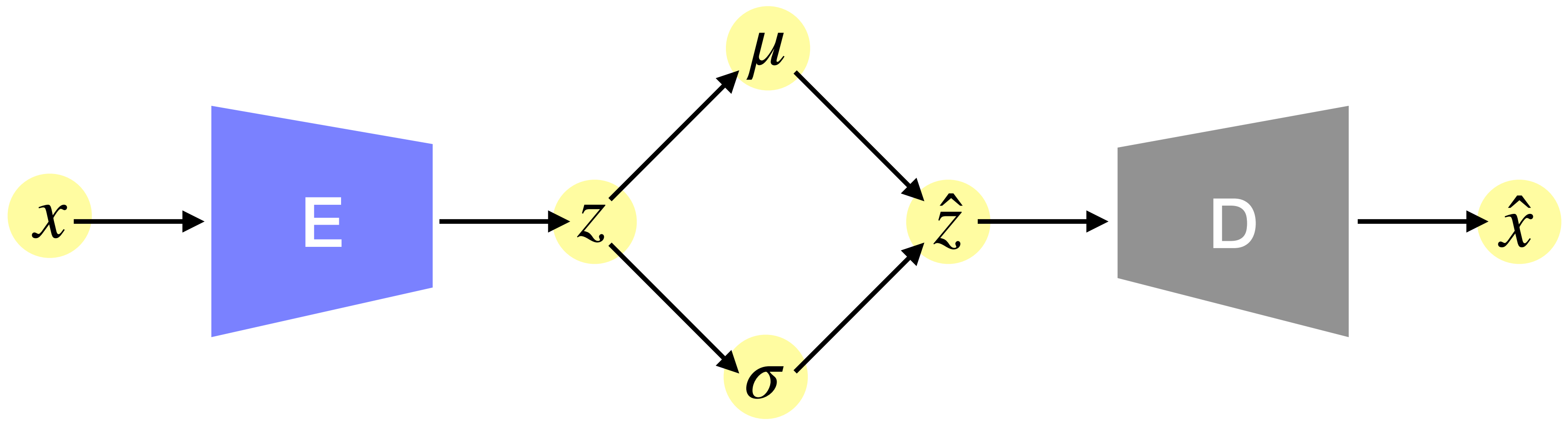
ManifoldGLOW: Extending Flow-based Generative Models to Manifolds

Background

- Most deep learning models work only on Euclidean space
- Generative models are gaining attention
- VAEs and GANs have their benefits as well as limitations

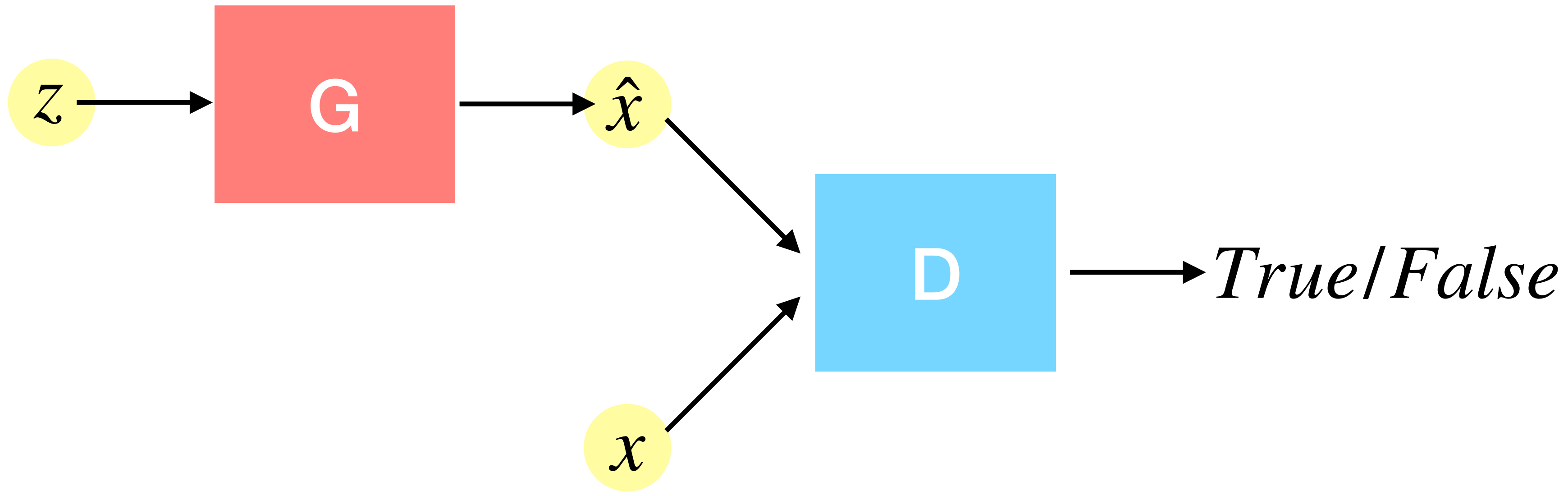
Generative models

- Variational autoencoder (VAE)



  - $l_i(\theta, \phi) = -\mathbf{E}_{z\sim q_\theta(z|x_i)}[logp_\phi(x_i|z)] + \mathbf{KL}(q_\theta(z|x_i)||p(z))$

- Generative adversarial network (GAN)



  - $\min_{G} \max_{D} V(D, G) = \mathbf{E}_{x\sim p_{data}(x)}[logD(x)] + \mathbf{E}_{z\sim p_z(z)}[log(1 - D(G(z)))]$

- Normalizing flows

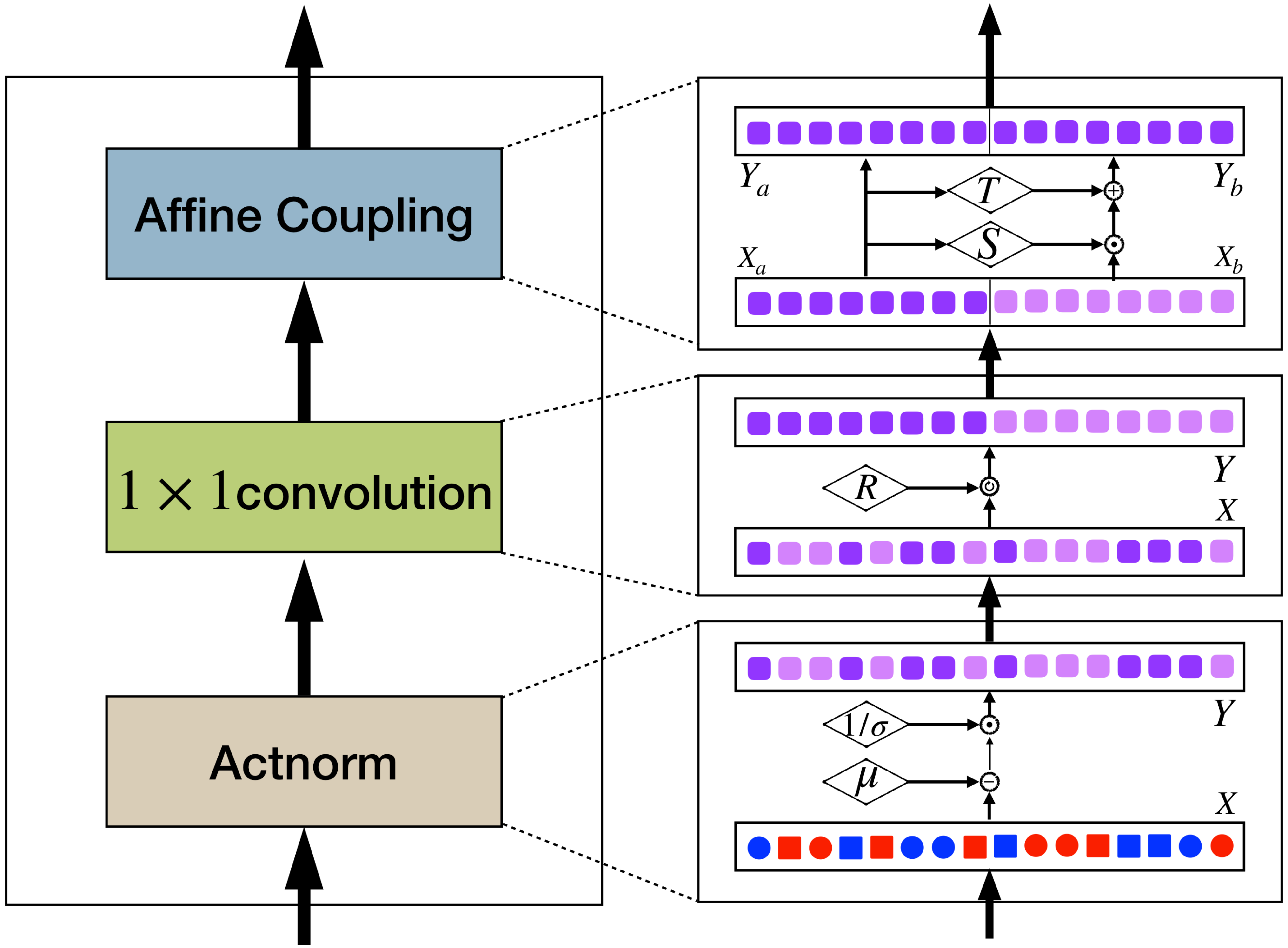
  - $\min l(\mathscr{D}) = \frac{1}{N}\sum_{i=1}^{N} - logp_\theta(x^{(i)}), z \sim p_\theta(z), x = g_\theta(z), z = g_\theta^{-1}(x)$

Normalizing flows

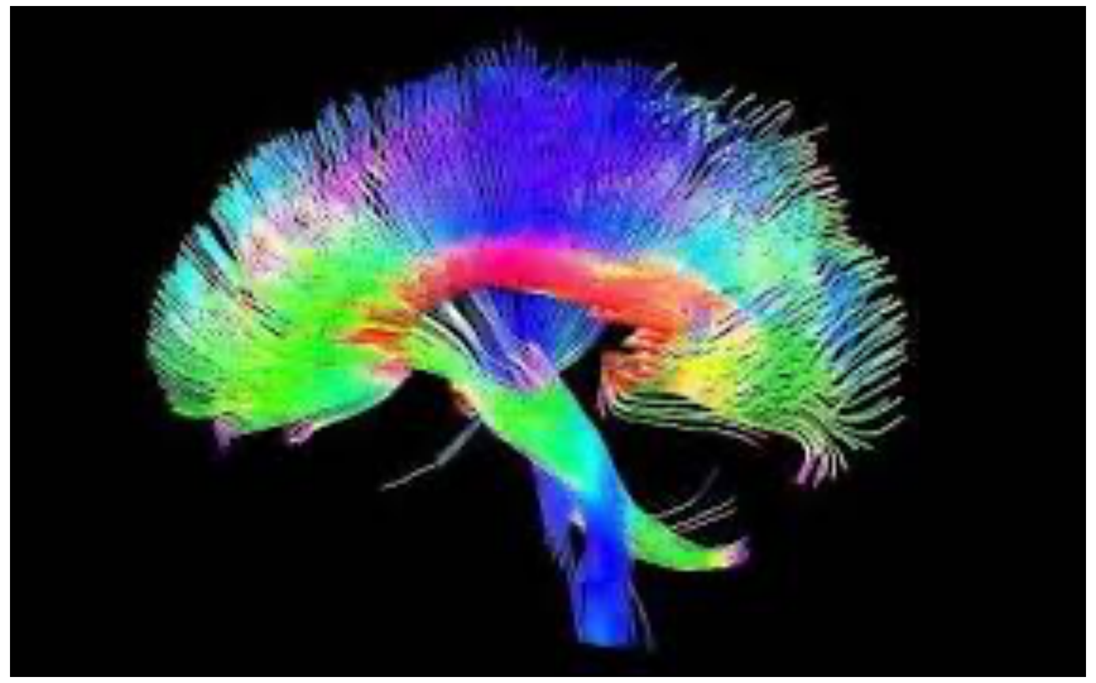
- $\mathbf{x} \leftrightarrow \mathbf{h}_1 \leftrightarrow \mathbf{h}_2 \ldots \leftrightarrow \mathbf{z}$
- $\log p_\theta(\mathbf{x}) = \log p(\mathbf{z}) + \log|\det(d\mathbf{z}/d\mathbf{x})| = \log p_\theta(\mathbf{z}) + \sum_{j=0}^{K} \log|\det(d\mathbf{h}_j/d\mathbf{h}_{j-1})|$
- If the Jacobian $d\mathbf{h}_j/d\mathbf{h}_{j-1}$ is a triangular matrix, $\log|\det(d\mathbf{h}_j/d\mathbf{h}_{j-1})| = \mathrm{sum}(\log|\mathrm{diag}(d\mathbf{h}_j/d\mathbf{h}_{j-1})|)$

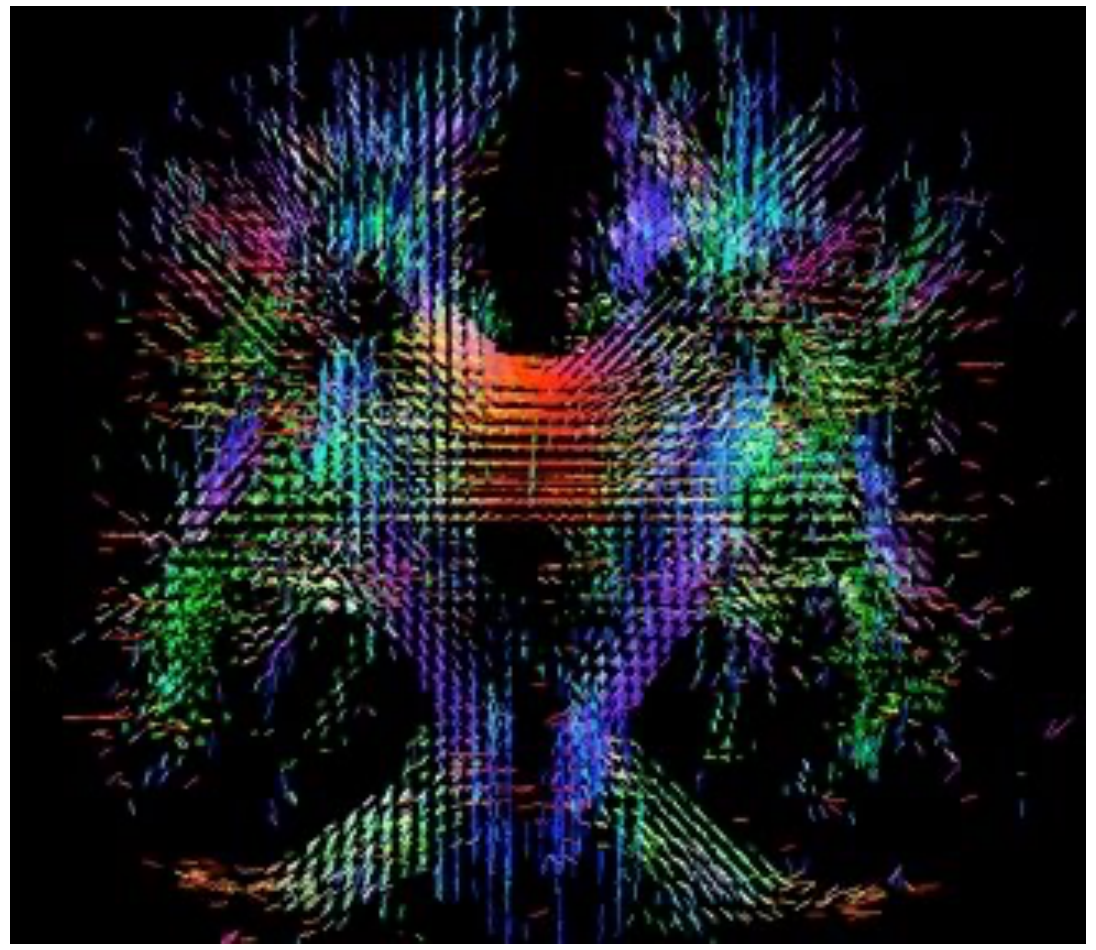
GLOW model

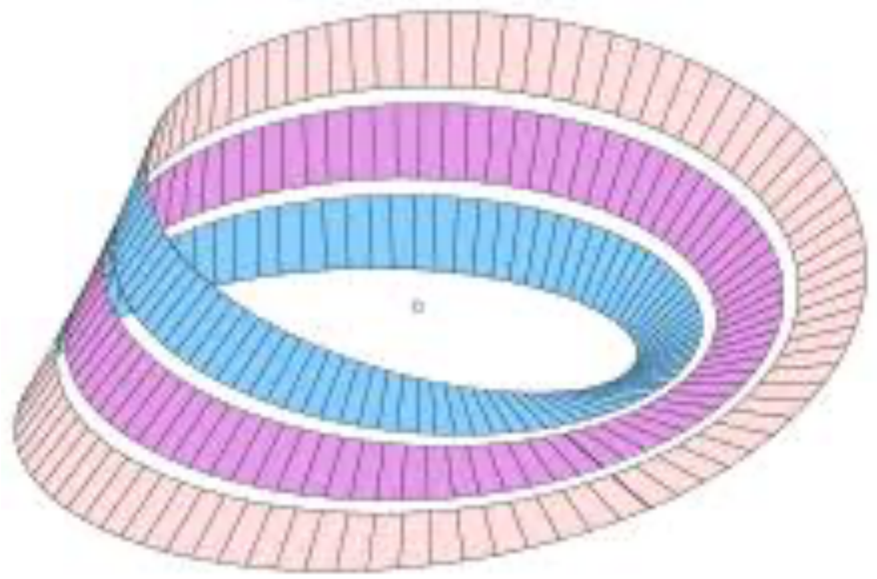
- Layers
  - Actnorm
  - Invertible 1 × 1 convolution
  - Affine Coupling
- Operators
  - Addition
  - Multiplication

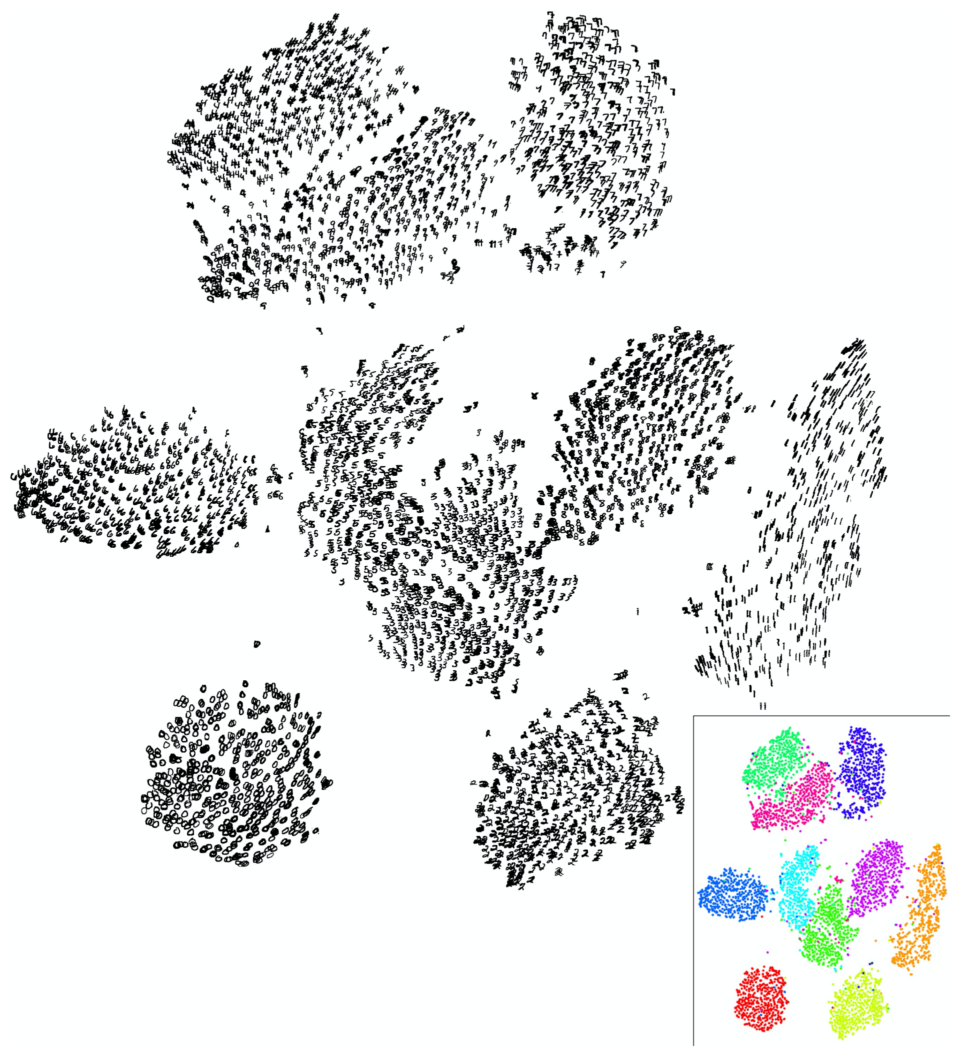


Manifold-values are important







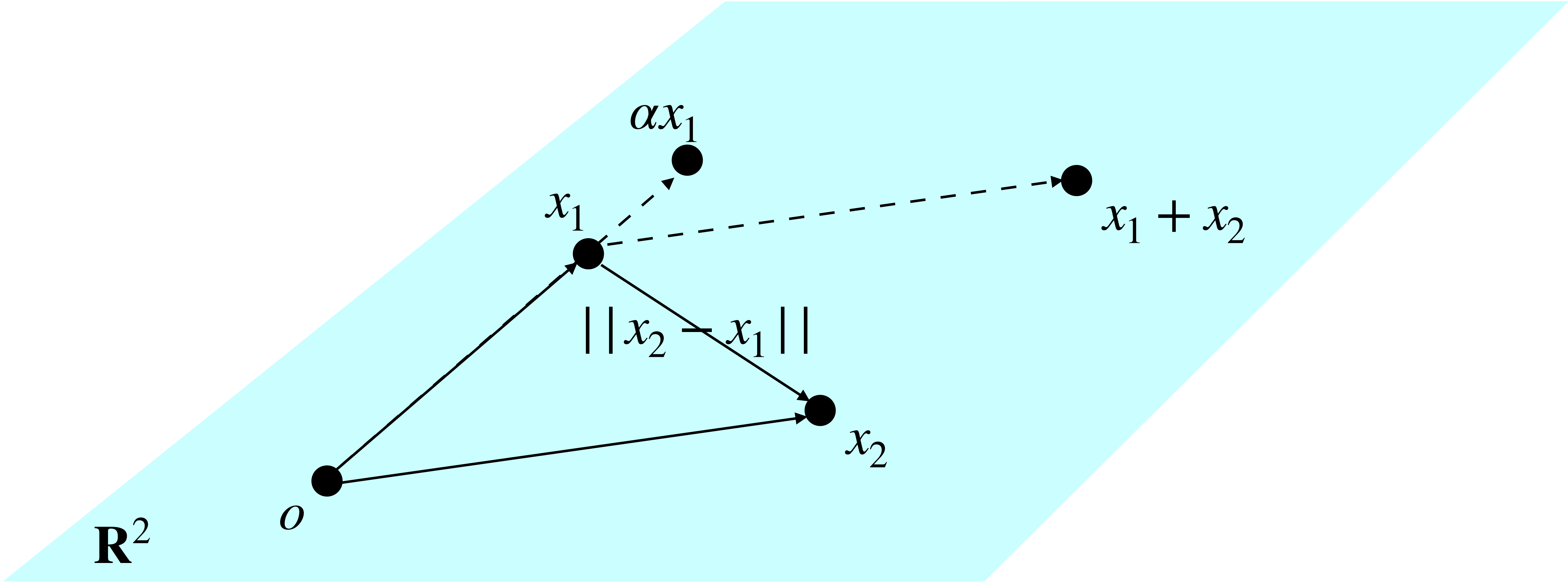


How to modify GLOW for manifolds?

- What is preserved in Manifold from Euclidean space?
- How to make minimum modifications on unpreserved parts from Euclidean space to Manifold?

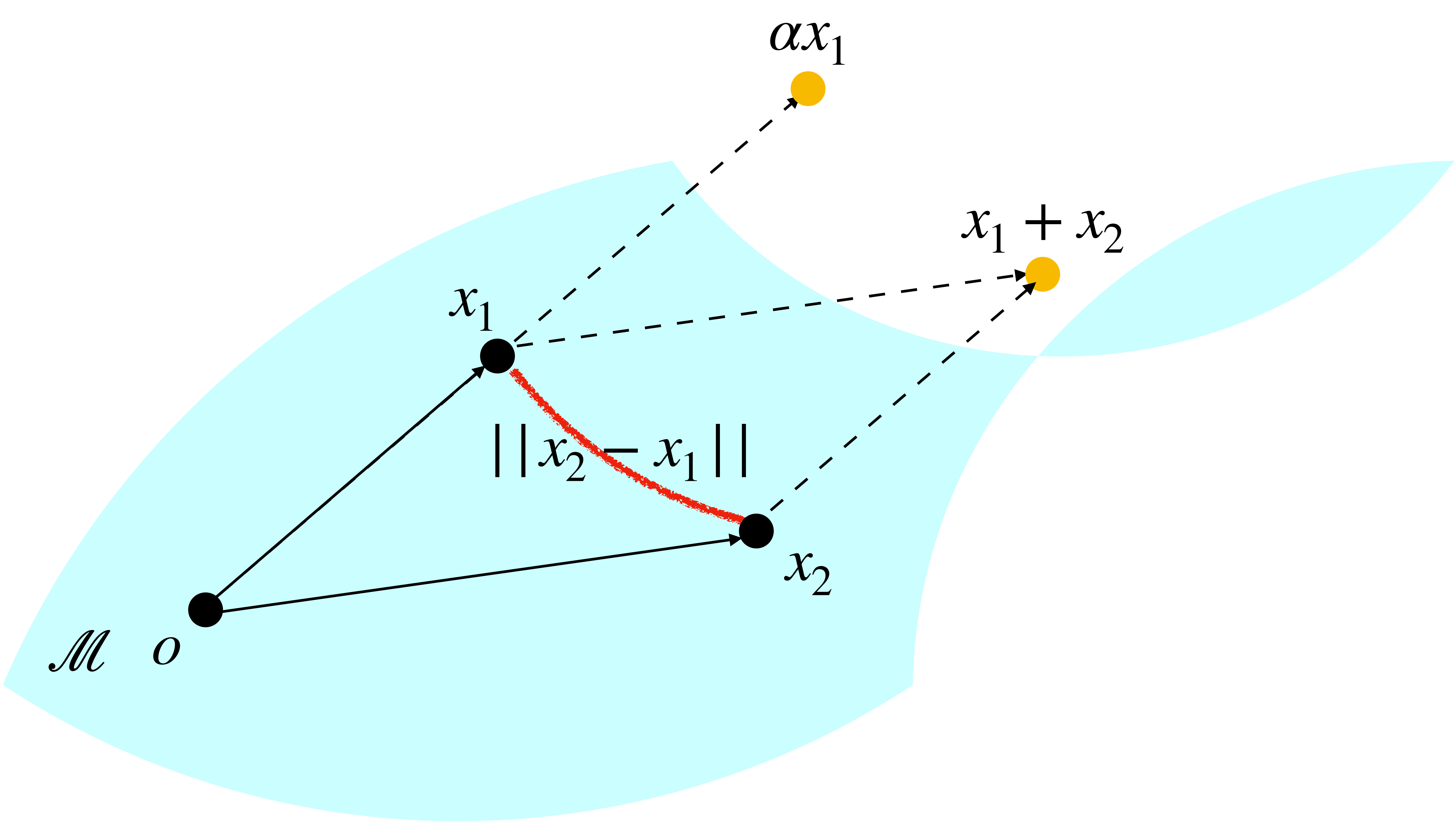
Euclidean space calculations

- Scalars and vectors
- Distance
- Basic operations:
  - Addition
  - Multiplication



Manifold space calculations

- Riemannian manifold
- Distance
- Basic operations, but harder:
  - Addition
  - Multiplication



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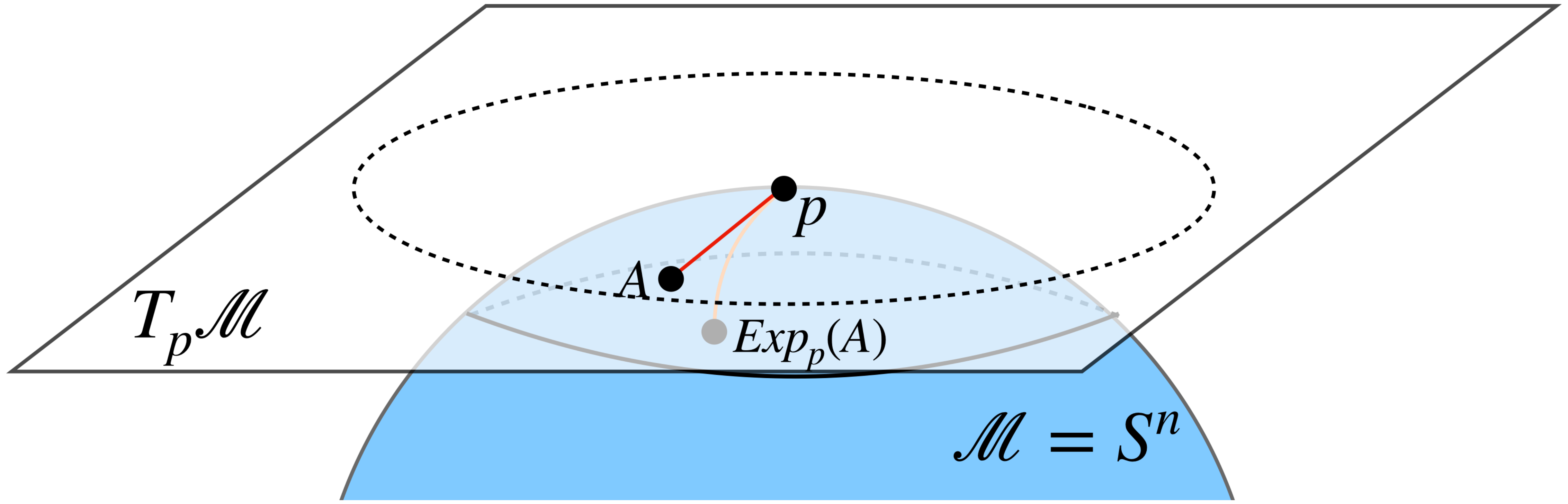
Which operators do we need?

- Tangent space at $p, T_p\mathscr{M}$
- Chart maps
  - An invertible map between a subset of the manifold and a simple space
- $\times$ defined on subspace of $\mathbf{R}^m$
- Group operations: A diffeomorphism ϕ, if $d(\phi(x), \phi(y)) = d(x, y)$



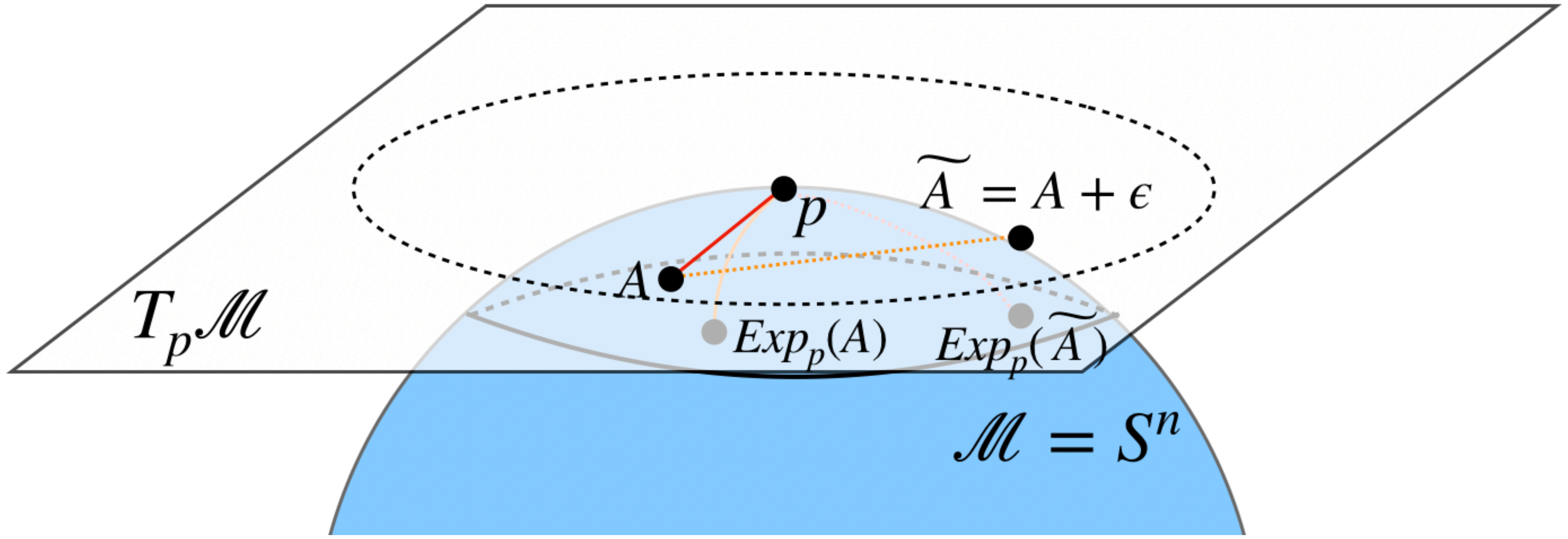
Derivative

- Differentiable function

$\widetilde{F} : \mathbf{R}^m \to \mathbf{R}^m$ and

$F : \mathcal{M} \to \mathcal{M}$,

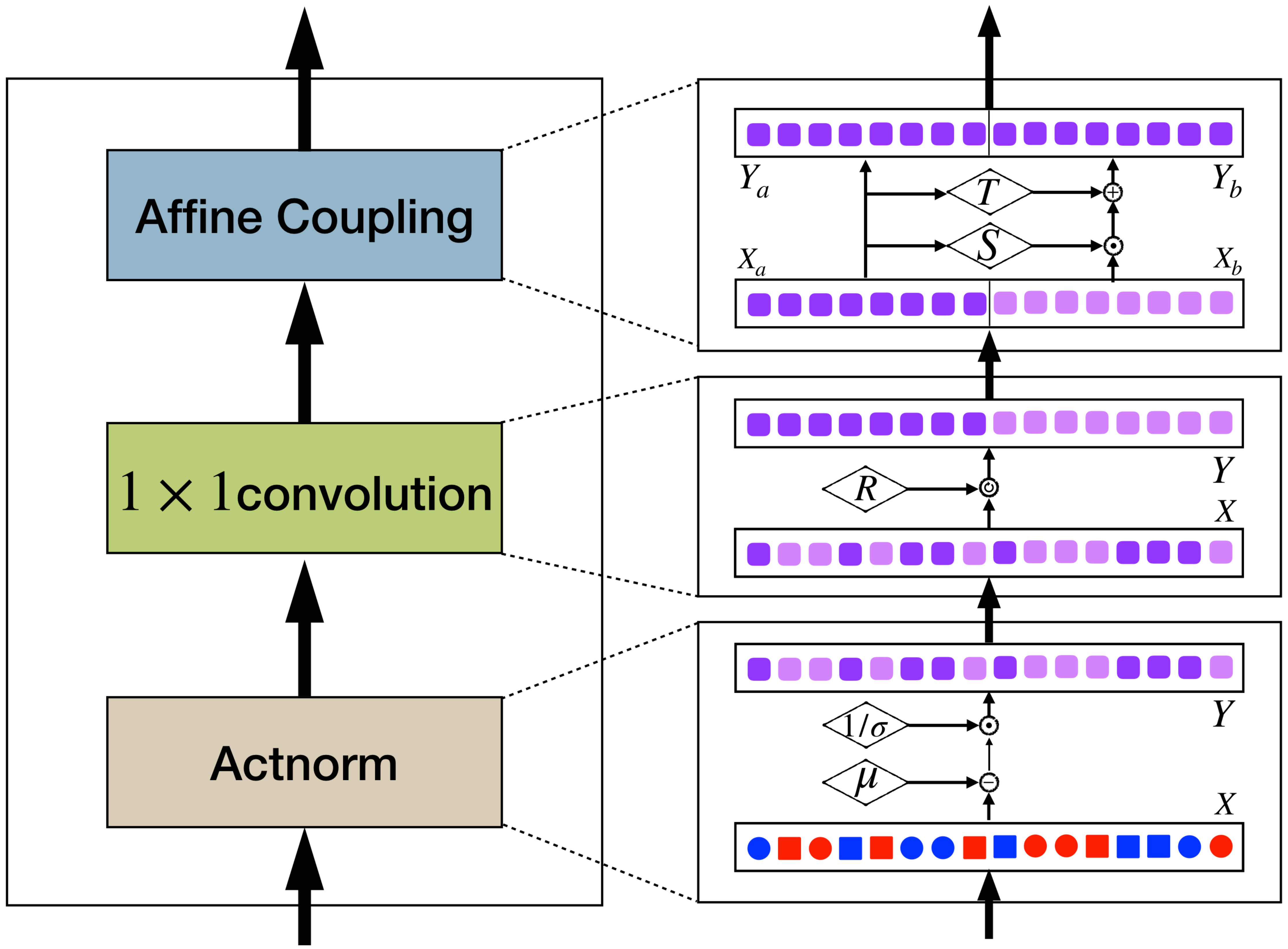
$$x \mapsto \Psi^{-1}\left(\widetilde{F}\left(\Phi(x)\right)\right)$$



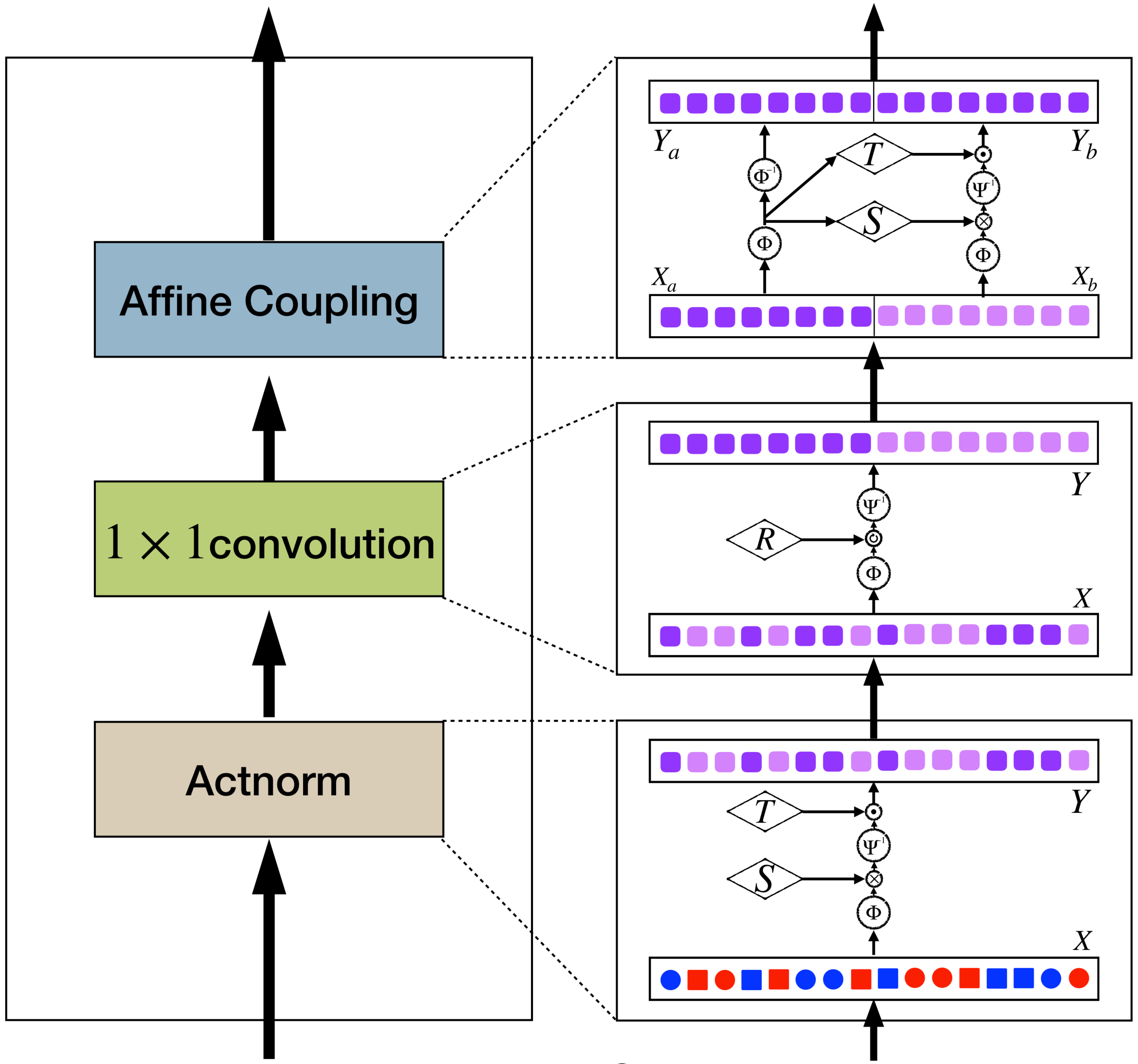
- Jacobian of F, denoted by $\frac{dy}{d\underset{\approx}{x}}$, is

$$\frac{dy}{d\underset{\approx}{x}} := \frac{\partial \Phi \circ \Psi^{-1}}{\partial \psi(x)} \frac{d\widetilde{F}}{d\widetilde{x}}\bigg|_{\Psi(x)}$$

manifoldGLOW



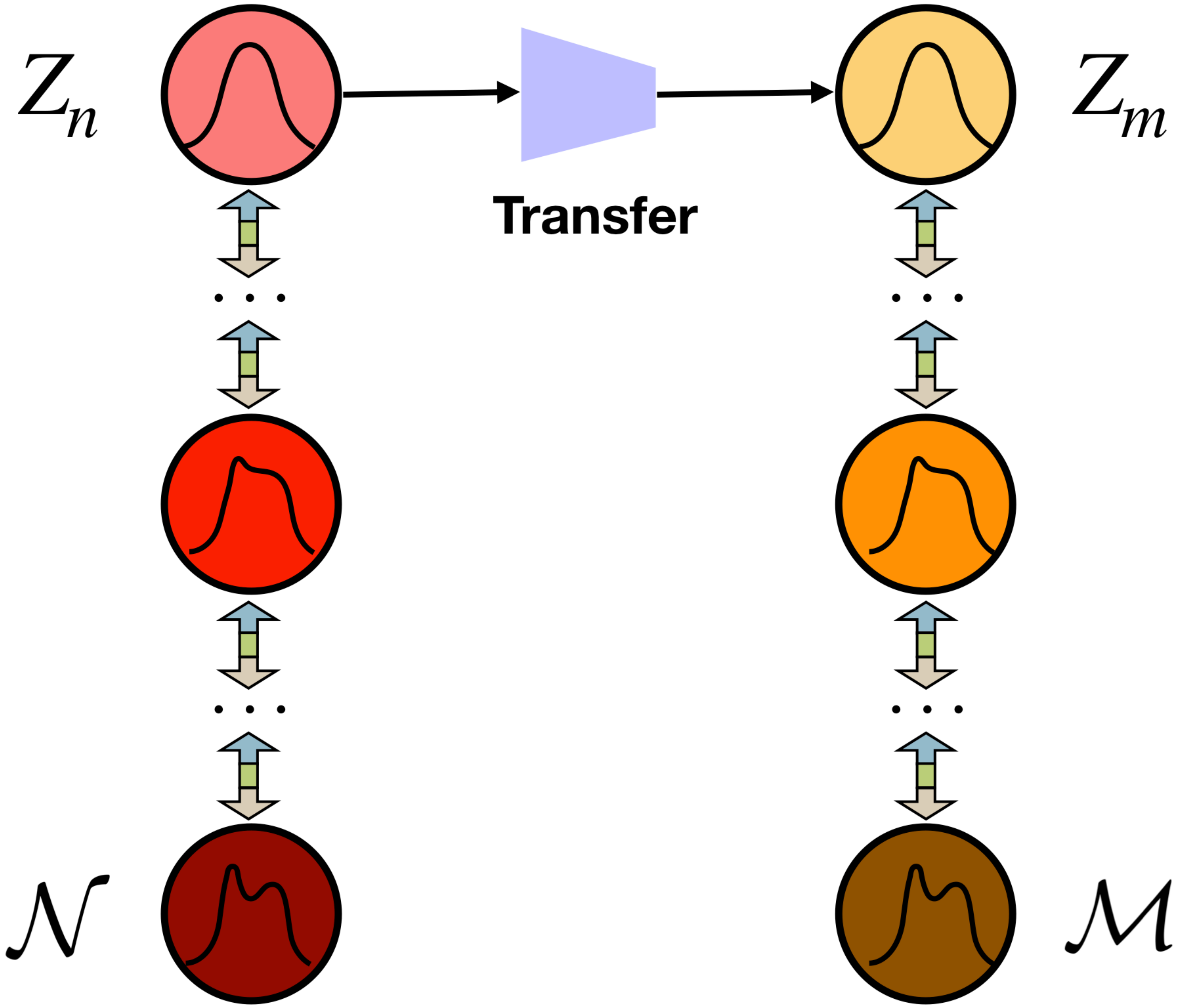
Glow



ManifoldGlow

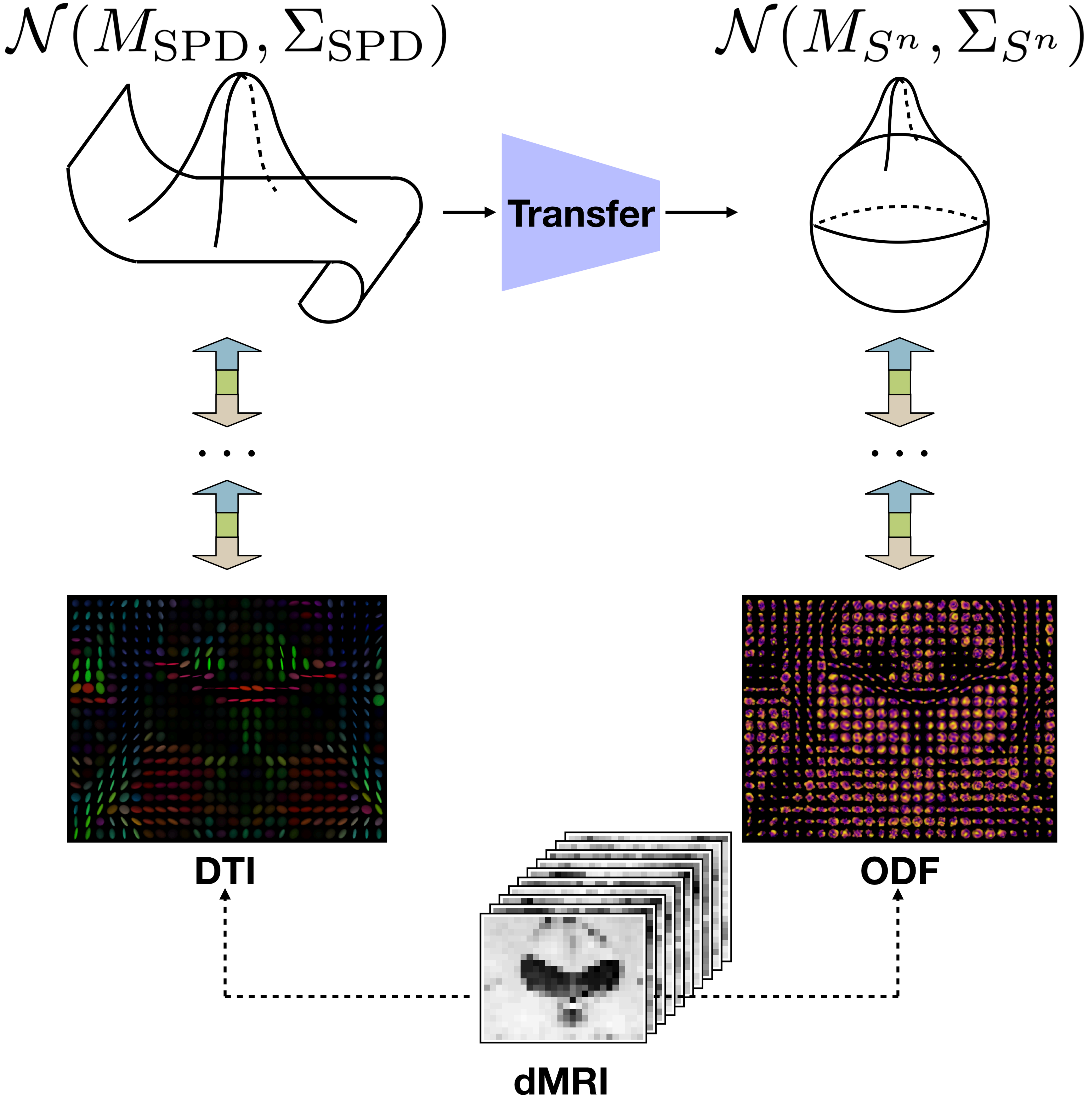
manifoldGLOW between two manifolds

- Use $\mathcal{N}$ as conditions when generating $\mathcal{M}$
- $P\left(Z_m; M_m, \Sigma_m\right)$
- $M_m = \Phi^{-1}\left(F_M(\Phi(Z_n))\right), \Sigma_m = F_S(\Phi(Z_n))$

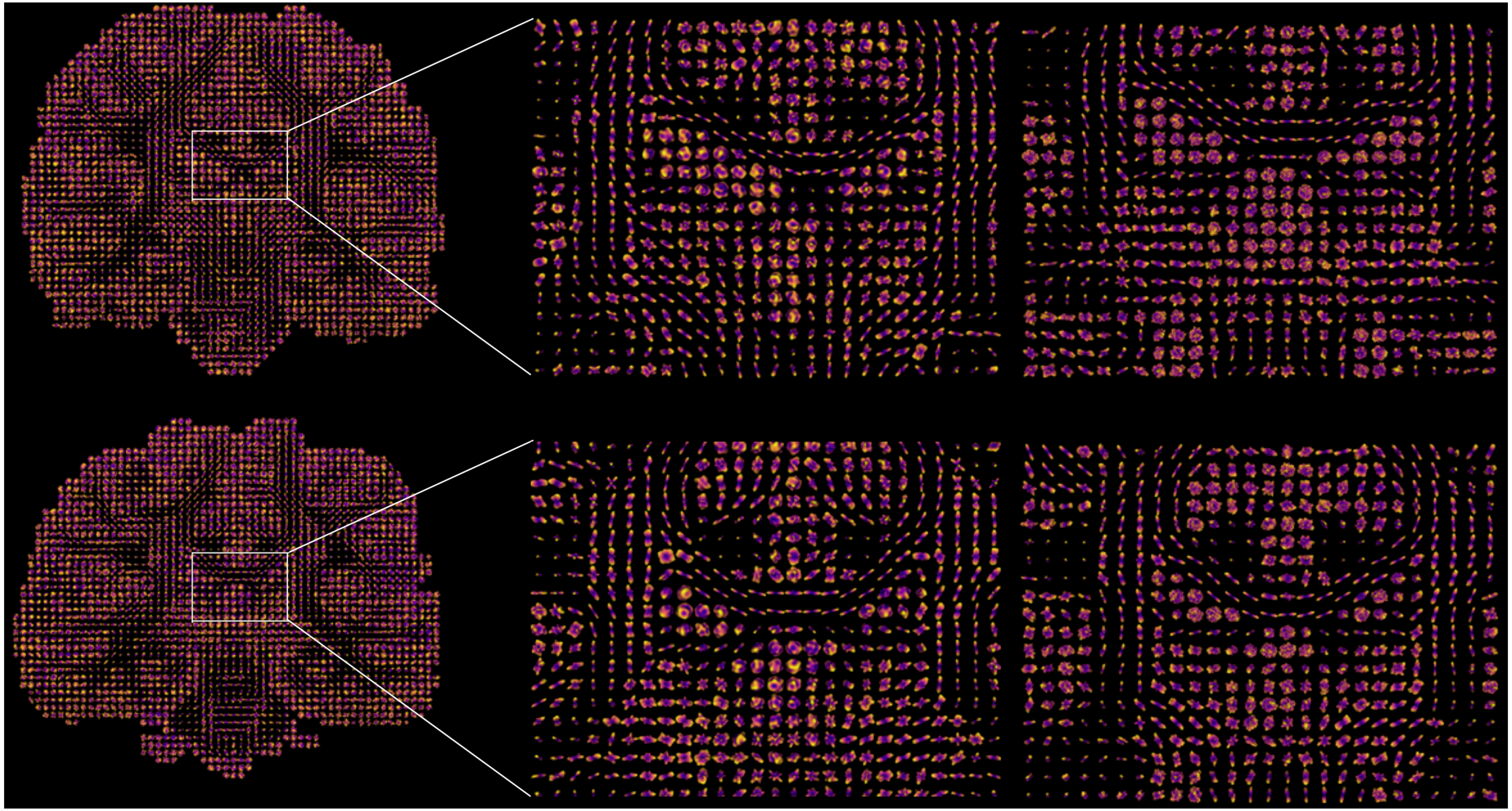


DTI to ODF

- Both are generated from dMRI
- No closed form to compute ODF from DTI
- Use 362 samples on sphere to represent ODF, ODF $\in \mathbf{S}^{361}$



DTI to ODF



Original ODF

Generated ODF

Thank you!