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THE LONGSTAFF – SCHWARTZ MODEL OF YIELD TERM STRUCTURE AND ITS EXPANSION

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Abstract

The Longstaff – Schwartz model is considered both in the space of latent transient states and in the space of observable (or estimated) state variables. Analytical expressions for yield curves to maturity and forward curves are obtained in both cases. The extension of the model to an arbitrary dimension of the state space is proposed. Within the framework of this expansion, a method is proposed for obtaining analytical solutions of equations with respect to the functions of the term structure of interest rates, when the initial equations for the dynamics of the short-term interest rate lead to non-linear systems of Riccati equations with respect to these functions, which do not allow an analytical solution to be obtained by known methods.This method is based on the obvious assertion that if the process of a short-term interest rate is specified, then the corresponding term structure of yield does not depend on how the space of variables of the state of the financial market is described. Numerical example is given.

Keywords: term structure of interest rates, short-term interest rate, Longstaff – Schwartz model, Riccati system of equations.

JEL Classification: G12

The term structure of zero-coupon yield is one of the claimed characteristics that is used in determining the value of financial assets. However, until now term structures in analytical form could be obtained only for affine systems of yield, as a rule, for single-factor models. Since single-factor models describe the situation in the financial market not accurately, models with more factors are emerging. One of the first such models is the two-factor Longstaff – Schwartz model (1992). It is based on the use of so-called latent (hidden) factors, i.e., state variables that are not directly observed on the market. It is considered that the dynamics of these factors is described by the "square root" processes of Cox – Ingersoll – Ross (CIR). Then, if it is possible to connect these processes with actually observed indicators, then it is possible to give an analysis of such model a real meaning too. This property can be used in more general circumstances when a model with an arbitrary number of latent factors is considered.

1. The Longstaff – Schwartz model

The well-known version of the two-factor model CIR is the Longstaff – Schwartz model [1]. In this model, two independent CIR processes are selected as the initial state variables, having the meaning of some unobserved economic factors:

$$\begin{aligned} dx &= (a - bx)dt + \sqrt{x}dW_1, \\ dy &= (d - ey)dt + \sqrt{y}dW_2. \end{aligned} \tag{1}$$

In what follows it is considered that the weighted sum of these factors forms a short-term interest rate

$$r(t) = \alpha x(t) + \beta y(t), \tag{2}$$

where α and β are nonnegative constants. Using the original equations and the Ito formula allows us to write that

$$dr(t) = \alpha\, dx(t) + \beta\, dy(t) =$$
$$= [\alpha a + \beta d - bx(t) - ey(t)]dt + \alpha\sqrt{x(t)}\, dW_1(t) + \beta\sqrt{y(t)}\, dW_2(t)\,.$$

We note that the stationary mathematical expectations of the processes $x(t)$ and $y(t)$ are determined by the equalities $E[x(t)] = a/b$ and $E[y(t)] = d/e$. For the same reasons the square $dr(t)$ has the form

$$(dr(t))^2 = \alpha^2 x(t)\, dt + \beta^2\, y(t)\, dt$$

The local (in time) variance of the change in the interest rate over a time interval $(t, t + dt)$ per unit time is

$$V(t) = \alpha^2 x(t) + \beta^2\, y(t). \tag{3}$$

It follows from (2) and (3) that there is a one-to-one relationship between the pairs (x, y) and (r, V) determined by the relations

$$r = \alpha x + \beta y, \quad V = \alpha^2 x + \beta^2 y,$$
$$x = \frac{\beta r - V}{\alpha(\beta - \alpha)}, \quad y = \frac{V - \alpha r}{\beta(\beta - \alpha)}\,. \tag{4}$$

Therefore, it is more natural to regard as state variables not unobservable indefinite "economic factors" (x, y), but having a quite definite meaning the interest rate r and its local variance V. However, it should be noted that if the domain of definition (x, y) is the entire first quadrant of the plane (X, Y), then the variables (r, V) take values only in the bounded domain of the first quadrant determined by the inequalities $\alpha r < V < \beta r$ (or $V/\beta < r < V/\alpha$), $0 < \alpha < \beta$, in which small rates can correspond only to small variances (or small variances have only small interest rates). Keeping in mind the existing linear transformation

$$\begin{pmatrix} r \\ V \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \alpha^2 & \beta^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{5}$$

Instead of the system of equations (1) by the Ito formula we obtain the corresponding system for (r, V)

$$d\begin{pmatrix} r \\ V \end{pmatrix} = K\left(\theta - \begin{pmatrix} r \\ V \end{pmatrix}\right)dt + \sigma(r,V)\; d\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \tag{6}$$

where

$$K = \frac{1}{\alpha - \beta}\begin{pmatrix} e\alpha - b\beta & b - e \\ \alpha\beta(e - b) & b\alpha - e\beta \end{pmatrix}, \;\; \theta = \begin{pmatrix} \alpha & \beta \\ \alpha^2 & \beta^2 \end{pmatrix}\begin{pmatrix} a/b \\ d/e \end{pmatrix},$$

$$\sigma(r,V) = \begin{pmatrix} \alpha & \beta \\ \alpha^2 & \beta^2 \end{pmatrix} \times \begin{pmatrix} \sqrt{\dfrac{(\beta r - V)}{\alpha(\beta - \alpha)}} & 0 \\ 0 & \sqrt{\dfrac{(V - \alpha r)}{\beta(\beta - \alpha)}} \end{pmatrix}.$$

To determine the differential equations for the term structure functions $A(\tau)$ and $B(\tau)$ we apply the technique described in [2]. To obtain the system of equations (4) – (5) from [2] we use the expansion in terms of the variables r and V of the elements of the stochastic differential equation (6). In the derivation of equations (5) from [2] the weight coefficients ϕi were chosen in such a way that the necessary condition $\lim_{\tau \to 0} y(\tau, x) = r$ was satisfied. So if (r, V) are taken as state variables then $\phi_r = 1$, and $\phi_V = 0$. If (x, y) are used as state variables then $\phi_x = \alpha$, and $\phi_y = \beta$. Note that

$$K\theta = \begin{pmatrix} \alpha a + \beta d \\ \alpha^2 a + \beta^2 d \end{pmatrix}, \;\; K_r = \frac{1}{\alpha - \beta}\begin{pmatrix} e\alpha - b\beta \\ \alpha\beta(e - b) \end{pmatrix}, \;\; K_V = \frac{1}{\alpha - \beta}\begin{pmatrix} b - e \\ b\alpha - e\beta \end{pmatrix}.$$

$$\sigma(r, V)\sigma(r, V)^{\mathrm{T}} = \beta_0 + \beta_r r + \beta_V V,$$

$$\beta_0 = 0, \quad \beta_r = -\alpha\beta\begin{pmatrix} 0 & 1 \\ 1 & \alpha + \beta \end{pmatrix}, \;\; \beta_V = \begin{pmatrix} 1 & \alpha + \beta \\ \alpha + \beta & \alpha^2 + \alpha\beta + \beta^2 \end{pmatrix}.$$

$$\sigma(r, V)\lambda(r, V)^{\mathrm{T}} = \xi + \eta_r r + \eta_V V,$$

$$\xi = 0, \quad \eta_r = \frac{1}{\beta - \alpha}\begin{pmatrix} \beta\,\lambda_r - \alpha\lambda_V \\ \alpha\beta(\lambda_r - \lambda_V) \end{pmatrix}, \quad \eta_V = \frac{1}{\alpha - \beta}\begin{pmatrix} \lambda_r - \lambda_V \\ \alpha\lambda_r - \beta\lambda_V \end{pmatrix}.$$

Here it is assumed that the market risk prices are determined by the relation $\lambda(r, V) = (\lambda_r, \lambda_V)\sigma(r, V)$, where λ_r and λ_V are constants. Substituting these expansions into equations (4) – (5) from [2], we find the system of equations for the term structure functions $A(\tau)$, $B_r(\tau)$ и $B_V(\tau)$:

$$A'(\tau) = -(\alpha a + \beta d)\, B_r(\tau) - (\alpha^2 a + \beta^2 d) B_V(\tau), \qquad A(0) = 0, \tag{7}$$

$$B_r'(\tau) = 1 - B(\tau)^{\mathrm{T}}(\eta_r + K_r) - B(\tau)^{\mathrm{T}}\beta_r B(\tau)/2 = \tag{8}$$

$$= 1 - \frac{1}{\alpha-\beta}((e\alpha - b\beta - \beta\lambda_r + \alpha\lambda_V) B_r(\tau) + \alpha\beta(e - b - \lambda_r + \lambda_V) B_V(\tau)) +$$

$$+ \alpha\beta B_V(\tau)(B_r(\tau) + (\alpha+\beta) B_V(\tau)/2), \quad B_r(0) = 0$$

$$B_V'(\tau) = -B(\tau)^{\mathrm{T}}(\eta_V + K_V) - \frac{1}{2} B(\tau)^{\mathrm{T}}\beta_V B(\tau) = \tag{9}$$

$$= -\frac{1}{\alpha-\beta}((b - e + \lambda_r - \lambda_V) B_r(\tau) + (b\alpha - e\beta + \alpha\lambda_r - \beta\lambda_V) B_V(\tau)) -$$

$$-\frac{1}{2} B_r(\tau)^2 - (\alpha+\beta) B_r(\tau) B_V(\tau) - \frac{1}{2}(\alpha^2 + \alpha\beta + \beta^2) B_V(\tau)^2, \quad B_V(0) = 0.$$

Unfortunately, the obtained system of ordinary differential equations for (B_r, B_V) refers to systems of Riccati equations, is nonlinear and is not known some methods for obtaining its analytical solution. At the same time, the problem of determining the term structure of the model under consideration when using the original non-observed state variables (x, y) has an analytical solution. Actually, for the system (1), the following data are obtained to compose equations (4) – (5) from [2]

$$K = \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}, \quad \theta = \begin{pmatrix} a/b \\ d/e \end{pmatrix}. \ K\theta = \begin{pmatrix} a \\ d \end{pmatrix}.$$

$$\sigma(x, y)\sigma^{\mathrm{T}}(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} y,$$

$$\beta_0 = 0, \ \beta_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \beta_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we assume that the market risk prices are determined by the relation $\lambda(x, y) = (\lambda_x, \lambda_y)\sigma(x, y)$, where λ_x and λ_y are constants then

$$\sigma(x, y)\lambda^{\mathrm{T}}(x, y) = \begin{pmatrix} \lambda_x x \\ \lambda_y y \end{pmatrix} = \begin{pmatrix} \lambda_x \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \lambda_y \end{pmatrix} y,$$

$$\xi = 0, \ \eta_x = \begin{pmatrix} \lambda_x \\ 0 \end{pmatrix}, \quad \eta_y = \begin{pmatrix} 0 \\ \lambda_y \end{pmatrix}.$$

The system of differential equations (4) – (5) from [2] for these data has the form

$$A'(\tau) = -a\, B_x(\tau) - d\, B_y(\tau), \quad A(0) = 0, \tag{10}$$

$$B_x'(\tau) = \alpha - (\lambda_x + b) B_x(\tau) - B_x^2(\tau)/2, \quad B_x(0) = 0 \tag{11}$$

$$B_y'(\tau) = \beta - (\lambda_y + e) B_y(\tau) - B_y^2(\tau)/2, \quad B_y(0) = 0. \tag{12}$$

The last two equations no constitute now a system, but are independent scalar Riccati equations having an analytic solution

$$B_x(\tau) = \alpha \left(\frac{\varepsilon_x}{e^{\varepsilon_x \tau} - 1} + J_x \right)^{-1}, \ \varepsilon_x = \sqrt{(b + \lambda_x)^2 + 2\alpha}, \ J_x = (\varepsilon_x + \lambda_x + b)/2. \tag{13}$$

$$B_y(\tau) = \beta \left(\frac{\varepsilon_y}{e^{\varepsilon_y \tau} - 1} + J_y \right)^{-1}, \ \varepsilon_y = \sqrt{(e + \lambda_y)^2 + 2\beta}, \ J_y = (\varepsilon_y + \lambda_y + e)/2. \tag{14}$$

Substitution of these expressions into the equation for the function $A(\tau)$ leads to the following result:

$$A(\tau) = -a\alpha \frac{J_x \tau - \ln[1 + (e^{\varepsilon_x \tau} - 1) J_x / \varepsilon_x]}{J_x (J_x - \varepsilon_x)} -$$

$$-\,d\beta\,\frac{J_y\tau-\ln[1+(e^{\varepsilon_y\tau}-1)J_y/\varepsilon_y]}{J_y(J_y-\varepsilon_Y)}. \tag{15}$$

Finally, the yield curve is calculated analytically

$$y(\tau\,|\,x,y)=\frac{xB_x(\tau)+yB_y(\tau)-A(\tau)}{\tau}= \tag{16}$$

$$=\frac{x\alpha}{\tau}\left(\frac{\varepsilon_x}{e^{\varepsilon_x\tau}-1}+J_x\right)^{-1}+\frac{y\beta}{\tau}\left(\frac{\varepsilon_y}{e^{\varepsilon_y\tau}-1}+J_y\right)^{-1}+$$

$$+\,a\alpha\,\frac{J_x\tau-\ln[1+(e^{\varepsilon_x\tau}-1)J_x/\varepsilon_x]}{J_x(J_x-\varepsilon_x)\tau}+d\,\beta\,\frac{J_y\tau-\ln[1+(e^{\varepsilon_y\tau}-1)J_y/\varepsilon_y]}{J_y(J_y-\varepsilon_Y)\tau}.$$

To obtain a forward curve, one can use formula (6) from [2], which gives

$$f(\tau\,|\,x,y)=x\frac{dB_x(\tau)}{d\tau}+y\frac{dB_y(\tau)}{d\tau}-\frac{dA(\tau)}{d\tau}= \tag{17}$$

$$=\;x\alpha+y\beta+(a-xb-x\lambda_x)B_x(\tau)+$$

$$+\,(d-ye-y\lambda_y)B_y(\tau)-x\,B_x^2(\tau)/2-y\,B_y^2(\tau)/2,$$

where the functions $B_x(\tau)$ and $B_y(\tau)$ are explicitly calculated by formulas (13) and (14).

Note that

$$\lim_{\tau\to 0}y(\tau\,|\,x,y)=\lim_{\tau\to 0}f(\tau\,|\,x,y)=x\alpha+y\beta=\;r, \tag{18}$$

$$\lim_{\tau\to\infty}y(\tau\,|\,x,y)=\lim_{\tau\to\infty}f(\tau\,|\,x,y)=\frac{a\alpha}{J_x}+\frac{d\beta}{J_y}. \tag{19}$$

At the same time, no matter what the state variables describe the behavior of the process of interest rates in the model in question, the yield to maturity for a certain term to maturity must be the same for any means of describing the process of interest rates, if the transformation of state variables defines a one-to-one correspondence of variables. In our case, for state variables (x, y) and (r, V) this hold because the matrix of the linear transformation (5) is nondegenerate. Consequently, the following equality must hold

$$y(\tau\,|\,x,y)=y(\tau\,|\,r,V).$$

Therefore, in order to obtain an analytic expression for the yield to maturity $y(\tau\,|\,r,V)$ it is sufficient to express in formula (13) the variable states x and y in terms of r and V by formula (4). Then after simplification we get

$$y(\tau\,|\,r,V)=\frac{1}{\tau}\left(\frac{\alpha^2B_y(\tau)-\beta^2B_x(\tau)}{\alpha\beta(\alpha-\beta)}r-\frac{\alpha B_y(\tau)-\beta B_x(\tau)}{\alpha\beta(\alpha-\beta)}V-A(\tau)\right), \tag{20}$$

where the term structure functions $A(\tau)$, $B_x(\tau)$ and $B_y(\tau)$ are calculated from formulas (13) – (15). The expression of yield to maturity (20) is more preferred than formula (16), since it uses variables r and V that can be observed or estimated, while in (16) unobserved variables x and y are used, which not have a clear financial interpretation.

Similarly, to obtain an analytic expression for the forward curve $f(\tau\,|\,r,V)$ it is sufficient in formula (17) to express the states x and y in terms of r and V by formulas (4). This gives

$$f(\tau\,|\,r,V)=a\,B_x(\tau)+d\,B_y(\tau)+$$

$$+\left(1+\beta\frac{(b+\lambda_x)B_x(\tau)+B_x(\tau)^2/2}{\alpha(\alpha-\beta)}-\alpha\frac{(e+\lambda_y)B_y(\tau)+B_y(\tau)^2/2}{\beta(\alpha-\beta)}\right)r\,-$$

$$-\left(\frac{(b+\lambda_x)B_x(\tau)+B_x(\tau)^2/2}{\alpha(\alpha-\beta)}-\frac{(e+\lambda_y)B_y(\tau)+B_y(\tau)^2/2}{\beta(\alpha-\beta)}\right)V\,. \tag{21}$$

Thus, the yield curves in space (r, V) can be calculated by formulas (20) – (21) in an analytical form or numerically with the help of "analytically insolvable" nonlinear equations (7) – (9). The calculations show that both these methods lead to completely coincident curves. Therefore, the functions $B_r(\tau)$ and $B_V(\tau)$ that satisfy the system of equations (8) – (9) can be written in the following analytic form

$$B_r(\tau) = \frac{\alpha^2 B_y(\tau) - \beta^2 B_x(\tau)}{\alpha\beta(\alpha - \beta)}, \quad B_V(\tau) = \frac{\beta B_x(\tau) - \alpha B_y(\tau)}{\alpha\beta(\alpha - \beta)}, \tag{22}$$

where the functions $B_x(\tau)$ and $B_y(\tau)$ are calculated by formulas (13) – (15). The validity of this assertion is easily verified by substituting expressions (22) into equations (8) and (9).

Note that the analytical expressions for the yield curve to maturity for the model under consideration were found by F. Longstaff and E. Schwartz (1992) in a different, more cumbersome form.

In Fig. 1 the curves $y(\tau \mid r, V)$ and $f(\tau \mid r, V)$ are presented for a risk-neutral case ($\lambda_x = \lambda_y = 0$) with the following parameter values

$$r = 0{,}06,\ V = 0{,}03,\ a = 0{,}3,\ b = 4,\ d = 0{,}5,\ e = 1{,}7,\ \alpha = 0{,}3,\ \beta = 0{,}7.$$

To represent the curves "entirely" for the entire interval of possible values of the terms to maturity $\tau \in (0, \infty)$ the non-linear transformation of terms to maturity is used: : $u = 1 - e^{-\rho\tau}$, which maps the positive semiaxis $(0, \infty)$ in the unit interval (0, 1). The numerical value $\rho = \ln 10/30 = 0{,}07675$ adopted in the calculations, corresponds to the fact that the terms to maturity from 0 to 30 years are displayed in the interval (0, 0.9). The curves start from the point $y(0 \mid r, V) = f(0 \mid r, V) = r = 0{,}06$ and converge when $u \to 1$ to the same limit value $y(1 \mid r, V) = f(1 \mid r, V) = 0{,}2079$. The solid line shows the yield curve to maturity $y(u)$, and the dotted curve shows the forward rate curve $f(u)$.

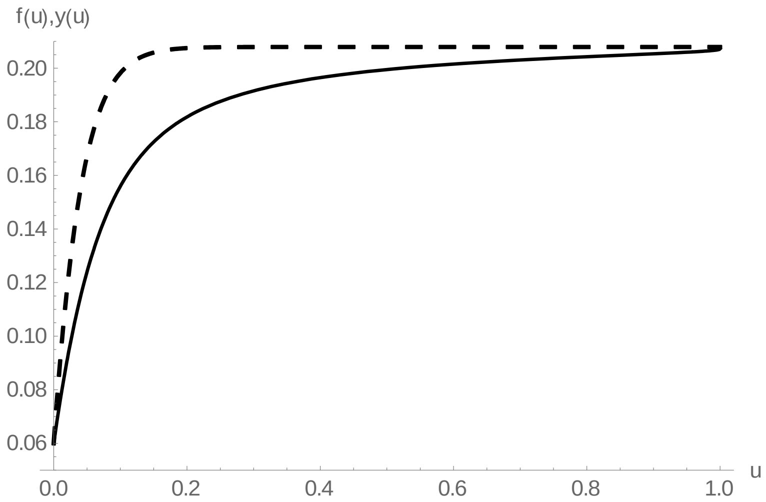

Fig. 1. The yield curve to maturity $y(u)$ and the forward rate curve $f(u)$.

2. Extension of the Longstaff – Schwartz model

The Longstaff – Schwartz model considered can be classified as a class of two-factor Cox – Ingersoll – Ross models. However, the obtained results allow us to extend this model to the case of n factors. Indeed, let $X = (x_1, x_2, ..., x_n)^{\mathrm{T}}$ denote the n-dimensional vector of some latent market state variables.

The equations of the dynamics of these state variables, extending for this case the Longstaff – Schwartz model (1), can be written in the form

$$dx_i = \kappa_i (a_i - x_i)dt + \sqrt{x_i}\, dW_i, \quad 1 \le i \le n. \tag{23}$$

Suppose that the latent variables X are connected by a nondegenerate linear transformation

$$Z = H X \tag{24}$$

with the vector of other variables of the state $Z = (z_1, z_2, ..., z_n)^{\mathrm{T}}$, whose components are observable and have a quite definite economic or statistical meaning. For example, z_1 is a short-term interest rate, z_2 is its instantaneous variance, etc. H is a nondegenerate matrix with its components h_{is}, $1 \le i, s \le n$. Then in accordance with the Ito stochastic analysis for the dynamics of the state variable vector Z, an equation analogous to equation (6) is obtained,

$$dZ = K(\theta - Z)dt + H\, D(\sqrt{H^{-1}Z})\, dW, \tag{25}$$

where $K = HD(\kappa)H^{-1}$, $\theta = Ha$. κ is a vector with components κ_i, a is a vector with components a_i, $D(\kappa)$ is a diagonal matrix on whose main diagonal the elements of the vector κ are located, $D(\sqrt{H^{-1}Z})$ is a diagonal matrix on whose main diagonal are the square roots of the elements of the vector $H^{-1}Z$.

Relative to the transformation matrix H, we make the following assumptions. It is convenient to identify the component z_1 of the state vector Z with the short-term interest rate r. So according to the representation (24) the first row of the matrix H must consist of elements ensuring the equality $z_1 \equiv r = \sum_{i=1}^{n} h_{1i}x_i$. In addition, the elements of the matrix H must be such that the necessary condition (18) for the yield curves $\lim_{\tau\to 0} y(\tau \mid X) = r$ is satisfied.

We denote the curve of yield to maturity in the case when the variable states are defined by the vector X as $y(\tau|X)$ and, respectively, the yield curve to maturity in the case when the state variables are defined by the vector Z as $y(\tau|Z)$. The nature of the dynamics of the variables of the state in both cases provides the fact that the yield curves will refer to the affine class, i.e. $\tau y(\tau|X) = A(\tau) + X^{\mathrm{T}}B(\tau)$ and $\tau y(\tau|Z) = a(\tau) + Z^{\mathrm{T}}b(\tau)$. Obviously, in whatever coordinates the current state of any particular process of the short-term interest rate is described, the yield curves for this process must coincide, i.e. $\tau y(\tau|X) = \tau y(\tau|Z)$ or $A(\tau) + X^{\mathrm{T}}B(\tau) = a(\tau) + Z^{\mathrm{T}}b(\tau)$.

Since these equalities must hold for any terms to maturity τ and any values of the state variables X and Z (and zero ones too), we have $a(\tau) = A(\tau)$, $X^{\mathrm{T}}B(\tau) = Z^{\mathrm{T}}b(\tau) = X^{\mathrm{T}}H^{\mathrm{T}}b(\tau)$. Thus, if the term structure functions $A(\tau)$ and $B(\tau)$ for the yield curve $y(\tau|X)$ are known, for the yield curve $y(\tau|Z)$ the time structure functions $a(\tau)$ and $b(\tau)$ are found from equalities

$$a(\tau) = A(\tau),\ \ b(\tau) = (H^{\mathrm{T}})^{-1}B(\tau). \tag{26}$$

To determine the differential equations with respect to the functions of the term structure $A(\tau)$ and $B(\tau)$ we again apply the approach described in [2]. For system (23) the following data are obtained for the composition of equations (4) – (5) from [2]

$$K = \begin{pmatrix} \kappa_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \kappa_n \end{pmatrix},\quad \theta = \begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix},\ K\theta = \begin{pmatrix} \kappa_1 a_1 \\ \ldots \\ \kappa_n a_n \end{pmatrix}.$$

$$\sigma(X)\sigma^T(X) = \begin{pmatrix} x_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & x_n \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 0 \end{pmatrix} x_1 + \ldots + \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix} x_n,$$

$$\beta_0 = 0,\ \ \beta_1 = \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 0 \end{pmatrix}, \ldots, \beta_n = \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix}.$$

If we assume that the market risk prices are determined by the relation $\lambda(X) = (\lambda_1\ \lambda_2\ \ldots\ \lambda_n)\ \sigma(X)$, where λ_1, λ_2, ..., λ_n are constants, then

$$\sigma(X)\lambda^{\mathrm{T}}(X) = \begin{pmatrix} \lambda_1 x_1 \\ \ldots \\ \lambda_n x_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \ldots \\ 0 \end{pmatrix} x_1 + \ldots + \begin{pmatrix} 0 \\ \ldots \\ \lambda_n \end{pmatrix} x_n,$$

$$\xi = 0,\ \ \eta_1 = \begin{pmatrix} \lambda_1 \\ \ldots \\ 0 \end{pmatrix},\ \ldots,\ \eta_n = \begin{pmatrix} 0 \\ \ldots \\ \lambda_n \end{pmatrix}.$$

Consequently, the system of differential equations (4) – (5) from [2] in this case has the form

$$A'(\tau) = -\sum_{i=1}^{n} a_i \kappa_i B_i(\tau),\ \ A(0) = 0, \tag{27}$$

$$B_i'(\tau) = h_{1i} - (\kappa_i + \lambda_i)B_i(\tau) - B_i^2(\tau)/2,\ B_i(0) = 0,\ 1 \le i \le n. \tag{28}$$

The equations obtained coincide up to the parameters with the equations (11) and (12) therefore their solutions have the same form as (13):

$$B_i(\tau) = h_{1i}\left(\frac{\varepsilon_i}{e^{\varepsilon_i\tau}-1}+J_i\right)^{-1},\ \varepsilon_i = \sqrt{(\kappa_i+\lambda_i)^2+2h_{1i}},\ J_i = (\varepsilon_i+\lambda_i+\kappa_i)/2. \tag{29}$$

Thus,

$$y(\tau|X) = \frac{1}{\tau}\sum_{i=1}^{n} h_{1i}\left(x_i\left(\frac{\varepsilon_i}{e^{\varepsilon_i\tau}-1}+J_i\right)^{-1}+a_i\kappa_i\frac{J_i\tau-\ln[1+(e^{\varepsilon_i\tau}-1)J_i/\varepsilon_i]}{J_i(J_i-\varepsilon_i)}\right). \tag{30}$$

The forward curve $f(\tau|X)$ is determined by the formula (17) by the following expression

$$f(\tau|X) = \sum_{i=1}^{n} x_i(\,h_{1i} - (\kappa_i+\lambda_i)B_i(\tau) - B_i^2(\tau)/2) + \sum_{i=1}^{n} a_i\kappa_i B_i(\tau), \tag{31}$$

where the functions $B_i(\tau)$ are calculated by formulas (29).

The curves $y(\tau|X)$ and $f(\tau|X)$ have the following limiting properties

$$\lim_{\tau\to 0} y(\tau\,|\,X) = \lim_{\tau\to 0} f(\tau\,|\,X) = \sum_{i=1}^{n} h_{1i}x_i = r, \tag{32}$$

$$\lim_{\tau\to\infty} y(\tau\,|\,X) = \lim_{\tau\to\infty} f(\tau\,|\,X) = \sum_{i=1}^{n}\frac{a_i\,\kappa_i\,h_{1i}}{J_i}. \tag{33}$$

Thus, after the yield curves $y(\tau|X)$ and $f(\tau|X)$ are determined in the space of variable X, the yield curves $y(\tau|Z)$ and $f(\tau|Z)$ Z) in the space of variables Z are determined with the help of relations (26) by formulas

$$y(\tau|Z) = Z^{\mathrm{T}}(H^{\mathrm{T}})^{-1}B(\tau)/\tau - A(\tau)/\tau, \tag{34}$$

$$f(\tau\,|\,Z) = Z^{\mathrm{T}}(H^{-1})^{\mathrm{T}}\frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}, \tag{35}$$

where the functions $A(\tau)$ and $B(\tau)$ are determined by relations (27) – (29).

Thus, if the observed (or estimated) market state variables Z have the dynamics described by equation (25), then the equation of the term structure of zero-coupon yield will have the form

$$-\frac{dA(\tau)}{d\,\tau} + Z^{\mathrm{T}}\frac{dB(\tau)}{d\,\tau} + (Z^{\mathrm{T}}-\theta^{\mathrm{T}})K^{\mathrm{T}}B(\tau) + \frac{1}{2}B(\tau)^{\mathrm{T}}H\,D(H^{-1}Z)\,H^{\mathrm{T}}B(\tau) -$$

$$-Z^{\mathrm{T}}H_1^{\mathrm{T}} = -\lambda^{\mathrm{T}}D(H^{-1}Z)B(\tau),$$

where $D(H^{-1}Z)$ is the diagonal matrix on the main diagonal of which the elements of the vector $H^{-1}Z$ are placed, H_1 is the first row of the matrix H. Assuming the components of the vector Z to be independent variables, we can obtain from this equality the following a system of equations for the term structure functions $a(\tau)$ and $b(\tau)$

$$a'(\tau) = -(K\theta)^{\mathrm{T}}b(\tau),\ \ a(0) = 0, \tag{36}$$

$$b_i'(\tau) = h_{1i} - \xi_i\,b(\tau) - b^{\mathrm{T}}(\tau)\,\zeta_i\,b(\tau)/2,\ \ b_i(0) = 0,\ \ 1 \le i \le n, \tag{37}$$

where $\xi_i = (K_i - D(G_i)\lambda)^{\mathrm{T}}$, K_i and G_i are the i-th columns of the matrices K and H^{-1}, respectively, $\zeta_i = HD(G_i)H^{\mathrm{T}}$, $D(G_i)$ is, as before, the diagonal matrix, on the main diagonal of which are the elements of the vector G_i. There are no direct methods for an analytic solution of the system of Riccati equations (37). However, as follows from the above analysis, the transition to state variables $X = H^{-1}Z$ allows us to analytically solve the problem of determining the yield curves $y(\tau|X)$ and $f(\tau|X)$ in the form (30) and (31), and then express the yield curves $y(\tau|X)$ and $f(\tau|X)$ in the analytical form by the formulas (34) – (35).

3. Numerical Example

In conclusion, let us consider a numerical example. For simplicity, we will solve the problem in a risk-neutral setting, when the market price of risk is $\lambda = 0$. Let the observation of processes in the financial market allow us to form the following system of stochastic differential equations of the type (6) for the description of the dynamics of the observed state variables $Z = (z_1, z_2, z_3)^{\mathrm{T}}$, for which $z_1 = r$ is the short-term interest rate,

$$dZ = \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 - z_1 \\ 2 - z_2 \\ 3 - z_3 \end{pmatrix} dt + \begin{pmatrix} \sqrt{z_1 - z_3} & \sqrt{z_1 - z_2} & \sqrt{z_3 + z_2 - z_1} \\ \sqrt{z_1 - z_3} & 0 & \sqrt{z_3 + z_2 - z_1} \\ 0 & \sqrt{z_1 - z_2} & \sqrt{z_3 + z_2 - z_1} \end{pmatrix} dW. \tag{38}$$

To composite the equations for the time structure functions $a(\tau)$ and $b(\tau)$ we write down the necessary elements of the system

$$K\theta = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; K_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \; K_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \; K_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix},$$

$$\sigma(Z)\,\sigma(Z)^{\mathrm{T}} = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_2 & z_3 + z_2 - z_1 \\ z_3 & z_3 + z_2 - z_1 & z_3 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \; \beta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; \beta_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Substituting these expressions into the system of equations (4) – (5) from [2], we obtain equations for $a(\tau)$ and $b(\tau)$ in the form

$$\begin{gathered} a'(\tau) = -8b_1(\tau) - 4b_2(\tau) - 5b_3(\tau), \quad a(0) = 0, \\ b_1'(\tau) = 1 - 4b_1(\tau) - 2b_2(\tau) - b_3(\tau) - 0{,}5b_1(\tau)^2 + b_2(\tau)b_3(\tau), \quad b_1(0) = 0. \\ b_2'(\tau) = b_1(\tau) - b_2(\tau) + b_3(\tau) - 0{,}5b_2(\tau)^2 - b_1(\tau)b_2(\tau) - b_2(\tau)b_3(\tau), \quad b_2(0) = 0. \\ b_3'(\tau) = 2b_1(\tau) + 2b_2(\tau) - b_3(\tau) + 0{,}5b_3(\tau)^2 - b_1(\tau)b_3(\tau) - b_2(\tau)b_3(\tau), \quad b_3(0) = 0. \end{gathered} \tag{39}$$

The system of differential equations for the components of the vector function $b(\tau)$ is nonlinear and methods for obtaining its solution in analytical form are not known, although its numerical solution can be found simply using, for example, the Wolfram Mathematica program.

To obtain an analytical solution following the discussed above idea of describing the dynamics of market processes in another space of state variables, consider the following linear transformation

$$Z = H\,X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 \\ x_2 + x_3 \end{pmatrix},$$

then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 - z_3 \\ z_1 - z_2 \\ z_3 + z_2 - z_1 \end{pmatrix}. \tag{40}$$

We do not determine the economic meaning of the variables X, but use this transformation only to obtain analytical solutions for yield curves, in the final expressions for yield curves in which the variables X themselves are not used. With such a transformation, according to the Ito stochastic analysis, the drift function in equation (38) for the state variables X becomes

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 - z_1 \\ 2 - z_2 \\ 3 - z_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - x_1 \\ 2 - x_2 \\ 1 - x_3 \end{pmatrix}.$$

Note that the volatility matrix in equation (38) can be written as

$$\begin{pmatrix} \sqrt{z_1 - z_3} & \sqrt{z_1 - z_2} & \sqrt{z_3 + z_2 - z_1} \\ \sqrt{z_1 - z_3} & 0 & \sqrt{z_3 + z_2 - z_1} \\ 0 & \sqrt{z_1 - z_2} & \sqrt{z_3 + z_2 - z_1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{z_1 - z_3} & 0 & 0 \\ 0 & \sqrt{z_1 - z_2} & 0 \\ 0 & 0 & \sqrt{z_3 + z_2 - z_1} \end{pmatrix}.$$

Hence, taking into account the relations (40) in the transformation of the state variables to X, the volatility matrix is transformed to the diagonal form

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{z_1-z_3} & \sqrt{z_1-z_2} & \sqrt{z_3+z_2-z_1} \\ \sqrt{z_1-z_3} & 0 & \sqrt{z_3+z_2-z_1} \\ 0 & \sqrt{z_1-z_2} & \sqrt{z_3+z_2-z_1} \end{pmatrix} = \begin{pmatrix} \sqrt{x_1} & 0 & 0 \\ 0 & \sqrt{x_2} & 0 \\ 0 & 0 & \sqrt{x_3} \end{pmatrix}.$$

Consequently, in the state space X the equation for the dynamics of market processes (38) takes the form of the three-factor Longstaff – Schwartz model (1)

$$dX = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-x_1 \\ 2-x_2 \\ 1-x_3 \end{pmatrix} dt + \begin{pmatrix} \sqrt{x_1}\,dW_1 \\ \sqrt{x_2}\,dW_2 \\ \sqrt{x_3}\,dW_3 \end{pmatrix}. \tag{41}$$

This means that system (38) turns into a set of independent equations.

$$dx_1 = 3(1 - x_1)dt + \sqrt{x_1}\, dW_1,$$
$$dx_2 = 2(2 - x_2)dt + \sqrt{x_2}\, dW_2,$$
$$dx_3 = (1 - x_3)dt + \sqrt{x_3}\, dW_3.$$

Taking into account the fact that $z_1 = x_1 + x_2 + x_3 = r$, and also using the properties of system (41) for constructing equations (4) – (5) from [2] for time structure functions $A(\tau)$ and $B(\tau) = (B_1(\tau)\ \ B_2(\tau)\ \ B_3(\tau))^{\mathrm{T}}$ in the space of state variables X, we also obtain a set of independent equations for functions $B(\tau)$ of type (11) – (12)

$$A'(\tau) = -\,3B_1(\tau) - 4B_2(\tau) - B_3(\tau), \quad A(0) = 0$$
$$B_1'(\tau) = 1 - 3B_1(\tau) - B_1(\tau)^2/2, \quad B_1(0) = 0.$$
$$B_2'(\tau) = 1 - 2B_2(\tau) - B_2(\tau)^2/2, \quad B_2(0) = 0.$$
$$B_3'(\tau) = 1 - B_3(\tau) - B_3(\tau)^2/2, \quad B_3(0) = 0.$$

which have analytical solutions of the type (13) – (14):

$$B_1(\tau) = \left(\frac{\sqrt{11}}{e^{\tau\sqrt{11}}-1} + \frac{3+\sqrt{11}}{2}\right)^{-1}, \quad B_2(\tau) = \left(\frac{\sqrt{6}}{e^{\tau\sqrt{6}}-1} + \frac{2+\sqrt{6}}{2}\right)^{-1}, \quad B_3(\tau) = \left(\frac{\sqrt{3}}{e^{\tau\sqrt{3}}-1} + \frac{1+\sqrt{3}}{2}\right)^{-1}. \tag{41}$$

Finally, on the basis of equalities (26) we obtain the following expressions for $a'(\tau)$, $b_1(\tau)$, $b_2(\tau)$ and $b_3(\tau)$, which are analytic solutions of the nonlinear equations (36) – (37)

$$a'(\tau) = -\,3B_1(\tau) - 4B_2(\tau) - B_3(\tau),$$
$$b_1(\tau) = B_1(\tau) + B_2(\tau) - B_3(\tau),$$
$$b_2(\tau) = -\,B_2(\tau) + B_3(\tau),$$
$$b_3(\tau) = -\,B_1(\tau) + B_3(\tau),$$

where $B_1(\tau)$, $B_2(\tau)$ and $B_3(\tau)$ are calculated by the formulas (41). The validity of these solutions is verified simply by substituting found in this way $b_1(\tau)$, $b_2(\tau)$ and $b_3(\tau)$ in equations (39). After determining the functions of the term structure, formulas (34) and (35) can be used to construct yield curves to maturity $y(\tau|Z)$ and forward curves $f(\tau|Z)$. Analytic expressions $y(\tau|Z)$ and $f(\tau|Z)$ are not written out here because of their cumbersomeness. In Fig. 2 they are presented in the form of graphs for the starting values of the components of the vector Z: $z_1 = r = 10$ (in %), $z_2 = 0$, $z_3 = 0$. For the graphs in Fig. 2, the variable of term to maturity is transformed as well as in Fig. 1, only the value of the parameter ρ here is 0.2303.

Заметим, что согласно свойствам (32) – (33) кривые $Y(u|Z)$ и $F(u|Z)$ стартуют из общей точки $Y(0|Z) = F(0|Z) = r = 10$ и при $u \to 1$ стремятся к общему значению

Note that, according to the properties (32) – (33), the curves $Y(u|Z)$ and $F(u|Z)$ start from the common point $Y(0|Z) = F(0|Z) = r = 10$ and for $u \to 1$ converge to a common value

$$Y(1|Z) = F(1|Z) = \frac{3}{J_1} + \frac{4}{J_2} + \frac{1}{J_3} = \frac{6}{3+\sqrt{11}} + \frac{8}{2+\sqrt{6}} + \frac{2}{1+\sqrt{3}} = 3{,}4799.$$

The forward curve $F(u|Z)$ changes faster in the vicinity of the starting point than the curve of yield to maturity $Y(u|Z)$ (in theory, its derivative is twice as large as the derivative of the curve of yield to maturity). The curves have minimums, and the curve of yield to maturity $Y(u|Z)$ has a minimum at the point of intersection with the curve $F(u|Z)$, as it corresponds to the theory.

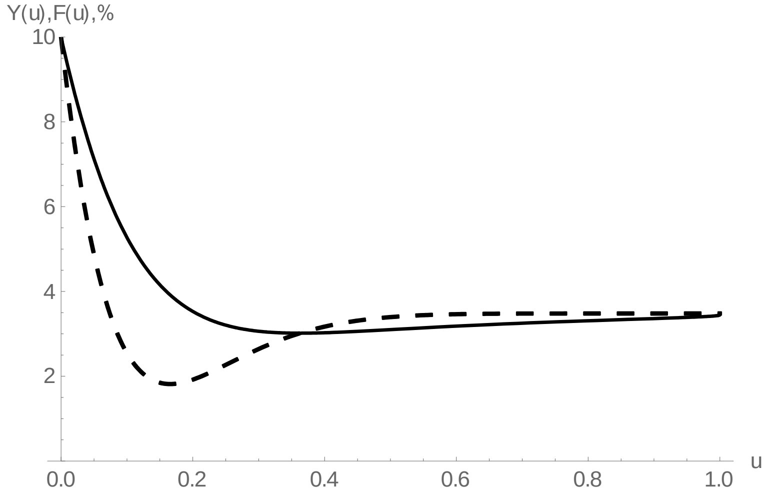

Fig.2. The yield curve to maturity $Y(u)$ and
the forward rate curve $F(u)$ for the case,
when the dynamics of state variables
is determined by equation (38).

Conclusion

In the paper on base of the analysis of the Longstaff – Schwartz model, its extended version is proposed for an arbitrary number of state variables that can be used to obtain an analytical solution of the equations with respect to the functions of the term structure of interest rates, which allows obtaining analytical expressions for curves of yield to maturity and forward interest rate curves. At the same time, an obvious principle is used: if the process of a short-term interest rate is specified, then the expressions for yield curves corresponding to this process do not depend on how the space of state variables of the financial market is described. It is shown that the application of this principle to the extension of the Longstaff – Schwartz model allows one to obtain analytical solutions of a system of nonlinear Riccati equations of arbitrary order, whose solution by analytical methods are not described in the literature. The proposed extension of the Longstaff – Schwartz model is free from the weakness inherent in this model, which consists in the fact that the transition from latent variables to real observable variables limits the range of possible values of real state variables. In the expansion there is no such effect.

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