

Maximum Likelihood

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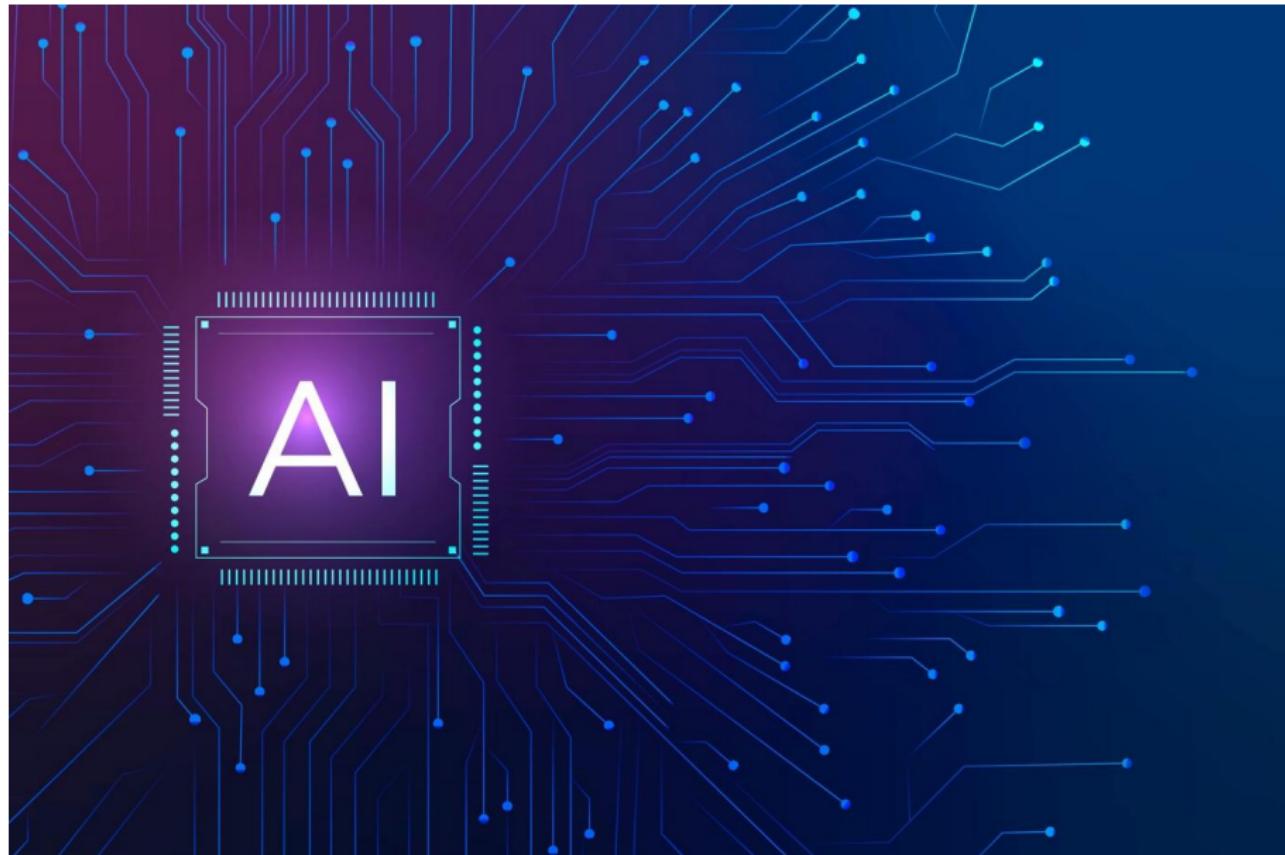
Scientific Reasoning



Deductive Reasoning



Inductive Reasoning



Where is Econometrics Going?

Covered topics

- Linear models
- OLS
- Endogeneity
- 2SLS, GMM

We will continue...



In the Era of AI

- Is econometrics a hard subject?
- Many steps of data analysis
- Automating routine tasks
 - Coding and data cleaning
 - Factual math derivation
- Domain knowledge: Interpretation
- Understanding the big picture
- What if AI suggests something completely new to you?
- Welcome to the beautiful new era

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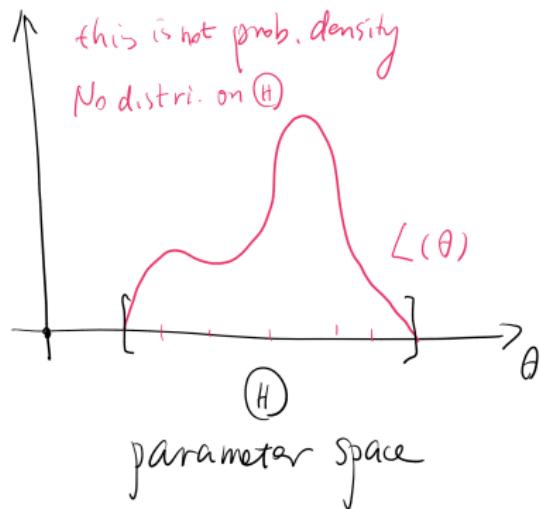
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Task

- Purpose: predict y with X
- Beyond continuous random variables
 - Binary
 - Multi-responses
 - Integer
 - Mixed type: censoring, truncation
 - Self-selection

Likelihood Approach

- Philosophy: The most likely outcome (Abductive reasoning).
- Distributional assumption
- Information theory



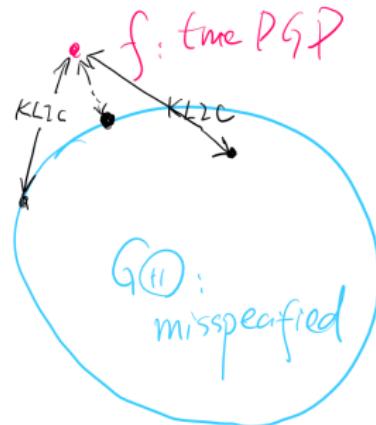
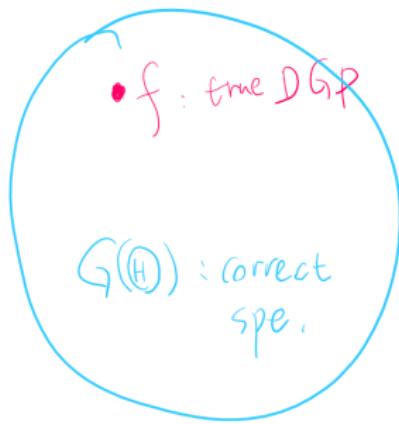
Principles

- Hero: Ronald Fisher (1890–1962)
- General framework
- Special cases of MLE
 - OLS
 - LIML
- Numerical optimization



Model Specification

- Nature: draws data Y from a parametric model f
- Human: specifies a family of models $g(y; \theta)$ and a parameter space Θ , which span a **model space**
 $G(\Theta) = \{g(y; \theta) : \theta \in \Theta\}.$



Model and Specification

Parametric model. The distribution of the data $\mathbf{Y} = (Y_1, \dots, Y_N)$ is known up to a finite dimensional parameter.

- **Semiparametric model:** If we know $Y \sim (\mu, \sigma^2)$, we can estimate μ, σ^2 by method of moments.
- **Parametric model:** If we assume $Y \sim N(\mu, \sigma^2)$, the model has only two parameters μ and σ^2 .

Likelihood Function

- For simplicity, let $\mathbf{Y} = (Y_1, \dots, Y_N)$ be i.i.d.
- The **likelihood** of the sample under a hypothesized value of $\theta \in \Theta$ is

$$L(\theta; \mathbf{Y}) = f(\mathbf{Y}; \theta) = \prod_{i=1}^N f(Y_i; \theta)$$

- Two perspectives:
 - (Probabilist) $f(\mathbf{Y}; \theta)$ is a function of \mathbf{Y} given the parameter θ
 - (Statistician; reverse engineering) $L(\theta; \mathbf{Y})$ is a function of θ given the data \mathbf{Y}

Section 1

Correct Specification

Log-likelihood

- log-likelihood

$$\ell_N(\theta) = \log L(\theta; \mathbf{Y}) = \sum_{i=1}^N \log f(Y_i; \theta)$$

is easier to compute.

- $\log(\cdot)$ is a monotonically increasing function
- The MLE estimator

$$\hat{\theta} = \arg \max_{\theta \in \Theta} \ell_N(\theta)$$

Why Maximization: Deep Justification

Theorem

If the model is correctly specified, then θ_0 is the maximizer.

- Kullback-Leibler information criterion (KLIC):

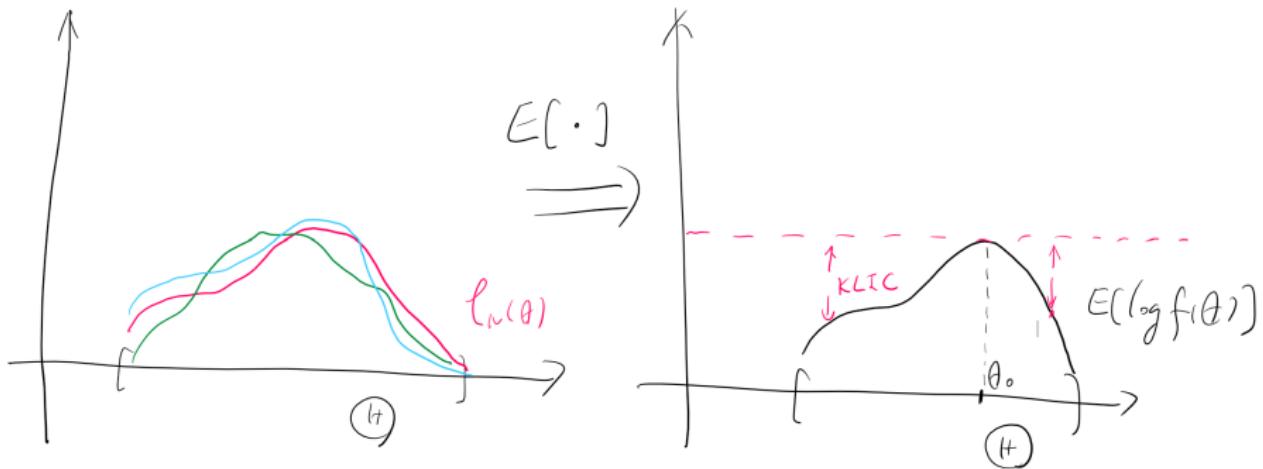
$$KLIC(f, g) = \int f(z) \log \frac{f(z)}{g(z)} dz$$

- $KLIC \geq 0$ because

$$\begin{aligned} & E[\log f(Y; \theta_0)] - E[\log f(Y; \theta)] \\ &= E[\log(f(Y; \theta_0)/f(Y; \theta))] \\ &= -E[\log(f(Y; \theta)/f(Y; \theta_0))] \\ &\geq -\log E[f(Y; \theta)/f(Y; \theta_0)] = 0 \end{aligned}$$

by the Jensen's inequality.

KLIC



- Numerical demon: Normal MLE

<https://www.kaggle.com/code/frankshi0/normal-mle>

Score and Hessian

- Score $s_N(\theta) = \sum_{i=1}^N \frac{\partial}{\partial \theta} \log f(Y_i; \theta)$ is a function of θ
- Efficient score $s_{i0} = \frac{\partial}{\partial \theta} \log f(Y_i; \theta_0)$ is evaluated at the true value θ_0

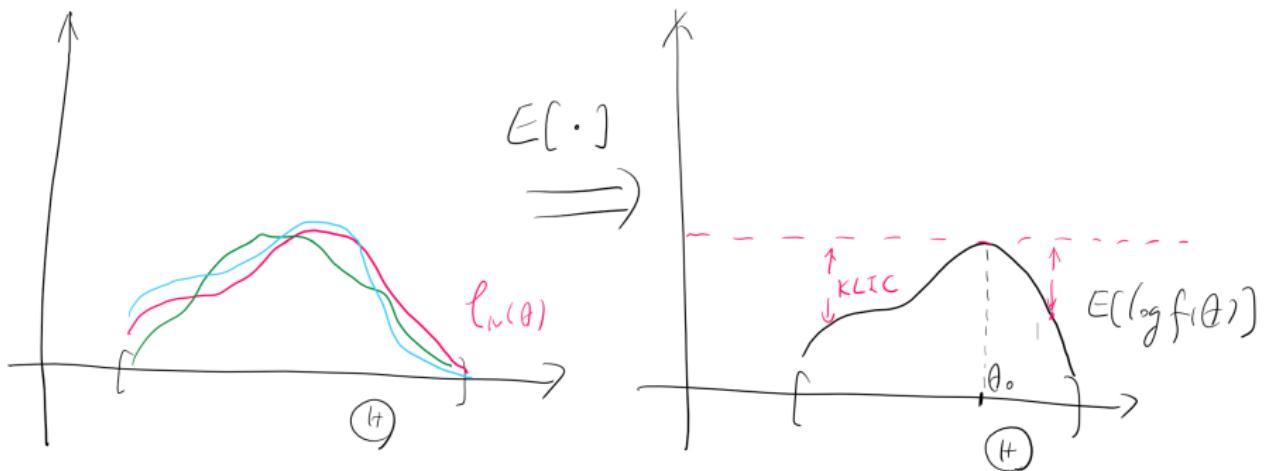
Theorem

If the model is correctly specified, the support of Y does not depend on θ , and θ_0 is in the interior of Θ , then $E[s_{i0}] = 0$.

MLE is equivalent to looking for roots of $s_N(\theta) = 0$.

- Hessian: $H_N(\theta) = - \sum_{i=1}^N \frac{\partial^2}{\partial \theta \partial \theta'} \log f(Y_i; \theta)$
- Expected Hessian: $H_0 = -E \left[\frac{\partial^2}{\partial \theta \partial \theta'} \log f(Y; \theta_0) \right]$

Score and Hessian: Illustration



Information Equality

- Fisher Information Matrix: $I_0 = E[s_{i0}s'_{i0}]$

Theorem

If the model is correctly specified, the support of Y does not depend on θ , and θ_0 is in the interior of Θ , then

$$I_0 = H_0.$$

- Information equality fails when the model is misspecified

Cramér-Rao Lower Bound

Theorem

Suppose the model is correctly specified, the support of Y does not depend on θ , and θ_0 is in the interior of Θ . If $\tilde{\theta}$ is unbiased estimator, then

$$\text{var}(\tilde{\theta}) \geq (NI_0)^{-1}.$$

- More general than OLS's "BLUE"
- A lower bound for variance of unbiased estimator
- When reached, an estimator is called **Cramér-Rao efficient**.

Example: Normal MLE

- Normal distribution $Y_i \sim N(\mu, \sigma^2)$ gives density

$$f(Y_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\gamma}} \exp\left(-\frac{(Y_i - \mu)^2}{2\gamma}\right)$$

where $\gamma = \sigma^2$ to simplify notations and derivatives.

- The log-likelihood is

$$\ell_N(Y; \theta) = -\frac{N}{2} \log \gamma - \frac{N}{2} \log 2\pi - \frac{1}{2\gamma} \sum_{i=1}^N (Y_i - \mu)^2$$

where $\theta = (\mu, \gamma)$.

MLE Estimator

Set the score equal to zero

$$s_N(\theta) = \begin{bmatrix} \frac{1}{\gamma} \sum_{i=1}^N (Y_i - \mu) \\ -\frac{N}{2\gamma} + \frac{1}{2\gamma^2} \sum_{i=1}^N (Y_i - \mu)^2 \end{bmatrix} = 0$$

and we solve the MLE estimator

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^N Y_i = \bar{Y}$$

$$\hat{\gamma} = \frac{1}{N} \sum_{i=1}^N (Y_i - \hat{\mu})^2 = \frac{1}{N} \sum_{i=1}^N (Y_i - \bar{Y})^2$$

Lower Bound

- The Hessian

$$H_N(\theta) = \begin{bmatrix} \frac{N}{\gamma} & \frac{1}{\gamma^2} \sum_{i=1}^N (Y_i - \mu) \\ \star & -\frac{N}{2\gamma^2} + \frac{1}{\gamma^3} \sum_{i=1}^N (Y_i - \mu)^2 \end{bmatrix}$$

- Expectation: $E[H_N(\theta_0)] = \begin{bmatrix} \frac{N}{\gamma} & 0 \\ 0 & \frac{N}{2\gamma^2} \end{bmatrix} = N \times H_0$

- Take inverse: $\begin{bmatrix} \frac{\gamma}{N} & 0 \\ 0 & \frac{2\gamma^2}{N} \end{bmatrix}$

- Information equality can be verified.

MLE for the Mean

- The sample mean

$$\text{var} \left(\frac{1}{N} \sum_{i=1}^N Y_i \right) = \frac{\sigma^2}{N}$$

reaches the Cramér-Rao lower bound

MLE for the Variance

- $E(S_N^2) = \sigma^2$ is unbiased, where

$$s_N^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2 = \frac{1}{N-1} Y' \left(I - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N' \right) Y$$

- It follows $s_N^2 = \frac{\sigma^2}{N-1} \cdot \chi^2(N-1)$ because

$$(N-1) \frac{s_N^2}{\sigma^2} = \left(\frac{Y}{\sigma} \right)' \left(I - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N' \right) \left(\frac{Y}{\sigma} \right) \sim \chi^2(N-1).$$

- As a result,

$$\text{var}(s_N^2) = \frac{\sigma^4}{(N-1)^2} \cdot 2(N-1) = \frac{2\sigma^4}{N-1} > \frac{2\sigma^4}{N}$$

Is not Cramér-Rao efficient

Return to Normal Regression

- The normal regression model is

$$Y_i = X'_i \beta + \varepsilon_i$$

- Under the assumption $\varepsilon_i | X_i \sim N(0, \gamma)$, the conditional distribution is

$$Y_i | X_i \sim N(X'_i \beta, \gamma).$$

- Parameter $\theta = (\beta, \gamma)$
- The joint likelihood

$$f(Y_i, X_i) = f(Y_i | X_i) f(X_i),$$

where the specification of $f(X_i)$ is irrelevant to θ .

Conditional Log-Likelihood

- The **conditional log-likelihood** of the sample is

$$\ell_N(\theta) = -\frac{N}{2} \log \gamma - \frac{N}{2} \log 2\pi - \frac{1}{2\gamma} \sum_{i=1}^N (Y_i - X'_i \beta)^2,$$

where the distribution of X_i is unspecified.

- The MLE estimator

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N \hat{\varepsilon}_i^2$$

where $\hat{\varepsilon}_i = Y_i - X'_i \hat{\beta}$.

Asymptotic Normality

- Under regularity conditions, $\hat{\theta} \xrightarrow{p} \theta_0$, and

$$\sqrt{N} (\hat{\theta} - \theta_0) \xrightarrow{d} N \left(0, H_0^{-1} I_0 H_0^{-1} \right)$$

- When the information equality holds, we have

$$\sqrt{N} (\hat{\theta} - \theta_0) \xrightarrow{d} N \left(0, I_0^{-1} \right),$$

or equivalently

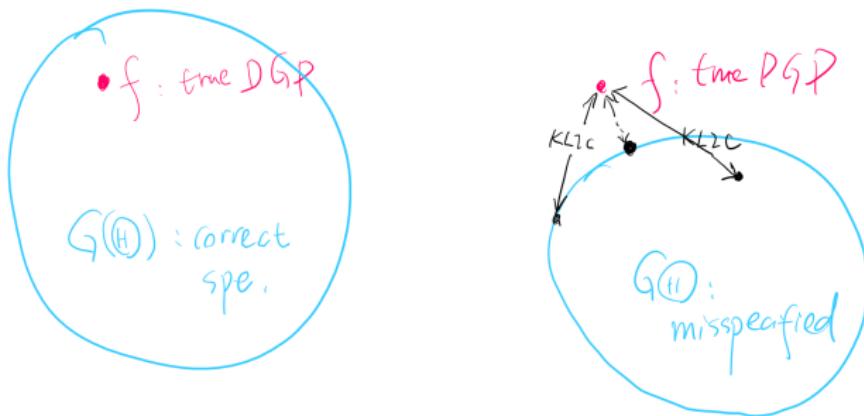
$$\hat{\theta} - \theta_0 \xrightarrow{a} N \left(0, \frac{I_0^{-1}}{N} \right),$$

- The variance $(NI_0)^{-1}$ is efficient!

Section 2

Mispecification

KLIC for Misspecified Models



- If $f \notin G(\Theta)$, the model is misspecified.

$$\begin{aligned} KLIC(f, g(y; \theta)) &= \int f(y) \log f(y) dy - \int f(y) \log g(y; \theta) dy \\ &= E[\log f(y)] - E[\log g(y; \theta)] > 0 \end{aligned}$$

Misspecified Model

- Misspecified: $\min_{\theta \in \Theta} KLIC(f, g(y; \theta)) > 0$
- MLE is still meaningful
- Pseudo-true parameter:

$$\theta^* = \arg \max_{\theta \in \Theta} E[\ell(\theta)]$$

the minimizer of $KLIC(f, g(y; \theta))$ in the parameter space Θ

- Under standard assumption, the MLE estimator $\hat{\theta} \xrightarrow{p} \theta^*$ and

$$\sqrt{N} (\hat{\theta} - \theta^*) \xrightarrow{d} N(0, H_*^{-1} I_* H_*^{-1})$$

Three Tests

Testing a hypothesis: $H_0 : R\theta_0 = q$ vs. $H_1 : R\theta_0 \neq q$.

- Wald test: unconstrained estimation, $\ell_N(\hat{\theta})$
- Lagrange multiplier (LM) test: constrained estimation under the null, $\ell_N(\tilde{\theta})$
- Likelihood ratio (LR) test: $\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})$
- All three are asymptotically equivalent
- Also work for nonlinear hypothesis
- Trade-off: computational convenience vs. finite sample performance

Wald Test

- Null hypothesis: $H_0 : R\theta_0 = q$, where R is $r \times k$ with rank r
- Uses the **unconstrained MLE** $\hat{\theta}$
- Test statistic:

$$W = N(R\hat{\theta} - q)' \left[R\hat{I}_0^{-1}R' \right]^{-1} (R\hat{\theta} - q)$$

where $\hat{I}_0 = \frac{1}{N} \sum_{i=1}^N \hat{s}_i \hat{s}_i'$ is the estimated information matrix

- Under H_0 :

$$W \xrightarrow{d} \chi^2(r)$$

- Reject H_0 if $W > \chi_r^2(\alpha)$, the $(1 - \alpha)$ quantile

Wald Test: Intuition

- Measures how far the unrestricted estimate $\hat{\theta}$ is from satisfying the constraint
- If H_0 is true, $R\hat{\theta}$ should be close to q
- The distance $(R\hat{\theta} - q)$ is standardized by its asymptotic variance
- **Advantages:**
 - Only requires unconstrained estimation
 - Easy to compute
 - Most commonly used in practice
- **Disadvantages:**
 - Not invariant to reparameterization
 - May have poor finite sample properties

Lagrange Multiplier (LM) Test

- Also called the **Score test**
- Uses the **constrained MLE** $\tilde{\theta}$ under $H_0 : R\theta = q$
- Test statistic:

$$LM = \frac{1}{N} s_N(\tilde{\theta})' \tilde{I}_0^{-1} s_N(\tilde{\theta})$$

where $s_N(\tilde{\theta}) = \sum_{i=1}^N \frac{\partial}{\partial \theta} \log f(Y_i; \tilde{\theta})$ is the score at $\tilde{\theta}$

- Under H_0 :

$$LM \xrightarrow{d} \chi^2(r)$$

- Reject H_0 if $LM > \chi_r^2(\alpha)$

LM Test: Intuition

- If H_0 is true, the score at $\tilde{\theta}$ should be close to zero
- If the constraint is violated, imposing it will push us away from the true maximum, making $s_N(\tilde{\theta})$ large
- **Advantages:**
 - Only requires constrained estimation
 - Useful when unconstrained estimation is difficult
 - Common in misspecification tests
- **Disadvantages:**
 - Need to solve constrained optimization
 - Less commonly used than Wald test

Likelihood Ratio (LR) Test

- Uses both $\hat{\theta}$ (unconstrained) and $\tilde{\theta}$ (constrained)
- Test statistic:

$$LR = 2[\ell_N(\hat{\theta}) - \ell_N(\tilde{\theta})]$$

- Under H_0 :

$$LR \xrightarrow{d} \chi^2(r)$$

- Reject H_0 if $LR > \chi^2_r(\alpha)$
- Measures the improvement in log-likelihood from relaxing the constraint

LR Test: Intuition

- If H_0 is true, imposing the constraint should not significantly reduce the likelihood
- If H_0 is false, the constrained model fits much worse
- **Advantages:**
 - Invariant to reparameterization
 - Often has better finite sample properties
 - Natural interpretation as likelihood comparison
- **Disadvantages:**
 - Requires both constrained and unconstrained estimation
 - More computationally demanding

Comparison of Three Tests

	Wald	LM	LR
Unconstrained $\hat{\theta}$	✓		✓
Constrained $\tilde{\theta}$		✓	✓
Computational cost	Low	Low	High
Invariance	No	Yes	Yes

- Asymptotically equivalent: $W \approx LM \approx LR$ under H_0
- In finite samples: typically $W \geq LR \geq LM$
- Choice depends on computational convenience and context

Summary

- Parametric models
- Specification of distribution family
- MLE
- Score, Hessian, information matrix
- Misspecification