

On the coming down from infinity of Shiga's coalescing Brownian Motions

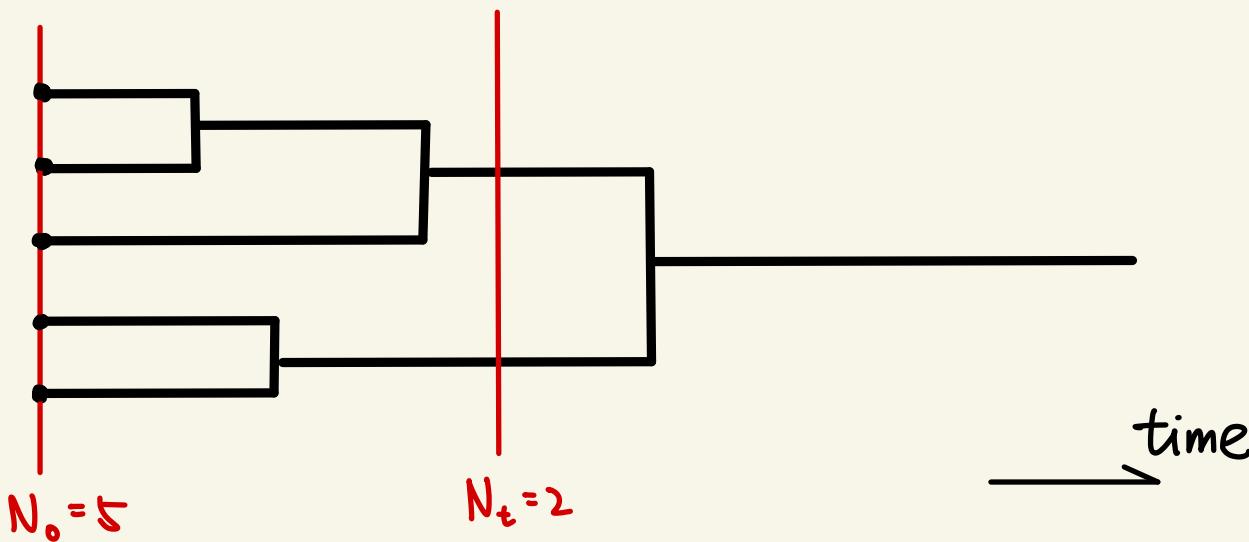
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kingman (1982)

kingman's coalescent

Consider a system of finitely many particles where each pair of particles coalesces into one particle with rate 1, independent of other pairs. Then the total number of particles $(N_t)_{t \geq 0}$ is a continuous time Markov chain on \mathbb{N} which jumps from n to $n-1$ with rate $n(n-1)/2$.



a graphical construction.

Aldous (1999)

kingman's coalescent can come down from $+\infty$, i.e. \exists a \mathbb{N} -valued Markov chain $(N_t^\infty)_{t \geq 0}$ which evolves as kingman's coalescent and satisfies $t N_t^\infty \rightarrow 2$ when $t \downarrow 0$ in probability.

equal in law.

Berestycki, Berestycki & Limic (2010)

There exists a process $(N_t^{(n)})_{t \geq 0, n \in \mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$, $(N_t^{(n)})_{t \geq 0}$ is a kingman's coalescent with $N_0^{(n)} = n$; and $\forall t \geq 0$, $N_t^{(n)} \leq N_t^{(n+1)}$. The increasing limit $N_t^\infty := \lim_{n \uparrow \infty} N_t^{(n)}$ satisfies $t N_t^\infty \rightarrow 2$ when $t \downarrow 0$ a.s. and in L^p for $p \geq 1$.

Constructed using partitions of \mathbb{N} .

Spatial Kingman's coalescent is a system of particles living on a Graph G s.t.

- At each site, particles coalesce according to Kingman's coalescent.
- Each particle moves as an independent random walk on G .

$N_t(v) :=$ number of particles at site v and time t .

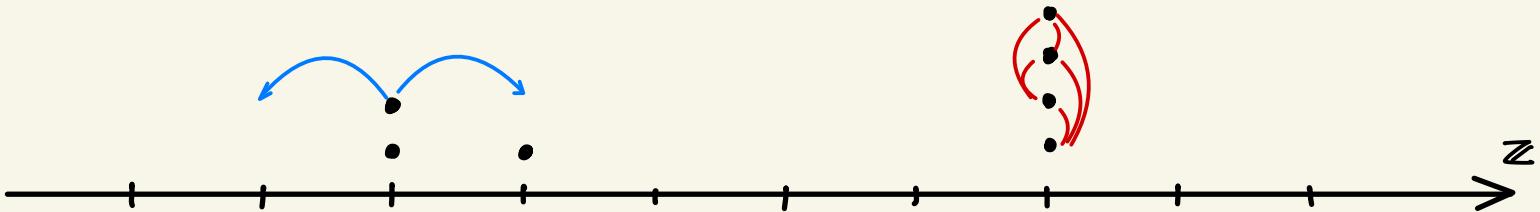
Then $(N_t)_{t \geq 0}$ is a Markov Process.

One can construct a process $(N_t^{(n)})_{t \geq 0, n \in \mathbb{N}}$ s.t.

- $\forall n \in \mathbb{N}$, $(N_t^{(n)})_{t \geq 0}$ is a spatial Kingman's coalescent with n initial particles located at a distinguished root site;
- $\forall t \geq 0$, $N_t^{(n)}(v) \leq N_t^{(n+1)}(v)$, $v \in G$, $n \in \mathbb{N}$.

Define $N_t^{(\infty)}(v) := \lim_{n \uparrow \infty} N_t^{(n)}(v)$ and the total population $N_t^{(\infty)}(G) := \sum_{v \in G} N_t^{(\infty)}(v)$.

Spatial movement



coalescent

Limic & Sturm (2006)

If G is a finite graph, then $\mathbb{P}(N_t^{(\infty)}(G) < +\infty) = 1, \forall t > 0$.

Angel, Berestycki & Limic (2012)

If G is an infinite connected graph with bounded degree,

then $\mathbb{P}(N_t^{(\infty)}(G) = +\infty) = 1, \forall t \geq 0$.



Is there an analogy of Kingman's coalescent in the continuum spatial setting?

The Wright-Fisher diffusion

Well-known duality

Let $(N_t)_{t \geq 0}$ be Kingman's coalescent with initial value $N_0 = n$.

Let $(X_t)_{t \geq 0}$ be the solution of the stochastic differential equation

$$dX_t = \frac{1}{2n} \sqrt{X_t(1-X_t)} dB_t; \quad X_0 = p \in [0,1].$$

Then $\mathbb{E}[(1-X_0)^{N_t}] = \mathbb{E}[(1-X_t)^{N_0}]$.

Brownian motion.



Is there an analogy of Kingman's coalescent
in the continuum spatial setting?

Yes!

Shiga (1988)

Shiga's coalescing Brownian motions.

Consider a system of finitely many Brownian particles on \mathbb{R} where each pair of particles coalesces into one particle with rate $1/2$ according to their intersection local time.

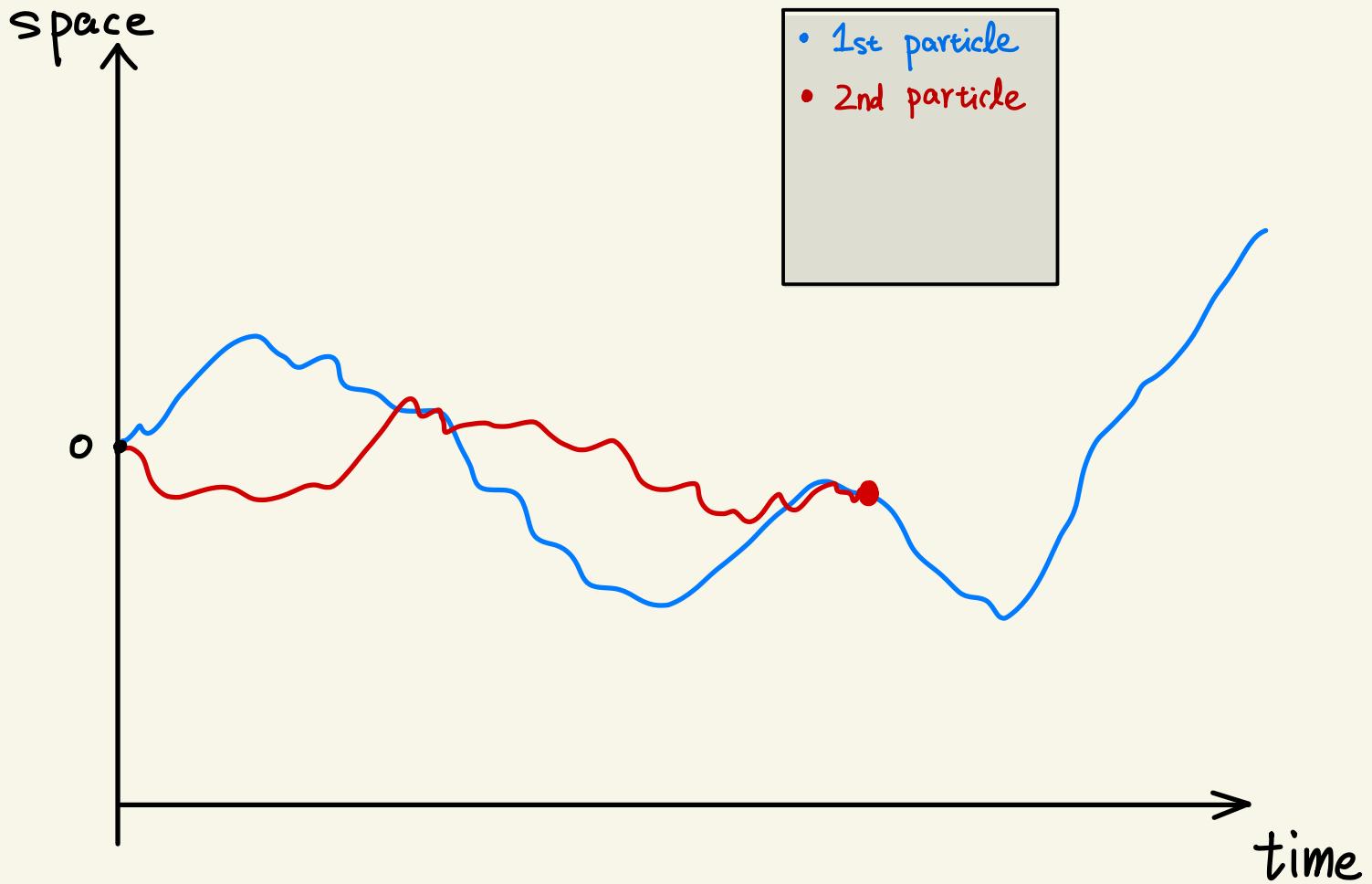
Denote by $(B_t^{(i)})_{i \in I_t}$ the positions of the particles at time t .

Let $(u_t(x))_{t \geq 0, x \in \mathbb{R}}$ be the solution to the stochastic partial differential equation

$$\partial_t u_t(x) = \frac{1}{2} \partial_x^2 u_t(x) + \sqrt{u_t(x)(1-u_t(x))} \dot{W}_{t,x}, \quad u_0 = f \in C(\mathbb{R}, [0, 1]).$$

$$\text{Then } \mathbb{E}\left[\prod_{i \in I_t} (1 - u_0(B_t^{(i)}))\right] = \mathbb{E}\left[\prod_{i \in I_0} (1 - u_t(B_0^{(i)}))\right].$$

Space-time white noise



?

Can we initiate shiga's coalescing Brownian motions with infinitely many initial particles? Yes!

Tribe's construction (1995)

of shiga's coalescing Brownian motions $(Z_t)_{t \geq 0}$ with initial configuration $(X_i)_{i \in \mathbb{N}} \subset \mathbb{R}$:

- Let $(B_t^{(i)})_{i \in \mathbb{N}}$ be a sequence of independent Brownian motions s.t. $\forall i \in \mathbb{N}, B_0^{(i)} = X_i$.
- Let $(e^{(i)})_{i \in \mathbb{N}}$ be a family of independent exponential random variable with mean 2.
- Let $(L_t^{(i,j)})_{t \geq 0}$ be the local time of $(B_t^{(i)} - B_t^{(j)})_{t \geq 0}$ at position 0 for $i, j \in \mathbb{N}$.
- Define $\varsigma_1 := +\infty$, and inductively $\forall i \geq 1$,

$$\varsigma_i := \inf \left\{ t \geq 0 : \sum_{j < i} L_{t \wedge \varsigma_j}^{(i,j)} \geq e^{(i)} \right\}. \quad (\text{Life time of } i\text{th particle})$$

- Define $Z_t^{(n)} := \sum_{i=1}^n \mathbf{1}_{t \in [0, \varsigma_i]} \delta_{B_t^{(i)}}, \quad t \geq 0;$ $Z_t := \sum_{i=1}^{+\infty} \mathbf{1}_{t \in [0, \varsigma_i]} \delta_{B_t^{(i)}}, \quad t \geq 0.$

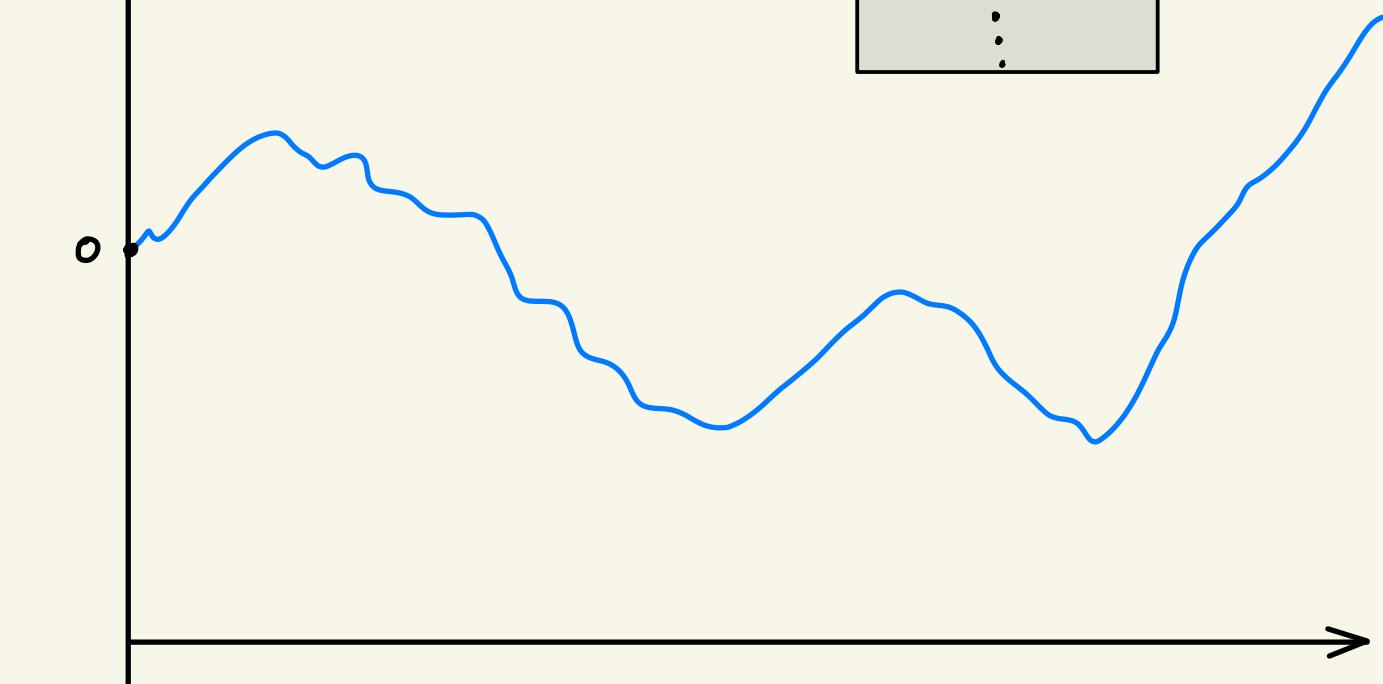
Space

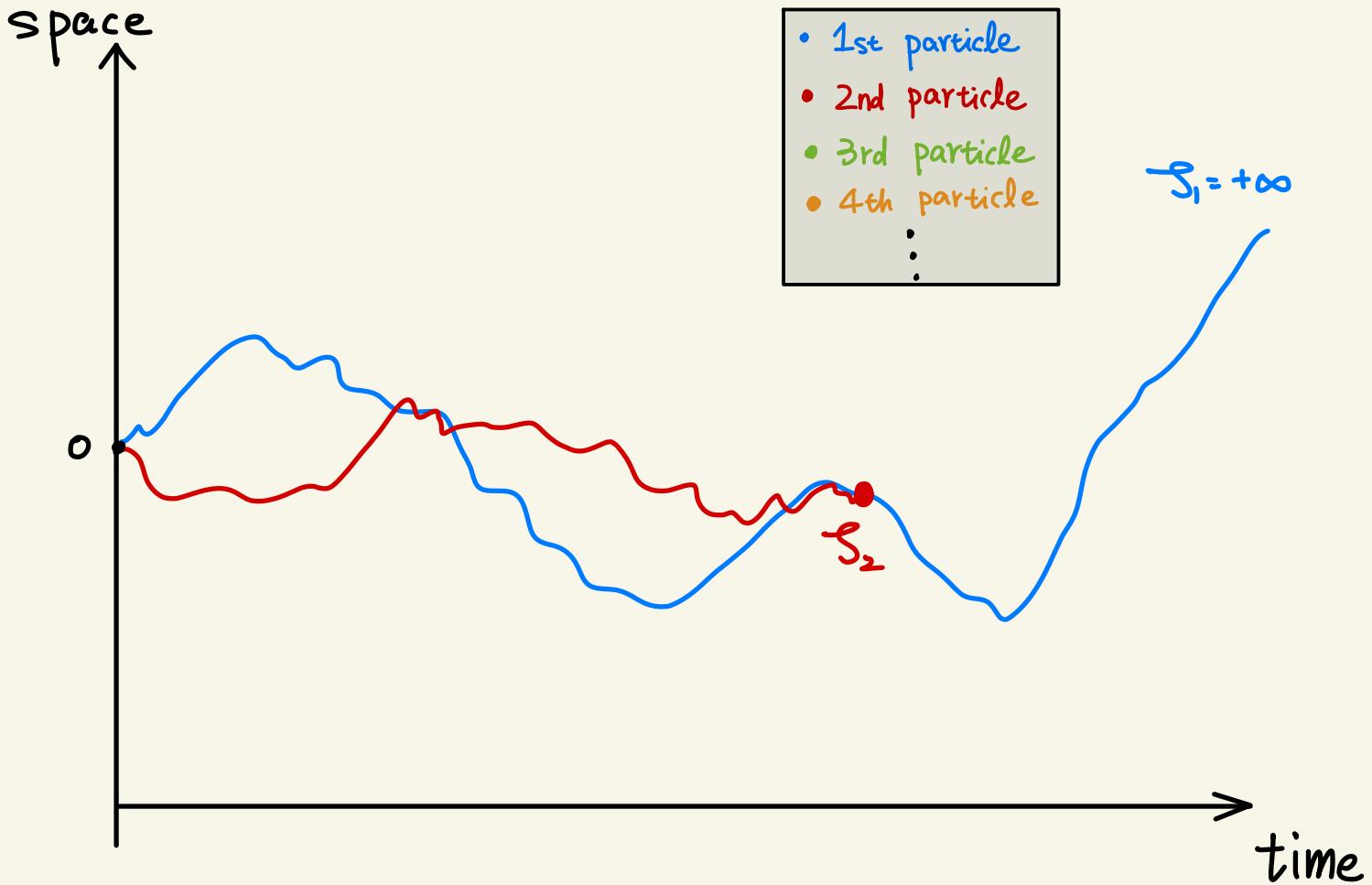
- 1st particle
- 2nd particle
- 3rd particle
- 4th particle
- ⋮

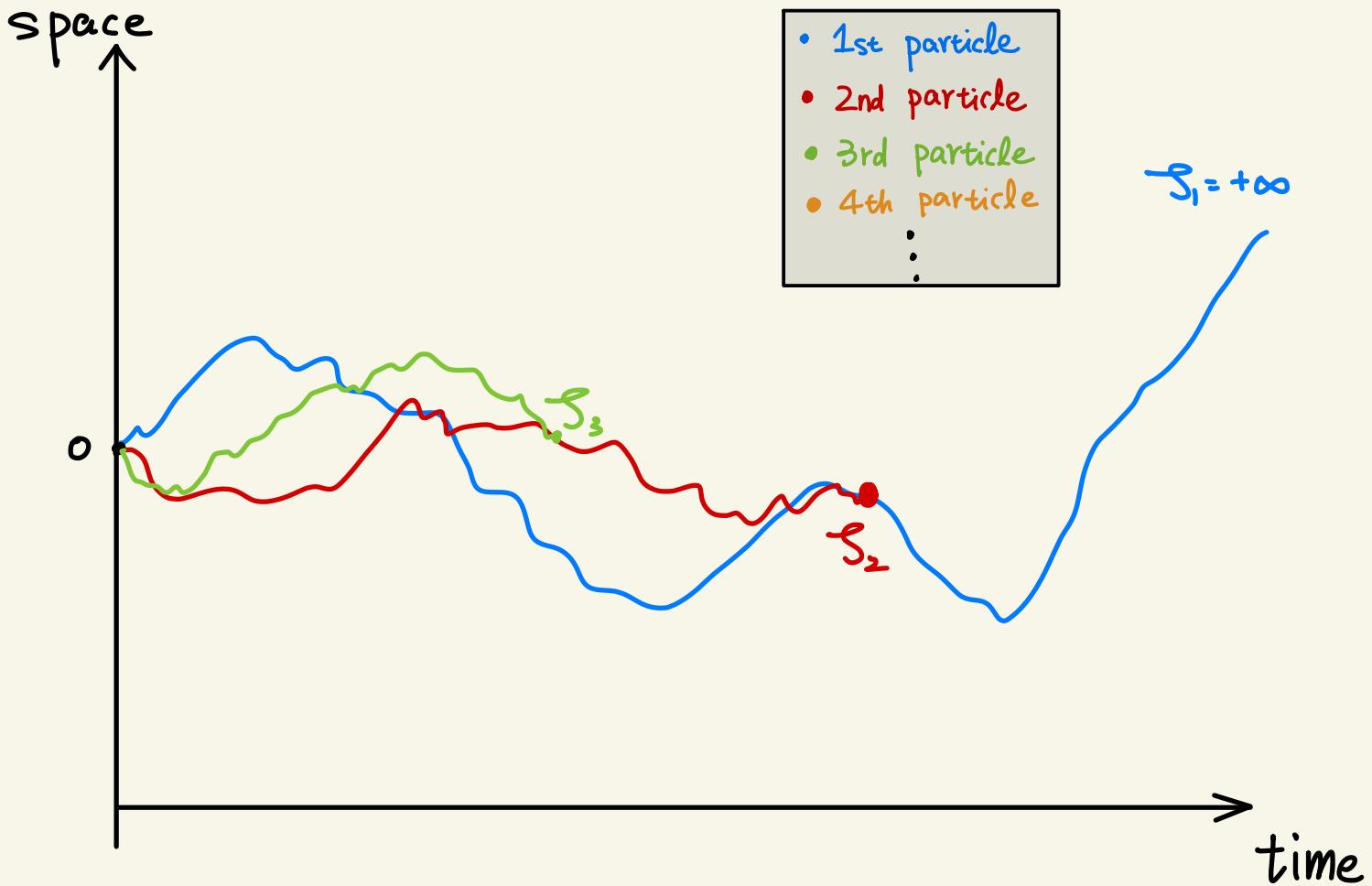
$$S_1 = +\infty$$

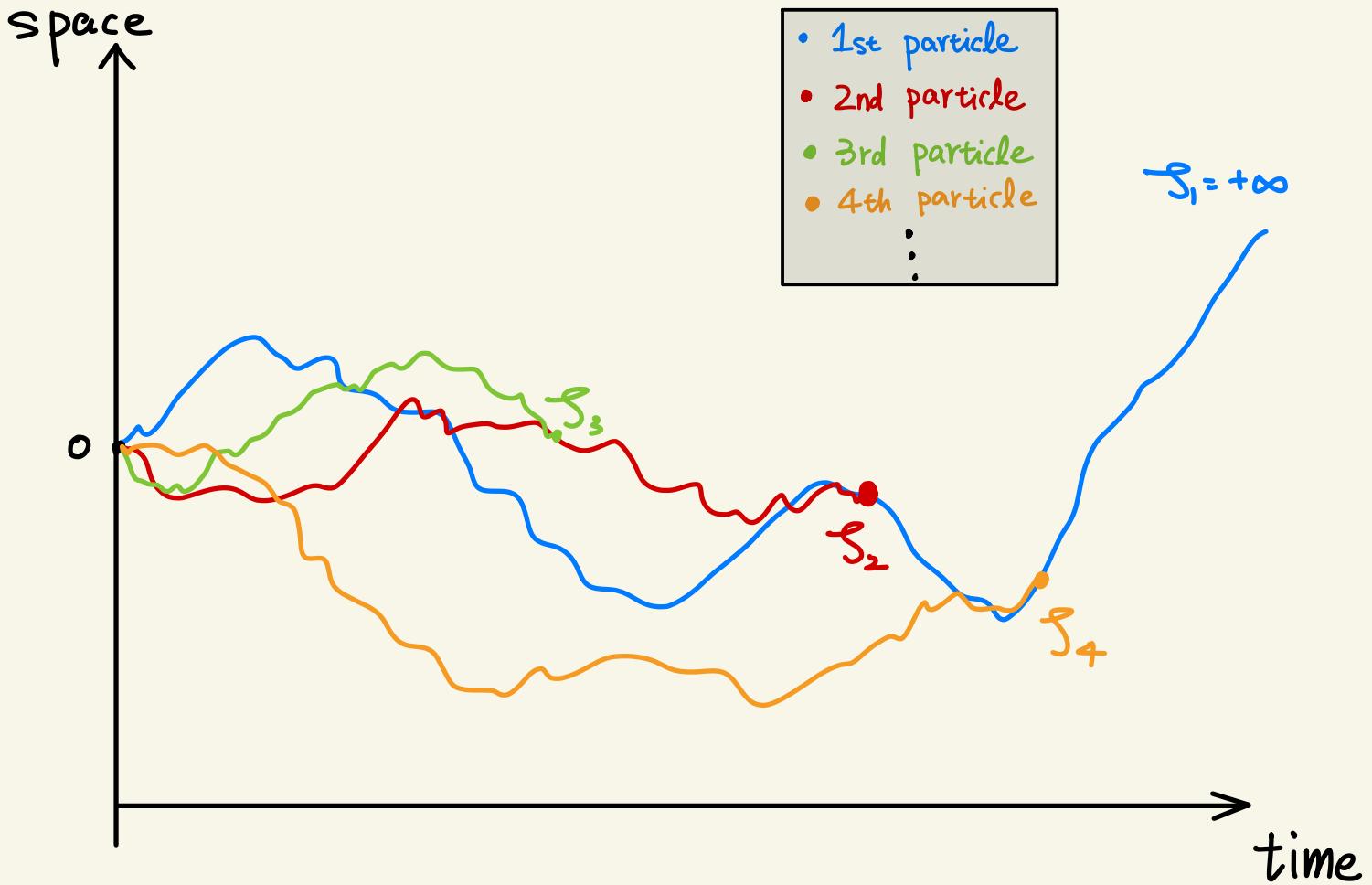
0

time











Does shiga's coalescing Brownian motions
have the property of coming down from infinity?

Hobson & Tribe (2005)

Consider shiga's coalescing Brownian motions on the unit length circle S with initial configuration $(X_i)_{i \in \mathbb{N}}$ sampled as i.i.d uniform r.v.
Then the total number of particles $Z_t(S)$ is finite for $\forall t > 0$ a.s.
Moreover, $t Z_t(S) \rightarrow 2$ in probability when $t \downarrow 0$.



Does shiga's coalescing Brownian motions on \mathbb{R}
have the property of coming down from infinity?

Barnes, Mytnik & S. (ongoing)

Consider shiga's coalescing Brownian motions on \mathbb{R}

with arbitrary (deterministic) initial configuration $(x_i)_{i \in \mathbb{N}} \subset \mathbb{R}$.

Let $U \subset \mathbb{R}$ be an arbitrary open interval.

- (1) If $\{x_i : i \in \mathbb{N}\} \cap U$ is bounded, then $P(Z_t(U) < +\infty, \forall t > 0) = 1$.
- (2) If $\{x_i : i \in \mathbb{N}\} \cap U$ is unbounded, then $P(Z_t(U) = +\infty, \forall t \geq 0) = 1$.

?

What are the rate of coming down from infinity for Shiga's coalescing Brownian motions?

The initial trace of the solution v .

Le Gall (1996)

\forall closed subset $\Lambda \subset \mathbb{R}$, and \forall non-negative Radon measure μ on Λ^c ,
 \forall there exists a unique non-negative $v = v^{(\Lambda, \mu)} \in C^{1,2}((0, +\infty) \times \mathbb{R})$ s.t.

$$\left\{ \begin{array}{l} \partial_t v = \frac{1}{2} \partial_x^2 v - \frac{1}{2} v^2 \quad , \quad \forall t > 0, x \in \mathbb{R}, \\ \Lambda = \left\{ y \in \mathbb{R} : \forall r > 0, \lim_{t \downarrow 0} \int_{y-r}^{y+r} v_{t,x} dx = +\infty \right\}, \\ \int_{\Lambda^c} \phi(x) \mu(dx) = \lim_{t \downarrow 0} \int_{\Lambda^c} \phi(x) v_{t,x} dx, \quad \forall \phi \in C_c(\Lambda^c). \end{array} \right.$$

Represented using the Brownian snake.

Let $(Z_t)_{t \geq 0}$ be Shiga's coalescing Brownian motions with initial configuration $(x_i)_{i \in \mathbb{N}}$.

Define $\Lambda := \{y \in \mathbb{R} : \forall r > 0, Z_0(y-r, y+r) = +\infty\}$ and $\mu = Z_0|_{\Lambda^c}$.

Barnes, Mytnik & S. (ongoing)

Let U be an arbitrary open interval.

If $\{x_i : i \in \mathbb{N}\} \cap U$ is bounded, then $E[Z_t(U)] < +\infty, \forall t > 0$;

Furthermore,

- if $\Lambda \cap \bar{U} = \emptyset$, then $\limsup_{t \downarrow 0} E[Z_t(U)] < +\infty$;
- if $\Lambda \cap \bar{U} \neq \emptyset$, then $\lim_{t \downarrow 0} E[Z_t(U)] = +\infty$, &

$$\frac{Z_t(U)}{\int_U v_{t,x}^{(\Lambda, \mu)} dx} \rightarrow 1 \quad \text{in } L' \text{ when } t \downarrow 0.$$

Corollary

If $\{X_i : i \in \mathbb{N}\}$ is unbounded, then $P(Z_t(\mathbb{R}) = +\infty, \forall t \geq 0) = 1$.

If $\{X_i : i \in \mathbb{N}\}$ is bounded, then $E[Z_t(\mathbb{R})] < +\infty, \forall t > 0$; $\lim_{t \downarrow 0} E[Z_t(\mathbb{R})] = +\infty$; &

$$\frac{Z_t(\mathbb{R})}{\int v_{t,x}^{(\lambda, \mu)} dx} \rightarrow 1 \quad \text{in } L' \text{ when } t \downarrow 0.$$

?

What more exactly is the rate of coming down from infinity?
(What is the behavior of $\int_{\mathbb{R}} V_{t,x}^{(-L,\mu)} dx$ when $t \downarrow 0$?)

Example 1

If $X_i = 0$ for every $i \in \mathbb{N}$, then

$$\sqrt{t} Z_t(\mathbb{R}) \rightarrow C := \int V_{1,x}^{(t \downarrow 0, \text{null})} dx \text{ in } L' \text{ when } t \downarrow 0.$$

Example 2

If $\{X_i : i \in \mathbb{N}\}$ is a dense subset of $[0, 1]$, then

$$\sqrt{t} Z_t(\mathbb{R}) \rightarrow 2 \text{ in } L' \text{ when } t \downarrow 0.$$

parallel result of Hobson & Tribe (2005)

?

For $\frac{1}{2} < \alpha < 1$, does there exist initial configuration $(x_i)_{i \in \mathbb{N}}$ so that the total population $Z_t(\mathbb{R})$ behaves like $t^{-\alpha}$ as $t \downarrow 0$?

$\forall A \subset \mathbb{R}$,

- define A 's γ -neighborhood

$$A_r := \{y \in \mathbb{R} : \exists x \in A, |y-x| < r\} \text{ for every } r > 0;$$

- we say A has Minkowski dimension $\delta \in [0, 1]$, if

$$\frac{\log \text{Leb}(A_r)}{\log r} \rightarrow 1 - \delta \quad \text{as } r \downarrow 0;$$

- when A has Minkowski dimension $\delta \in [0, 1]$, we say it is Minkowski measurable with Minkowski content $k \in (0, +\infty)$, if

$$\frac{\text{Leb}(A_r)}{r^{1-\delta}} \rightarrow k \quad \text{as } r \downarrow 0.$$

Barnes, Mytnik & S. (ongoing)

Suppose that $(x_i)_{i \in \mathbb{N}}$ is bounded, without isolated points ($\mu = 0$).

Suppose that Λ has Minkowski dimension $\delta \in [0, 1]$. Then

$$\frac{\log Z_t(R)}{\log t} \rightarrow -\frac{1+\delta}{2} \quad \text{in probability as } t \downarrow 0.$$

Conjecture

Further suppose that Λ is Minkowski measurable

with Minkowski content $k \in (0, +\infty)$, then

$\exists C(\delta) > 0$, depending only on δ , s.t.

$$t^{\frac{1+\delta}{2}} Z_t(R) \rightarrow C(\delta) k \quad \text{in } L' \text{ as } t \downarrow 0.$$

Shiga's coalescing Brownian motions.

$$\xleftarrow{\text{Shiga's duality}} \text{Shiga (1988)}$$

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \sqrt{u(1-u)} W$$

“Similarity”

Rate of Coming down from infinity

$$\partial_t \tilde{u} = \frac{1}{2} \partial_x^2 \tilde{u} + \sqrt{\tilde{u}} W$$

Konno & Shiga (1988)
Reimers (1989)

density
super-Brownian motion

Le Gall (1994)
genealogical structure

Brownian snake

$$\xrightarrow{\text{Probabilistic representation}} \text{Le Gall (1996)}$$

$$\begin{cases} \partial_t v = \frac{1}{2} \partial_x^2 v - \frac{1}{2} v^2 \\ \text{Singular points of } v_0 = \lambda \\ v_0|_{\lambda^c} = \mu \end{cases}$$

2 Similarity between u & \tilde{u} ?

Consider the case when $x_i = 0$ for every $i \in \mathbb{N}$.

Let u & \tilde{u} be the weak solution to SPDEs ($0 < \varepsilon < 1$)

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \sqrt{u(1-u)} \dot{W}, \quad u_0 \equiv \varepsilon,$$

$$\partial_t \tilde{u} = \frac{1}{2} \partial_x^2 \tilde{u} + \sqrt{\tilde{u}} \dot{W}, \quad \tilde{u}_0 \equiv \varepsilon.$$

Shiga's duality $\Rightarrow \mathbb{E}\left[(1-\varepsilon)^{\tilde{Z}_t(\mathbb{R})}\right] = \mathbb{P}(u_{t,0}=0).$

Standard result for Super-Brownian motion

$$\Rightarrow \exp(-\varepsilon \int v_{t,x}^{(f_0,0)} dx) = \mathbb{P}(\tilde{u}_{t,0}=0).$$

We can argue using SPDE tools that

$$|\mathbb{P}(u_{t,0}=0) - \mathbb{P}(\tilde{u}_{t,0}=0)| \lesssim \varepsilon$$

for small ε and t .

This is based on my joint ongoing work with



Clayton Barnes

&



Leonid Mytnik

Thanks !!