Quasi-stationary distributions for subcritical superprocesses

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Definition of the superprocess

Let us first give the definition of the superprocess.

- Let the underlying space E be a Polish space.
- Let the spatial motion $\{(\xi_t)_{0\leq t<\zeta}; (\Pi_x)_{x\in E}\}$ be an E-valued Borel right process with a random lifetime $\zeta \in (0,\infty]$.
- Let the branching mechanism ψ be a real function on $E\times[0,\infty)$ such that for any $x\in E$ and $z\geq 0$,

$$\psi(x,z) = -\beta(x)z + \sigma(x)^2 z^2 + \int_{(0,\infty)} (e^{-zu} - 1 + zu)\pi(x,du)$$

  where β and σ are bounded measurable functions on E and $(u\wedge u^2)\pi(x,du)$ is a bounded kernel from E to $(0,\infty)$.
- Let $\mathcal{M}_f(E)$ be the space of all finite Borel measures on E equipped with the topology of weak convergence.

Definition of the superprocess

- Denote by $\mu(f)$, $\langle f, \mu\rangle$ or $\langle \mu, f\rangle$ the integration of a function f and a measure μ whenever the integral is well-defined.
- Say a function f on $\mathbb{R}_+ \times E$ is locally bounded if
$$\sup_{s\in[0,t],x\in E} |f(s,x)| < \infty, \quad t \in \mathbb{R}_+.$$
- Let the cumulant semigroup $(V_t)_{t\geq 0}$ be given by the following. For any bounded non-negative function f on E, there exists a unique locally bounded non-negative Borel function $(t,x) \mapsto V_t f(x)$ on $[0,\infty) \times E$ such that for any $t \geq 0$ and $x \in E$,
$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \psi(\xi_s, V_{t-s}f(\xi_s))ds\Big] = \Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}].$$

Definition of the superprocess

- Let the (ξ, ψ)-superprocess $\{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}_f(E)}\}$ be given as an $\mathcal{M}_f(E)$-valued Markov process such that for each bounded non-negative measurable function f on E, $t \geq 0$ and finite measure μ on E, $\mathbb{P}_\mu[e^{-\langle f, X_t\rangle}] = e^{-\langle V_t f, \mu\rangle}$.
- (ξ, ψ)-superprocess exists (Watanabe (1968), Ikeda, Nagasawa and Watanabe (1968, 1969), Dawson (1975, 1977)).

Definition of Yaglom limit, QLDs and QSDs

Let us now introduce the concept of Yaglom limit, QLDs and QSDs for the superprocess.

- Denote $\mathbf{0}$ the null measure on E. Write $\mathcal{M}_f^o(E) = \mathcal{M}_f(E) \setminus \{\mathbf{0}\}$. Any probability measure $\mathbf{P}$ on $\mathcal{M}_f^o(E)$ will also be understood as its unique extension on $\mathcal{M}_f(E)$ with $\mathbf{P}(\{\mathbf{0}\}) = 0$.
- We say a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is the Yaglom limit of the superprocess X if for any $\mu \in \mathcal{M}_f^o(E)$,

  $\mathbb{P}_\mu(X_t \in \cdot | \|X_t\| > 0) \xrightarrow[t\to\infty]{d} \mathbf{Q}(\cdot).$

Definition of Yaglom limit and QSDs

- For any probability measure $\mathbf{P}$ on $\mathcal{M}_f(E)$, define $(\mathbf{P}\mathbb{P})[\cdot] := \int_{\mathcal{M}_f(E)} \mathbb{P}_\mu[\cdot]\mathbf{P}(d\mu).$
- We say a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a quasi-limit distribution (QLD) of X, if there exists a probability measure $\mathbf{P}$ on $\mathcal{M}_f^o(E)$ such that $(\mathbf{P}\mathbb{P})(X_t \in B | \|X_t\| > 0) \xrightarrow[t\to\infty]{} \mathbf{Q}(B), \quad B \in \mathcal{B}(\mathcal{M}_f^o(E)).$
- We say a probability measure $\mathbf{Q}$ on $\mathcal{M}_f^o(E)$ is a quasi-stationary distribution (QSD) of X, if $(\mathbf{Q}\mathbb{P})(X_t \in B | \|X_t\| > 0) = \mathbf{Q}(B), \quad t \geq 0, B \in \mathcal{B}(\mathcal{M}_f^o(E)).$

Motivation

Motivation

We want to investigate those sets: $\{\text{Yaglom limit of X}\}, \{\text{QLDs of X}\}$, and $\{\text{QSDs of X}\}$.

Here are some basic facts (Méléard and Villemonais (2012)):

- $\#\{\text{Yaglom limit of X}\} \leq 1$.
- $\{\text{Yaglom limit of X}\} \subset \{\text{QLDs of X}\} = \{\text{QSDs of X}\}$.
- For any $\mathbf{Q} \in \{\text{QSDs of X}\}$, there exists an $r \in (-\infty, 0)$ such that $(\mathbf{Q}P)(X_t \neq \mathbf{0}) = e^{rt}$. We say r is the decay rate of $\mathbf{Q}$.

Criticality of superprocesses

Let us first discuss the criticality of the superprocesses.

- The mean semigroup $(P_t^\beta)_{t\geq 0}$ of X is given by $P_t^\beta f(x) := \Pi_x\left[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)\mathbf{1}_{t<\zeta}\right]$ where $f \in \mathcal{B}_b(E), t \geq 0$ and $x \in E$.
- **Assumption 0:** There exist a constant $\lambda < 0$, a bounded strictly positive Borel function ϕ on E, and a probability measure ν with full support on E such that for each $t \geq 0$,
  $P_t^\beta \phi = e^{\lambda t}\phi, \quad \nu P_t^\beta = e^{\lambda t}\nu, \quad \nu(\phi) = 1.$
- The assumption $\lambda < 0$ says that the mean of $(X_t(\phi))_{t\geq 0}$ decay exponentially with rate $-\lambda > 0$, and in this case the superprocess X is called subcritical.

Intrinsic ultracontractive and non-persistent

We will also need the following two assumptions:

- Denote by $L_1^+(\nu)$ the collection of non-negative Borel functions on E which are integrable w.r.t. ν.
- **Assumption 1:** For all $t > 0, x \in E$ and $f \in L_1^+(\nu)$, it holds that $P_t^\beta f(x) = e^{\lambda t}\phi(x)\nu(f)(1 + C_{t,x,f})$ for some real $C_{t,x,f}$ with $\sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}| < \infty$ and $\lim_{t\to\infty} \sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}| = 0$.
- **Assumption 2:** There exists $T \geq 0$ such that $\mathbb{P}_\nu(\|X_t\| = 0) > 0$ for all $t > T$.

One simple consequence of Assumption 1 is the following.

- $\mathbb{P}_\mu(\|X_t\| > 0) > 0$ for each $t \geq 0$ and $\mu \in \mathcal{M}_f^o(E)$. Hence we can condition the superprocess X on survival up to a given time t.

Main Results

Theorem (Liu, Ren, Song and S. (2020))

If Assumptions 0-2 hold, then there exists a probability measure $\mathbf{Q}$ *on* $\mathcal{M}_f^o(E)$ *such that for each* $\mu \in \mathcal{M}_f^o(E)$,

$$\mathbb{P}_\mu(X_t \in \cdot \,|\, \|X_t\| > 0) \xrightarrow[t\to\infty]{d} \mathbf{Q}(\cdot).$$

Theorem (Liu, Ren, Song and S. (2020))

Suppose that Assumptions 0-2 hold. Then (1) for each $r \in (-\infty, \lambda)$*, there is no QSD for* X *with decay rate* r*; and (2) for each* $r \in [\lambda, 0)$*, there exists a unique QSD* $\mathbf{Q}_r$ *for* X *with decay rate* r*. Moreover, for any* $r \in [\lambda, 0)$ *and non-negative Borel function* h *on* E,

$$\Big(\int_{\mathcal{M}_f^o(E)} (1 - e^{-w(h)})\mathbf{Q}_r(dw)\Big)^{1/r} = \Big(\int_{\mathcal{M}_f^o(E)} (1 - e^{-w(h)})\mathbf{Q}(dw)\Big)^{1/\lambda}.$$

Literature

- Yaglom limit for Galton-Watson processes are studied by Yaglom (1947), Heathcote, Seneta and Vere-Jones (1967), and Joff (1967).
- QSDs for Galton-Watson processes are studied by Hoppe and Seneta (1976).
- Yaglom limit for Multitype Galton-Watson processes are studied by Hoppe (1975), Hoppe and Seneta (1978), Joffe (1967).
- For Yaglom limit and QSDs for branching Markov processes, see Asmussen and Hering's book: *branching processes* (1983) and the references therein.
- Yaglom limit for continuous-state branching process (a degenerated superprocess) where studied by Li (2000) and Lambert (2007).
- QSDs for continuous-state branching processes are studied by Lambert (2007).

Thanks!