Wright-Fisher stochastic heat equations with irregular drifts

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Based on joint work with **Clayton Barnes** and **Leonid Mytnik**

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Regularization by noise

- Consider the differential equation:

$$\begin{cases} \mathrm{d}X_t = b(X_t)\mathrm{d}t = |X_t|^\alpha \mathrm{d}t, \quad t > 0, \\ X_0 = 0, \end{cases}$$

where $\alpha \in (0,1)$.

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- The other solution $X_t = C_\alpha t^{\frac{1}{1-\alpha}}, t \geq 0$.

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- Zvonkin's transform is not available for SPDEs.

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where, with $\alpha > 0$ and $\beta > 0$,

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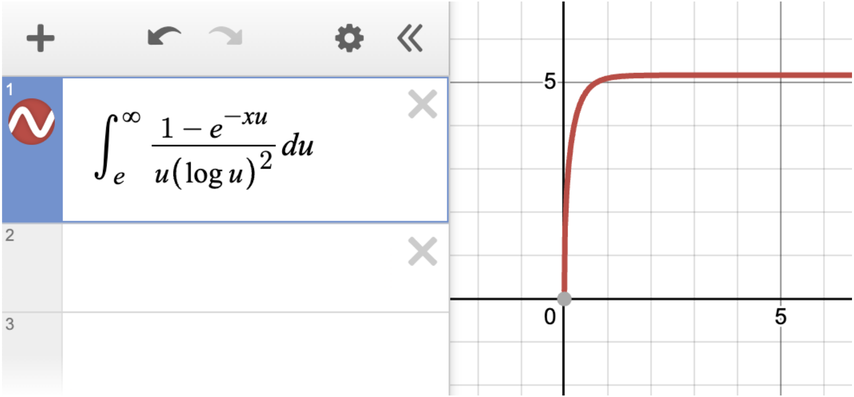
Weak regularization by multiplicative noise

- The shape of a “critical” drift $b(x)$:



Wright-Fisher Stochastic Heat Equations (Wright-Fisher SHE)

- Quasi-linear heat equation perturbed by the Wright-Fisher space-time white noise

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- $\dot{W}$ is the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, i.e. a centered Gaussian process with $\mathbb{E}[\dot{W}_t(x)\dot{W}_s(y)] = \delta_0(t-s)\delta_0(x-y)$.

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    - is degenerate at $u = 0$ and $u = 1$.
- Challenging open problems:
    - the strong uniqueness?
    - the solution theory in higher dimensions?
- **Question:** How strong is the regularization effect of the Wright-Fisher white noise?

Motivation

- Shiga (1988, Math. Appl.):
  Wright-Fisher SHE = scaling limit of "genetic stepping stone model."
    - $b(u) = c_1(1-u) - c_2 u + c_3 u(1-u)$.
    - $c_1 \geq 0$ and $c_2 \geq 0$ are mutation rates.
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- Brunet-Derrida (1997, Phys. Rev. E), Mueller-Mytnik-Quastel (2011, Invent. Math.): The Wright-Fisher SHE is related to the Brunet-Derrida particle systems (branching processes with competition).

Weak existence

Shiga (1994, Can. J. Math.)

If $f \in \mathcal{C}(\mathbb{R}, [0,1])$, $b(\cdot)$ is continuous and $b(0) \geq 0 \geq b(1)$, then there exists a $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}, [0,1]))$-valued process $(u_t)_{t\geq 0}$ which is a probabilistically-weak, and PDE-weak, solution to

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Sketch of the Proof

Let u^n solves $\partial_t u_t^n = \frac{\Delta}{2} u_t^n + b_n(u^n) + \sigma_n(u^n)\dot{W}$ where $b_n(z)$ and $\sigma_n(z)$ are sequences of Lipschitz functions uniformly approximating $b(z)$ and $\sqrt{z(1-z)}$. Then $\{u^n\}_{n \in \mathbb{N}}$ is tight. Any subsequential weak limit u is a solution to the Wright-Fisher SHE with drift b. $b(0) \geq 0 \geq b(1)$ is used to make sure u only takes values in $[0,1]$. The continuity of $b(\cdot)$ is used to make sure that $b_n(u^n)$ converges to $b(u)$.

Weak Uniqueness: Duality Method

- Weak uniqueness = uniqueness in law.

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- Both Shiga (1988, Math. Appl.) and Athreya-Tribe (2000, AP) used the duality argument.

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- Suppose that the random field u and the particle system X are independent.
- We say the moment duality holds between u and X if

$$\mathbb{E}\left[\prod_{i\in I_0} u_t(X_0^i)\right] = \mathbb{E}\left[\prod_{i\in I_t} u_0(X_t^i)\right], \quad t \geq 0.$$

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- The formula characterizes the distributions of both u and X, provided they are Markovian.
- For example, we can take $\{(X_t^i)_{t\geq 0} : i = 1, \ldots, n\}$ to be a sequence of independent Brownian motions, and u to satisfy the heat equation $\partial_t u = \frac{\Delta}{2} u$.

Weak Uniqueness: The Girsanov transformation

Mueller-Mytnik-Ryzhik (2021, CMP)

The weak uniqueness holds provided

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- When the red part holds, we say the initial value f has a compact interface.
- The main tool is Girsanov transformation.

Quantification of the regularization effect

- Consider

$$(2) \qquad \begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}, \end{cases}$$

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- Mueller-Mytnik-Ryzhik (2021, CMP): $\alpha \in [\frac{1}{2}, 1]$ & f has compact interface $\implies$ weak uniqueness.
- **Question:** What happens when $\alpha \in (0, \frac{1}{2})$? What happens when f doesn't have compact interface?

Propagation speed

Barnes-Mytnik-S. (2024+, to appear in AIHP)

Suppose that $\alpha \in [\frac{1}{2}, 1]$ and that $f \in \mathcal{C}(\mathbb{R}_+, [0,1])$ has compact interface. Let u satisfy

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \epsilon\sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

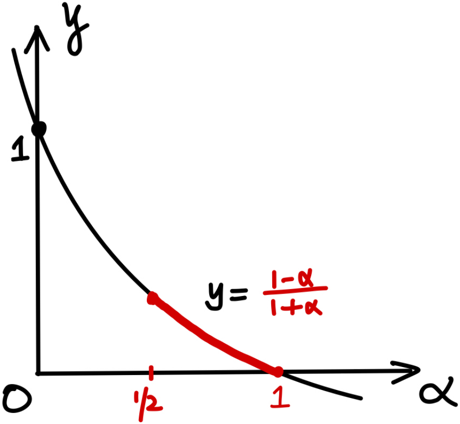
Then,

$$\text{the front of } u_t \; := \sup\{x : u_t(x) > 0\}$$

propagates with a deterministic speed $V(\epsilon) \asymp \epsilon^{-2\frac{1-\alpha}{1+\alpha}}$ for small ϵ.

Propagation speed

- Here is an image of the exponent $\frac{1-\alpha}{1+\alpha}$:



Main Result

- Recall AT's condition:

$$b(u) = \sum_{k=0}^{\infty} b_k u^k, \text{ and } b_1 < -\sum_{k\in\{0\}\cup\mathbb{N}\setminus\{1\}} |b_k| R^{k-1} \text{ for some } R > 1.$$

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Barnes-Mytnik-S. (2024, arXiv)

The weak existence and weak uniqueness holds for the 1-d Wright-Fisher SHE provided the initial value $f \in \mathcal{C}(\mathbb{R}, [0,1])$, and the drift term

$$b(u) = \sum_{k \in \{0, \infty\} \cup \mathbb{N}} b_k u^k = \sum_{k=0}^{\infty} b_k u^k + b_\infty \mathbf{1}_{\{1\}}(u)$$

$$\text{with } b_1 \leq - \sum_{k \in \{0,\infty\} \cup \mathbb{N} \setminus \{1\}}^{\infty} |b_k| R^{k-1} \text{ for some } R \geq 1.$$

Corollaries

Corollary 1 (expected)

The weak uniqueness holds for the SPDE

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}, \end{cases}$$

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when $\alpha \in (0,1]$ and $f \in \mathcal{C}(\mathbb{R}, [0,1])$.

- This is expected, since the weak uniqueness holds for the SDE

$$\mathrm{d}X_t = X_t^{\alpha}(1 - X_t)\mathrm{d}t + \sqrt{X_t(1-X_t)}\mathrm{d}B_t; \quad X_0 = x \in [0,1].$$

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- What if the Hölder coefficient $\alpha = 0$?

Corollaries

Corollary 2 (unexpected)

The weak existence and the weak uniqueness holds for the SPDE

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + \mathbf{1}_{\{u>0\}}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}, \end{cases}$$

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- The solution u in Corollary 2 is a new model. It does **not** satisfy $\partial_t u = \frac{\Delta}{2} u + (1-u) + \sqrt{u(1-u)}\dot{W}$.
- The weak existence is non-trivial, because the drift is discontinuous.

Perspectives

- Consider the 1-D quasi-linear stochastic heat equation

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$\sigma(u)$ \ $b(u)$	0	Lipschitz	Hölder	Discontinuous	Measurable
deterministic: 0	well-posed	well-posed	non-unique for some drift	non-unique for some drift	non-unique for some drift
additive: 1	well-posed	well-posed	well-posed	well-posed	well-posed
Lipschitz & non-degenerate	well-posed	well-posed	well-posed	well-posed	well-posed
3/4-Hölder continuous & non-degenerate	well-posed	weakly well-posed	weakly well-posed	?	?
Feller noise: $\sqrt{u}$	weakly well-posed	weakly well-posed for some drift	?	?	?
Wright-Fisher noise: $\sqrt{u(1-u)}$	weakly well-posed	weakly well-posed	weakly well-posed for some drift	weakly well-posed for some drift	?

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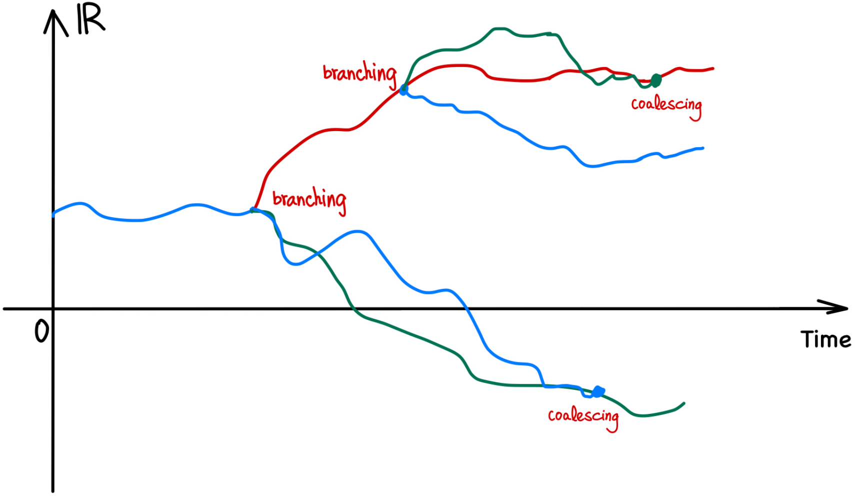
- Two parameters:
  - *Branching rate* $\mu > 0$.
  - *Offspring distribution* $(p_k)_{k\in\{0,\infty\}\cup\mathbb{N}}$.
- Three dynamics:
  - *Spatial movement:* Particle move as independent Brownian motions.
  - *Branching:* Each particle branches into a random number of particles with the rate μ. The offspring number is sampled according to the distribution $(p_k)_{k\in\{0,\infty\}\cup\mathbb{N}}$.
  - *Coalescing:* Each pair of particles coalesces as one particle with rate $1/2$ according to their intersection local time.

An illustration of the dual particle system



Explosion in CBBM

- To build a duality relation between CBBMs and the Wright-Fisher SHEs, we take
$$\mu := \sum_{k\in\{0,\infty\}\cup\mathbb{N}\setminus\{1\}}^{\infty} |b_k|$$
and $p_1 := 0$, $p_k := |b_k|/\mu$ for $k \in \{0,\infty\} \cup \mathbb{N} \setminus \{1\}$.

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- The definition of the particle system needs more justification!

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The total population $Z_t(\mathbb{R}) < \infty$ for every $t > 0$
$\iff$ $Z_0(\cdot)$ is compactly supported.
Moreover, in this case

$$\left(\int v_t(x)\mathrm{d}x\right)^{-1} Z_t(\mathbb{R}) \xrightarrow{L^1} 1, \quad t \downarrow 0$$

where $(v_t(x))_{t\geq 0, x\in\mathbb{R}}$ is the unique non-negative solution to the 1d PDE $\partial_t v_t = \frac{\Delta}{2} v_t - v_t^2/2$ with initial value $v_0 = Z_0$.

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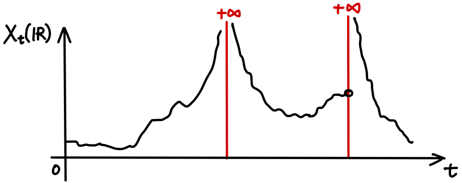
Barnes-Mytnik-S. (2024, arXiv)

If $X_0(\mathbb{R}) < \infty$, then $X_t(\mathbb{R})$ is reflecting from ∞.



Thanks!