030	0.0007
$\alpha_{5,1}$	-0.0025	0.0005
$\alpha_{5,2}$	0.0017	< 0.0001
$\alpha_{5,3}$	-0.0019	0.0009
$\alpha_{5,4}$	-0.0030	0.0007
$\alpha_{5,5}$	0.0094	0.0019
ψ_1	-0.8051	3.4164
ψ_2	1.2718	4.2793
ψ_3	1.2962	7.2933
ψ_4	1.0542	2.3792
ψ_5	1.6191	3.3827
Loglikelihood	$2.7103 \cdot 10^4$	-

Table 8: Parameter results 5 factor model

Measurement error	Value	stddev
$\sqrt{h_1}$	0.0061	< 0.0001
$\sqrt{h_2}$	0.0042	0.0001
$\sqrt{h_3}$	0.0028	0.0001
$\sqrt{h_4}$	0.0017	0.0001
$\sqrt{h_5}$	0.0006	< 0.0001
$\sqrt{h_6}$	0.0001	< 0.0001
$\sqrt{h_7}$	0.0000	< 0.0001
$\sqrt{h_8}$	0.0004	< 0.0001
$\sqrt{h_9}$	0.0005	< 0.0001
$\sqrt{h_{10}}$	0.0002	< 0.0001
$\sqrt{h_{15}}$	0.0007	< 0.0001
$\sqrt{h_{20}}$	0.0016	0.0001
$\sqrt{h_{25}}$	0.0010	0.0001
$\sqrt{h_{30}}$	0.0021	< 0.0001
$\sqrt{h_{35}}$	0.0019	< 0.0001
$\sqrt{h_{40}}$	0.0023	< 0.0001
$\sqrt{h_{45}}$	0.0022	< 0.0001
$\sqrt{h_{50}}$	0.0029	< 0.0001

Table 9: Measurement errors per maturity

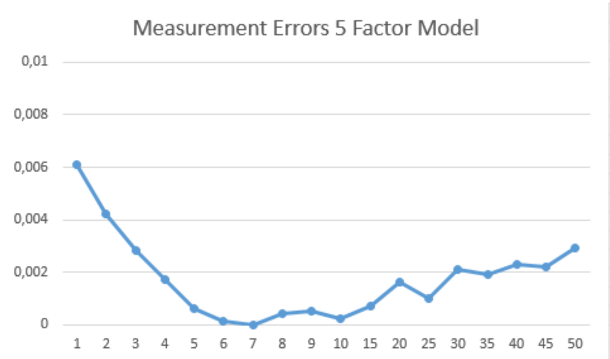
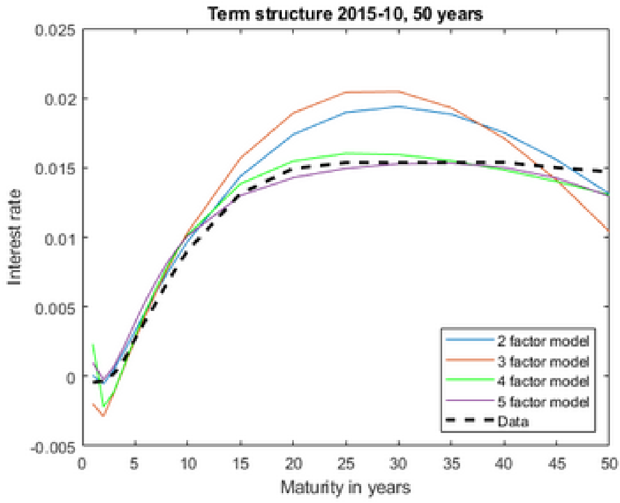
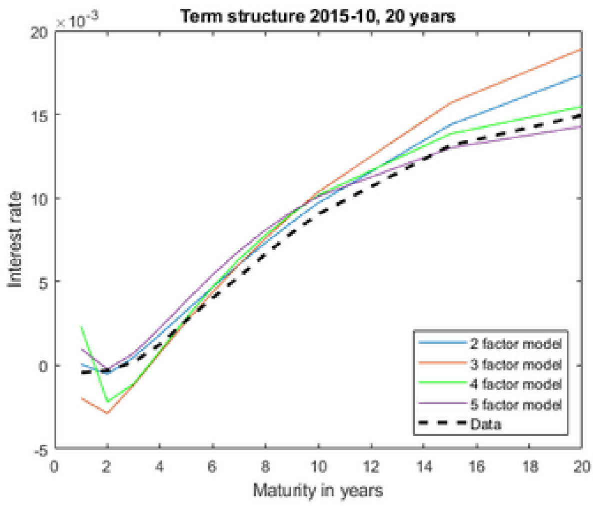


Figure 8: Graph of measurement errors plotted against maturity, 5 factor model

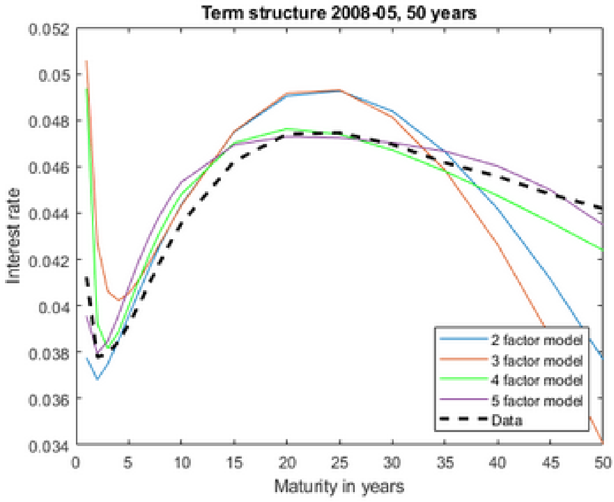
5.5 Comparison

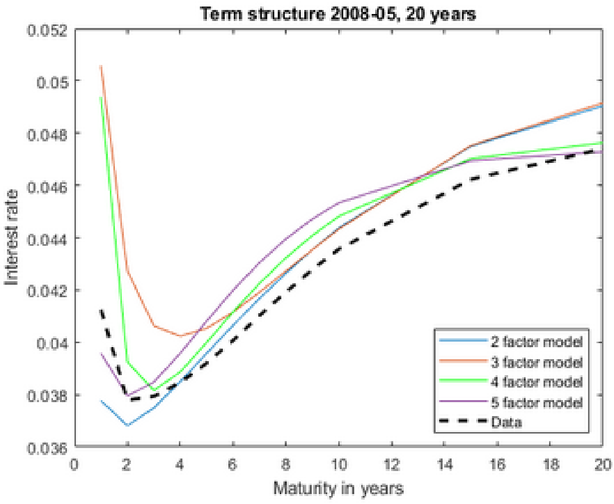
The results presented in the previous subsections seem plausible when compared with other results from similar research. The kappas have meaningful values and are significant, as well as the alphas. The standard deviation of the measurement errors seem plausible as well and model seems to value them very accurately. The psis however are not significant and do not give much information. This occurrence can be found in other literature as well and does not contribute greatly to the correctness of the model. The models can be compared graphically using the filtered x processes that the Kalman Filter provides and plotting them together with the actual data retrieved from the Bundesbank. To compare, 3 time periods are used. First after the crises, then during the crisis and lastly before the crises happened. We choose to do the three time periods as the term structures had different shapes in all periods.





Overall, the model fits the economically stable times better than the crises times. Comparing the results of the four factor Vasicek model with the three factor and two factor model, the four factor model fits the

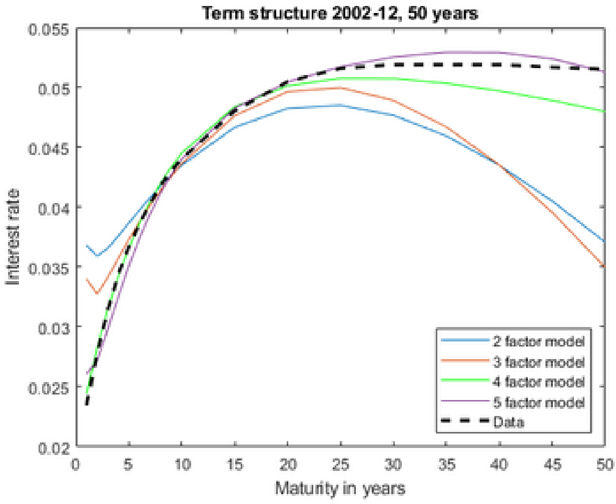


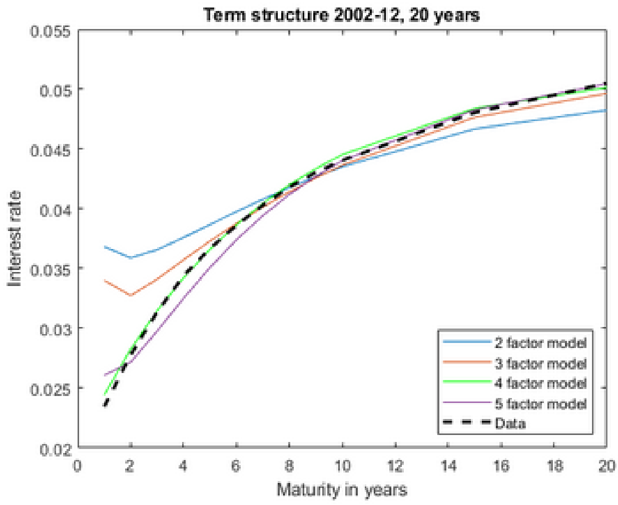


long end of the term structure better. This can be explained by a negative mean reversion appearing in the four factor model. As can be seen in Figure 4, the long end of the term structure decreases slightly. This is captured by a negative mean reversion. In the three factor and two factor model, this occurrence is not powerful enough to over shadow a value of the factors. Apparently other curvatures in the term structure are more important and do not show up in the model. In the four factor model the effect is included in the model. As can be seen from the graphs, the two factor model only fits the 0-20 year maturity part of the term structure relatively well. The same holds for the three factor model. However, the model seems to outperform the two factor model in all three time points. Additionally, the likelihood value increases significantly when substituting the three factor model for the two factor model. When performing a Log-likelihood Ratio Test (LRT), we see that the test strongly rejects the null-hypothesis that the two models perform as well. The LRT is performed by taking

$$\text{Likelihood-Ratio} = 2\left(\ln\left(L1\right) - \ln\left(L2\right)\right),$$

where $\ln(L1)$ represents the Log Likelihood value for model 1, and $\ln(L2)$ the Log Likelihood for model





2. The Likelihood ratio is χ^2 distributed. When the ratio exceeds the χ^2-value, with the degrees of freedom corresponding to the amount of extra variables, the Null Hypothesis that Model 1 performs the same as Model 2 is rejected (Dickey and Fuller, 1981). It is only the four-factor model that seems to fit the long term part of the graph sufficiently well. This might be because the four factor model includes a negative κ. The five factor model has similar values for κ as the four factor model. From the likelihood value we can see that the five factor model does not improve the model in a great way. However, the LRT score is 32 and with a χ^2 distribution with 11 degrees of freedom, which is the amount of extra parameters used for the five factor model, the Null-Hypothesis is rejected. The χ^2 table can be found in the appendix. It is to note that comparing the difference in likelihood with the other models and the four factor and the five factor, the difference seems relatively small. Additionally, the cut off point is not extremely far away from 32. A summary is given in the table below.

Models compared	ΔLikelihood	LTR Score	χ^2 0.1	χ^2 0.05	χ^2 0.01
2 factor vs. 3 factor	2668	5336	12.017	14.067	18.475
3 factor vs. 4 factor	4262	8524	14.631	16.919	21.666
4 factor vs. 5 factor	16	**32**	17.275	19.675	**24.725**

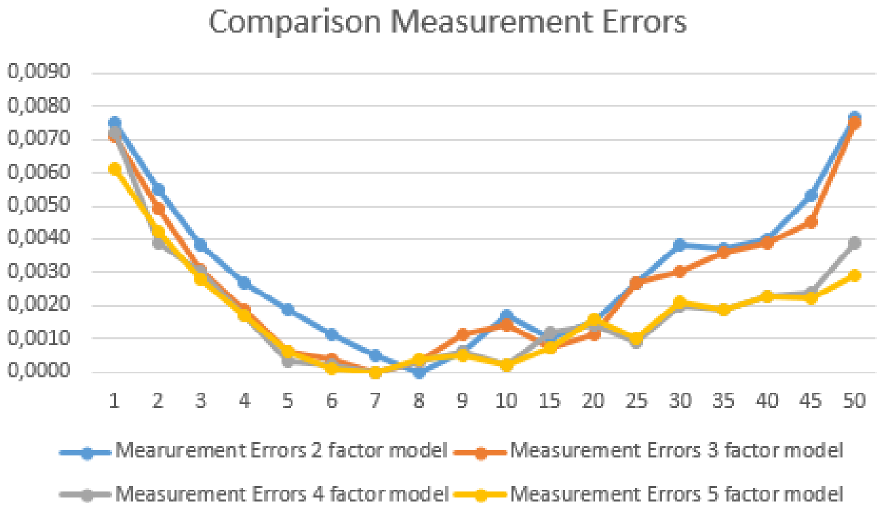


Figure 9: Measurement errors of all different factor Vasicek models used.

The standard deviations of the measurement errors tend to decrease when more factors are added, as is to be expected. The measurement errors of the models used in above sections are depicted in Figure 9. As can be seen, the values do not decrease in the near maturities, most evident in the 1 year maturity. The measurement errors remain high when more factors are added. This can be explained by that the short term maturity rates are strongly correlated by the regulation of the ECB. Therefore, it is highly likely that the short term rates do not follow a diffusion process, but rather a jump process. This explains why the Vasicek model can not model these rates sufficiently, independent of the amount of factors added. In the very long term end of the graph, like 45 and 50 years of maturity, a big decrease in the measurement errors can be seen when adding the fourth factor and a further decrease when adding the fifth factor. For the two and three factor model it can be seen that the model focuses on decreasing the measurement error standard deviation of a maturity per factor. From the four factor model onward, this effect is not that observable as the two and three factor models. This could be explained by the long term interest rates being hard to fit for the Vasicek model, similarly to the very short term maturity rates. However, it is seen in the graphs that the four and five factor model do seem to fit the long term interest rates well.

6 Conclusion

To conclude, the Vasicek model seems to work well for the very long term interest rates in times where there is no financial crises present. However, the model needs more factors than usually used in the literature when the Vasicek model is discussed. A four factor model has the minimum amount of factors such that it captures the dynamics of the very long term rates. Strictly speaking the five factor model outperforms the four factor model. However, the cut off point is close to the test value of the LTR. Additionally, the parameter values do not differ extremely when comparing the five factor model to the four factor model. There is a downside of adding more factors to the model, because the difficulty to solve the optimization problem of the algorithm used becomes harder for a computer as more parameters are involved in the model and can therefore increase the running time to a large extend. According to this research, it depends on the problem that the model is applied to if a four or five factor model is more preferable. To better compare the models, the models would have to be applied to more data sets in further research. Overall, the results of the four and five factor models are a relatively good fit with the data up to 50 years of maturity. This gives new insights when choosing a model to create a term structure for very long maturities or when choosing a 'last liquid point', because as seen in this research, when using at least a four factor model the term structure with maturities up to 50 years can be modeled with the Vasicek model and can give probable results.

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Appendix

List of Figures

List of Tables

Kalman Filter Equations

All multi-factor Kalman Filter Equations

We state the summary of all equation used for the multi factor Kalman filter.

Initialization is done by

$$x_0 = \mathbb{E}(x_\infty)$$

and

$$P_0 = \mathrm{Var}(x_\infty)$$

The prediction made at time t is

$$x_{t|t-h} = \phi \widehat{x}_{t-h}$$

and

$$P_{t|t-h} = \phi \widehat{P}_{t-h} \phi' + Q$$

Now the predictions are obtained the likelihood contribution can be calculated by using the formula

$$\ln(L_t) = -\frac{1}{2}(\ln|V_t| + u_t' V_t^{-1} u_t)$$

where

$$u_t = y_t - A - BR_{t|t-h}$$

and

$$V_t = BP_{t|t-h}B' + H$$

After the steps stated above the values for P and x are updated. This is done by first calculating

$$K_t = P_{t|t-h}B'V_t^{-1}$$

$$L_t = I - K_t B.$$

These equations can update x and P by the formulas

$$\widehat{x}_t = x_{t|t-h} + K_t u_t$$

and

$$\widehat{P}_t = L_t P_{t|t-h}.$$

χ^2 Table

	P										
DF	**0.995**	**0.975**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.002**	**0.001**
1	0.0000393	0.000982	1.642	2.706	3.841	5.024	5.412	6.635	7.879	9.550	10.828
2	0.0100	0.0506	3.219	4.605	5.991	7.378	7.824	9.210	10.597	12.429	13.816
3	0.0717	0.216	4.642	6.251	7.815	9.348	9.837	11.345	12.838	14.796	16.266
4	0.207	0.484	5.989	7.779	9.488	11.143	11.668	13.277	14.860	16.924	18.467
5	0.412	0.831	7.289	9.236	11.070	12.833	13.388	15.086	16.750	18.907	20.515
6	0.676	1.237	8.558	10.645	12.592	14.449	15.033	16.812	18.548	20.791	22.458
7	0.989	1.690	9.803	12.017	14.067	16.013	16.622	18.475	20.278	22.601	24.322
8	1.344	2.180	11.030	13.362	15.507	17.535	18.168	20.090	21.955	24.352	26.124
9	1.735	2.700	12.242	14.684	16.919	19.023	19.679	21.666	23.589	26.056	27.877
10	2.156	3.247	13.442	15.987	18.307	20.483	21.161	23.209	25.188	27.722	29.588
11	2.603	3.816	14.631	17.275	19.675	21.920	22.618	24.725	26.757	29.354	31.264
12	3.074	4.404	15.812	18.549	21.026	23.337	24.054	26.217	28.300	30.957	32.909
13	3.565	5.009	16.985	19.812	22.362	24.736	25.472	27.688	29.819	32.535	34.528
14	4.075	5.629	18.151	21.064	23.685	26.119	26.873	29.141	31.319	34.091	36.123
15	4.601	6.262	19.311	22.307	24.996	27.488	28.259	30.578	32.801	35.628	37.697
16	5.142	6.908	20.465	23.542	26.296	28.845	29.633	32.000	34.267	37.146	39.252
17	5.697	7.564	21.615	24.769	27.587	30.191	30.995	33.409	35.718	38.648	40.790
18	6.265	8.231	22.760	25.989	28.869	31.526	32.346	34.805	37.156	40.136	42.312
19	6.844	8.907	23.900	27.204	30.144	32.852	33.687	36.191	38.582	41.610	43.820
20	7.434	9.591	25.038	28.412	31.410	34.170	35.020	37.566	39.997	43.072	45.315
21	8.034	10.283	26.171	29.615	32.671	35.479	36.343	38.932	41.401	44.522	46.797
22	8.643	10.982	27.301	30.813	33.924	36.781	37.659	40.289	42.796	45.962	48.268
23	9.260	11.689	28.429	32.007	35.172	38.076	38.968	41.638	44.181	47.391	49.728
24	9.886	12.401	29.553	33.196	36.415	39.364	40.270	42.980	45.559	48.812	51.179
25	10.520	13.120	30.675	34.382	37.652	40.646	41.566	44.314	46.928	50.223	52.620
26	11.160	13.844	31.795	35.563	38.885	41.923	42.856	45.642	48.290	51.627	54.052
27	11.808	14.573	32.912	36.741	40.113	43.195	44.140	46.963	49.645	53.023	55.476
28	12.461	15.308	34.027	37.916	41.337	44.461	45.419	48.278	50.993	54.411	56.892
29	13.121	16.047	35.139	39.087	42.557	45.722	46.693	49.588	52.336	55.792	58.301
30	13.787	16.791	36.250	40.256	43.773	46.979	47.962	50.892	53.672	57.167	59.703

Figure 10: χ^2 Table, retrieved from Medcalc (2019)