

Noise effect on the speed of stochastic reaction-diffusion equation with Hölder drift

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Stochastic reaction-diffusion equation

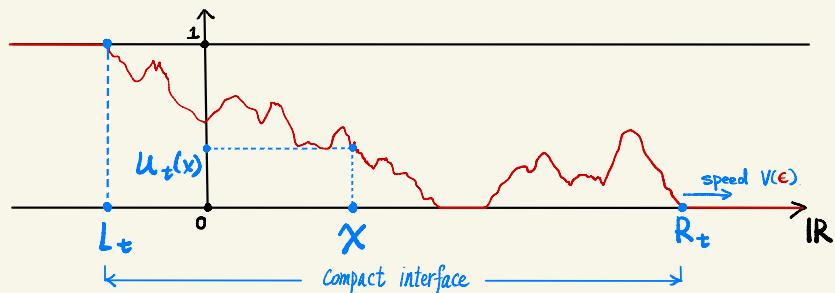
$$\textcircled{1} \quad \partial_t u = \frac{1}{2} \partial_x^2 u + u(1-u) + \epsilon \sqrt{u(1-u)} \dot{W} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}; \quad \epsilon > 0.$$

heat equation noise strength
 FKPP drift Wright-Fisher white noise

Shiga (1988), Mueller & Sowers (1995)

The weak existence and weak uniqueness of SPDE ① hold in

$C_I := \{ \text{continuous } [0,1]\text{-valued functions on } \mathbb{R} \text{ with compact interface} \}$.



Four equivalent solution concepts:

- Mild form
- Schwartz distribution
- Martingale problem
- Dual process of branching-coalescing Brownian motion.

Walsh (1984), Iwata (1987)

— Shiga (1988)

Mueller & Sowers (1995), Conlon & Doering (2005)

$\forall \epsilon > 0, \exists \text{ a deterministic } V(\epsilon) \in \mathbb{R} \text{ s.t. } \frac{R_t}{t} \xrightarrow{t \rightarrow \infty} V(\epsilon) \text{ a.s.}$

N -branching random walk

Bérard & Gouéré (2010).

Branching-coalescing Brownian motion

Mueller, Mytnik & Quastel (2011).

N -branching Brownian motion

Maillard (2012).

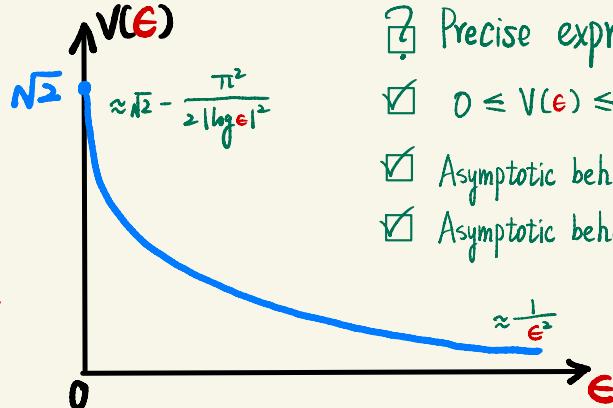
Branching Brownian motion in a strip

Berestycki, Berestycki & Schweinsberg (2013).

L -branching Brownian motion

Pain (2016).

Universality

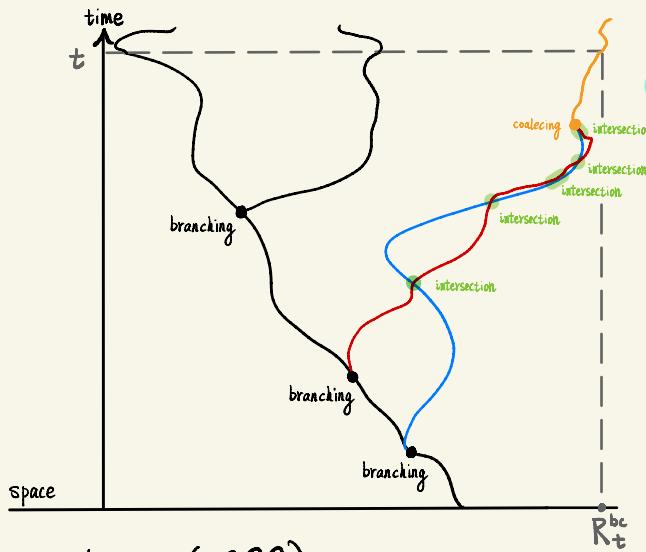


Brunet & Derrida conjecture (1997), Mueller, Mytnik & Quastel (2011)

$$V(\epsilon) = \sqrt{N^2} - \frac{\pi^2}{2|\log \epsilon|^2} + O\left(\frac{\log |\log \epsilon|}{|\log \epsilon|^3}\right), \quad \epsilon \rightarrow 0.$$

Conlon & Doering (2005), Mueller, Mytnik & Ryzhik (2021)

$$\lim_{\epsilon \rightarrow +\infty} \epsilon^2 V(\epsilon) = 1.$$



Branching-coalescing Brownian motion

Independently

- \forall particle moves as Brownian motion.
- \forall Particle branches into 2 particles with rate 1.
- \forall pair of particles coalesce into 1 particle with rate ϵ^2 according to their intersection local time.

Notation:

$$I_t = \{ \text{particles alive at time } t \}.$$

$$X_t^i = \text{Position of Particle } i \in I_t \text{ at time } t.$$

Shiga (1988)

If the branching-coalescing Brownian motion is independent of the weak solution $u_t(x)$,

$$\text{then } \mathbb{E} \prod_{i \in I_0} [1 - u_t(X_0^i)] = \mathbb{E} \prod_{i \in I_t} [1 - u_0(X_t^i)].$$

LHS

\rightarrow finite moments of u_t $\xrightarrow{\text{determine}}$ distribution of u_t \longrightarrow weak uniqueness.

$$\text{taking } u_0 = 1_{(-\infty, 0)}$$

$$\rightarrow P(R_t^{bc} > x) = \mathbb{E} u_t(x) \longrightarrow \text{propagation of branching-coalescing Brownian motion.}$$

Benguria & Depassier (1996): In application, people are interested in reaction-diffusion equation $\partial_t u = \frac{1}{2} \partial_x^2 u + f(u)$ with more general reaction term.

And in many cases when f is Lipschitz, the system has finite propagation speed.

Aguirre & Escobedo (1986): If $f(u) = u^p(1-u)$ with $0 < p < 1$, then the system doesn't have finite propagation.

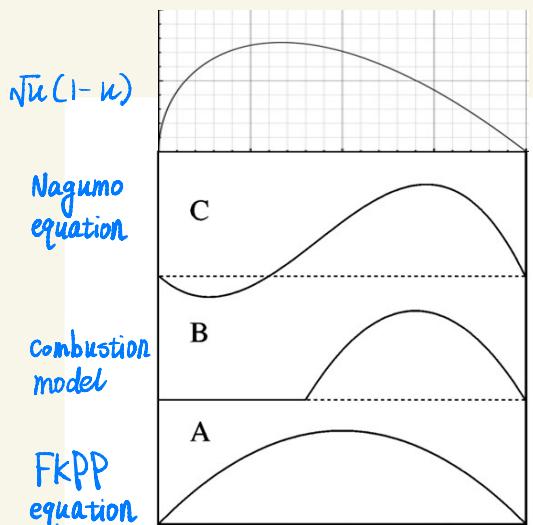


FIG. 1. The three basic types of reaction terms that arise in different applications.

Athreya & Tribe (2000):

A large class of stochastic reaction-diffusion equations

$$\partial_t u = \frac{1}{2} \partial_x^2 u + f(u) + \sigma(u) \dot{W}$$
 with Lipschitz f and σ^2 , have duality relation to some branching-coalescing type particle system.

Conjecture

Shiga's duality holds between $\partial_t u = \frac{1}{2} \partial_x^2 u + u^p(1-u) + \epsilon \sqrt{u(1-u)} \dot{W}$ and branching-coalescing Brownian motion whose offspring law has generating function $u - (1-u)^p u$. ($0 < p < 1$)

Image of $f(u)$ @ Benguria & Depassier (1996)

Examples: types A, B, C & $f(u) = u^p(1-u)$ with $\frac{1}{2} \leq p \leq 1$.

② $\partial_t u = \frac{1}{2} \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)} \dot{w}$ where $f \in C([0,1])$ and $\sup_{u \in (0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$

Mueller, Mytnik & Ryzhik (2021)

- $\forall \epsilon > 0$, the weak existence and weak uniqueness of SPDE ② hold in $C_I := \{ \text{continuous } [0,1]\text{-valued functions on } \mathbb{R} \text{ with compact interface} \}$.
- $\forall \epsilon > 0, \exists$ a deterministic $V_f(\epsilon) \in \mathbb{R}$ s.t. $\frac{R_t}{t} \xrightarrow[t \rightarrow \infty]{} V_f(\epsilon)$ a.s.

Idea: $\partial_t u = \frac{1}{2} \partial_x^2 u + \epsilon \sqrt{u(1-u)} \dot{w}$.

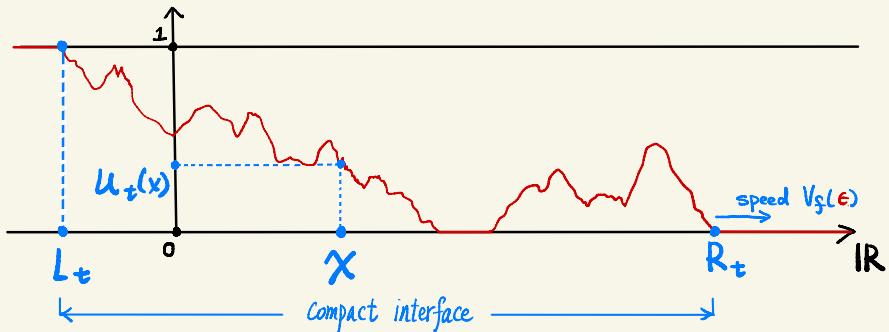
② Girsanov transformation

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$$

$$M_t = \int_{L_t}^{R_t} \int_0^t \frac{f(u)}{\sqrt{u(1-u)}} dW$$

bounded

Finite propagation $\xleftarrow[\text{due to}]{} \text{Noise } \epsilon > 0$



Mueller, Mytnik & Ryzhik (2021)

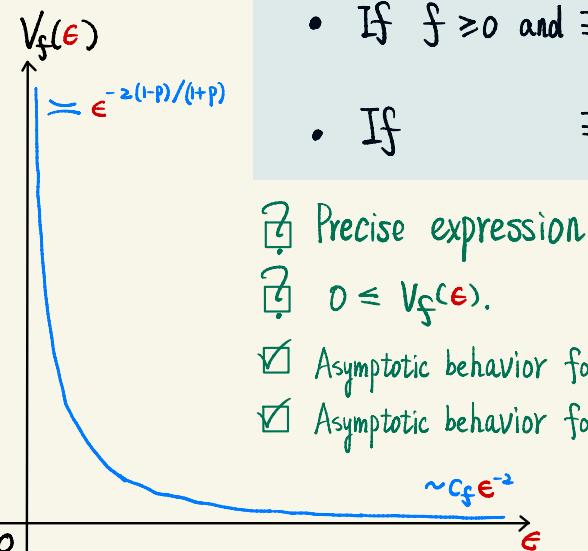
$\forall f \in C([0,1])$ satisfying a "slightly stronger" condition than $\sup_{u \in (0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$,

\exists a deterministic $C_f \in \mathbb{R}$ s.t. $\lim_{\epsilon \rightarrow +\infty} \epsilon^2 V_f(\epsilon) = C_f$.

Mytnik, Barnes & S. (2021+)

$\forall f \in C([0,1])$ s.t. $\sup_{u \in (0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$:

- If $f \geq 0$ and $\exists \frac{1}{2} \leq p \leq 1$ s.t. $\liminf_{u \downarrow 0} u^{-p} f(u) > 0$, then $\liminf_{\epsilon \downarrow 0} \epsilon^{\frac{1-p}{1+p}} V_f(\epsilon) > 0$;
- If $\exists \frac{1}{2} \leq p \leq 1$ s.t. $\limsup_{u \downarrow 0} u^{-p} f(u) < +\infty$, then $\limsup_{\epsilon \downarrow 0} \epsilon^{\frac{1-p}{1+p}} V_f(\epsilon) < +\infty$.



☐ Precise expression.

☐ $0 \leq V_f(\epsilon)$.

☐ Asymptotic behavior for small ϵ .

☐ Asymptotic behavior for large ϵ .

For example, if $f(u) = u^p(1-u)$ with $\frac{1}{2} \leq p \leq 1$, then

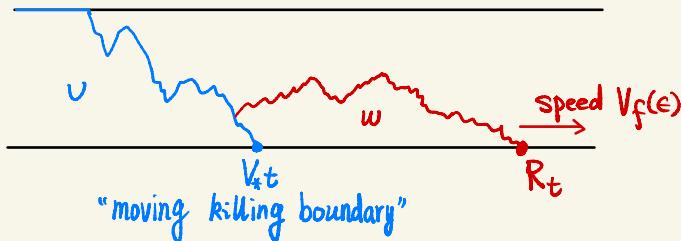
$\exists c, C > 0$ & $\epsilon_0 > 0$ s.t.

$$c \epsilon^{-\frac{1-p}{1+p}} \leq V_f(\epsilon) \leq C \epsilon^{-\frac{1-p}{1+p}}, \quad 0 < \epsilon \leq \epsilon_0.$$

Same exponent, can not be improved.

Step 1: Choose a killing boundary.

Decompose $u = v + w$ where $\begin{cases} \partial_t v = \frac{1}{2} \partial_x^2 v + f(v) + \epsilon \sqrt{v(1-v)} \dot{W}', \\ v = 0 \end{cases}, \quad \begin{array}{l} x < V_* t \\ x \geq V_* t \end{array}$



If we choose: left to be chosen

- $V_* \gg V_f(\epsilon)$ $\xrightarrow{\text{then}}$ w will be small,
- $V_* \ll V_f(\epsilon)$ $\xrightarrow{\text{then}}$ w will be large.

Idea: If $\exists V_*$ st. w is neither too small nor too large,
Then $V_* \approx V_f(\epsilon)$. Criticality

An insight: The balanced value for V_* can be predicted by finding the solution (F, V_*) so that

$$\begin{cases} F'(0-) = \epsilon^2, \\ F(-\infty) = 1, \\ F(t, x) := F(x - V_* t), \quad \forall (t, x), \\ \partial_t F = \frac{1}{2} \partial_x^2 F + f(F), \quad x < V_* t, \\ F = 0, \quad x \geq V_* t. \end{cases}$$

phase plane (PDE) argument

$$V_* \sim \epsilon^{-2 \frac{1-p}{1+p}}.$$

Only a prediction.

Still need to analyze w when $V_* \sim \epsilon^{-2 \frac{1-p}{1+p}}$
on next page...

Step 2: Analyze W .

$A_t = \int_0^t \dot{A}_s ds$ is the total mass of v killed at the boundary up to time t .

From $\partial_t u = \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)} \dot{W}$

subtract $\partial_t v = \partial_x^2 v + f(v) + \epsilon \sqrt{v(1-v)} \dot{W}'$

We get $\partial_t w = \partial_x^2 w + f(u) - f(v) + \epsilon \sqrt{u(1-u)} \dot{W} - \epsilon \sqrt{v(1-v)} \dot{W}' + \dot{A}_t \delta_{Vxt}(x)$.

Approximately $\partial_t w \approx \partial_x^2 w + c_f w^p + \epsilon \sqrt{w} \dot{W}'' + \dot{A}_t \delta_{Vxt}(x)$

Girsanov transformation $\partial_t w \approx \partial_x^2 w + \epsilon \sqrt{w} \dot{W}'' + \dot{A}_t \delta_{Vxt}(x)$

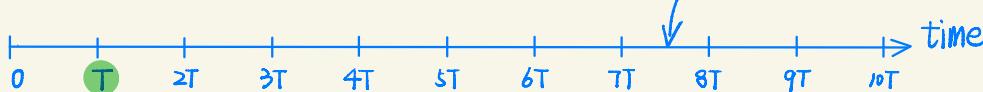
critical super-Brownian motion with immigration



Girsanov transformation doesn't preserve long time behavior.

properties of support.

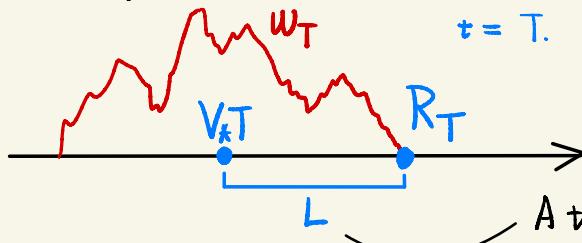
So we perform the transformation on each of the small intervals $[nT, (n+1)T]$.



left to be chosen
on next page...

$\left\{ \begin{array}{l} \text{can't be too large, otherwise: Transformed } w \not\approx \text{Original } w. \\ \text{can't be too small, otherwise: Small time random fluctuation } \xrightarrow{\text{Covers}} \text{info. of propagation speed.} \end{array} \right.$

Step 3 : Choose T .



A typical distance for the support of w to travel in a time interval with length T .

We want to choose T s.t.:

- L can be explained by the thermal diffusivity, i.e. $L \sim \sqrt{T}$.
- w does not give excess speed, i.e. $L/T \approx V_* \sim e^{-2\frac{H^2}{1+P}}$.
- T is as small as possible $\xleftarrow{\text{in order}}$ Transformed $w \approx$ Original w .

$$\Rightarrow T \sim e^{4\frac{1-P}{1+P}}.$$

Further questions:



$\lim_{u \downarrow 0} u^{-p} f(u)$ exists $\Rightarrow \lim_{\epsilon \downarrow 0} \epsilon^{\frac{1-p}{1+p}} V_f(\epsilon)$ exists.



Duality for $\partial_t u = \frac{1}{2} \partial_x^2 u + u^p(1-u) + \epsilon \sqrt{u(1-u)} \dot{W}$ ($0 < p < 1$).



Weak uniqueness for $\partial_t u = \frac{1}{2} \partial_x^2 u + u^p(1-u) + \epsilon \sqrt{u(1-u)} \dot{W}$ ($0 < p < \frac{1}{2}$).



Speed of L_t = Speed of R_t ($\stackrel{\text{duality}}{=}$ Speed of Branching-coalescing Brownian motion).



$(U_t(R_t+x) : x \in \mathbb{R})$ $\xrightarrow[t \rightarrow +\infty]{\text{weakly}}$ a stationary (random) wave profile.



Speed of N-BBM & L-BBM with heavy-tailed offspring law (universality).

This talk is based on ongoing joint work with



&



Leonid Mytnik

Clayton Barnes .

Manuscript will soon available on arXiv.

Thanks !!