Wright-Fisher stochastic heat equations with irregular drifts

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July, 2025

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Regularization by noise

- Consider the differential equation:

$$\begin{cases} \mathrm{d}X_t = b(X_t)\mathrm{d}t = |X_t|^\alpha \mathrm{d}t, & t > 0, \\ X_0 = 0, \end{cases}$$

  where $\alpha \in (0,1)$.
- The drift $b(x) = |x|^\alpha$ is not Lipschitz at 0
  $\implies$ non-uniqueness of the solutions.
- One solution $X_t \equiv 0$.
- The other solution $X_t = C_\alpha t^{\frac{1}{1-\alpha}}, t \geq 0$.

Regularization by non-degenerate noise

Zvonkin (1974, Mat. Sb. (N.S.)), Veretennikov (1979, Mat. Sb. (N.S.))

Suppose that

- b is a bounded measurable function, and
- B is a Brownian motion,

then there exists a unique strong solution to the SDE

$$\begin{cases} \mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t, \quad t > 0, \\ X_0 = x \in \mathbb{R}. \end{cases}$$

- Zvonkin's transform is not available for SPDEs.

Partial regularization effect by degenerate noise

- Uniqueness in law for one-dimensional SDE can be analyzed by Feller's test.
- For example, consider non-negative solutions to the SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sqrt{2X_t}\mathrm{d}B_t; \quad X_0 = 0$$

where, with $\alpha > 0$ and $\beta > 0$,

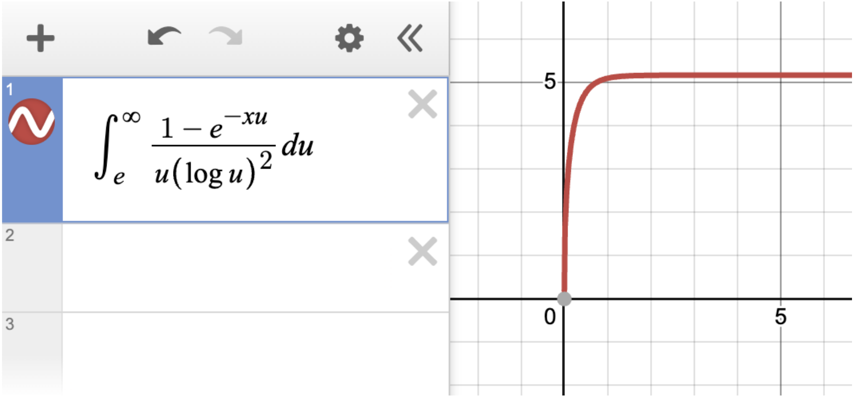
$$b(x) := \int_e^\infty \frac{1 - e^{-xu}}{\alpha u (\log u)^{1+\beta}} \mathrm{d}u, \quad x \geq 0.$$

Clement (2019, Electron. J. Probab.)

- If $\beta > 1$, the uniqueness in law holds;
- If $\beta = 1$ and $\alpha \geq 1$, the uniqueness in law holds;
- If $\beta = 1$ and $\alpha < 1$, the uniqueness in law fails;
- If $\beta < 1$, the uniqueness in law fails.

Regularization by multiplicative noise

- The shape of a "critical" drift $b(x)$:



Wright-Fisher Stochastic Heat Equations (Wright-Fisher SHE)

- Quasi-linear heat equation perturbed by the Wright-Fisher space-time white noise

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + b(u) + \sqrt{|u(1-u)|}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

- $\dot{W}$ is the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, i.e. a centered Gaussian process with $\mathbb{E}[\dot{W}_t(x)\dot{W}_s(y)] = \delta_0(t-s)\delta_0(x-y)$.
- The noise coefficient $\sqrt{|u(1-u)|}$
  - is non-Lipshitz at $u = 0$ and $u = 1$; and
  - is degenerate at $u = 0$ and $u = 1$.
- Challenging open problems:
  - the strong uniqueness?
  - the solution theory in higher dimensions?
- Question: How strong is the regularization effect of the Wright-Fisher white noise?

Motivation

- Shiga (1988, Math. Appl.): Wright-Fisher SHE = scaling limit of "genetic stepping stone model."
  - $b(u) = c_1(1-u) - c_2 u + c_3 u(1-u)$.
  - $c_1 \geq 0$ and $c_2 \geq 0$ are called the mutation rates.
  - $c_3 \in \mathbb{R}$ is called the selection rate.
- Mueller-Tribe (1995, Probab. Theory Related Fields), Durrett-Fan (2016, Ann. Appl. Probab.): Wright-Fisher SHE = scaling limit of (biased) voter model.
  - $b(u) = c_3 u(1-u)$.
  - Unbiased $\implies c_3 = 0$.
- Brunet-Derrida (1997, Phys. Rev. E), Mueller-Mytnik-Quastel (2011, Invent. Math.): The Wright-Fisher SHE is the key to the proof of the Brunet-Derrida conjecture.

Weak existence

- Fix intitial value $f \in \mathcal{C}(\mathbb{R}, [0,1])$.

Shiga (1994, Can. J. Math.)

If $b(\cdot)$ is continuous and $b(0) \geq 0 \geq b(1)$, then the weak existence holds for the SPDE

$$(*) \quad \begin{cases} \partial_t u = \frac{\Delta}{2} u + b(u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

That is, there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, a space-times white noise $\dot{W}$, and an adapted continuous $\mathcal{C}(\mathbb{R}, [0,1])$-valued process $(u_t)_{t\geq 0}$, such that $(*)$ holds in an analytically weak sense.

- Question: What if $b(\cdot)$ is discontinuous? Even the existence is not clear.

Uniqueness in law: Duality Method

- uniqueness in law = the probability law induced by $(u_t)_{t\geq 0}$ on the path sapce $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}, [0,1]))$ is unique.

Shiga (1988, Math. Appl.)

The uniqueness in law of $(*)$ holds provided $b(u) = c_1(1-u) - c_2 u + c_3 u(1-u)$ where $c_1 \geq 0, c_2 \geq 0$ and $c_3 \in \mathbb{R}$.

Athreya-Tribe (2000, Ann. Probab.)

The uniqueness in law of $(*)$ holds provided

$$b(u) = \sum_{k=0}^{\infty} b_k u^k, \text{ and } b_1 < -\sum_{k=0, k\neq 1}^{\infty} |b_k| R^{k-1} \text{ for some } R > 1.$$

- Both Shiga (1988) and Athreya-Tribe (2000) used the duality method.
- The drifts are Lipshitz functions.

Uniqueness in law: Girsanov transformation

Mueller-Mytnik-Ryzhik (2021, Comm. Math. Phys.)

The uniqueness in law holds provided b is continuous,

$$\sup_{u\in(0,1)} \frac{|b(u)|}{\sqrt{u(1-u)}} < \infty, \text{ and } f(x) = 1 - f(-x) = 0 \text{ for large enough } x.$$

- When the red part holds, we say the initial value f has a compact interface.
- The main tool is Girsanov transformation.
- The drift can be a non-Lipshitz Hölder continuous function.

Main Result

- Recall the condition in Athreya-Tribe (2000):

$$b(u) = \sum_{k=0}^{\infty} b_k u^k, \text{ and } b_1 < - \sum_{k \in \{0\} \cup \mathbb{N} \setminus \{1\}} |b_k| R^{k-1} \text{ for some } R > 1.$$

Barnes-Mytnik-S. (2025+, to appear in Probab. Theory Related Fields)

The weak existence and uniqueness in law holds for $(*)$ provided

$$b(u) = \sum_{k \in \{0, \infty\} \cup \mathbb{N}} b_k u^k = \sum_{k=0}^{\infty} b_k u^k + b_\infty \mathbf{1}_{\{1\}}(u)$$

$$\text{with } b_1 \leq - \sum_{k \in \{0, \infty\} \cup \mathbb{N} \setminus \{1\}}^{\infty} |b_k| R^{k-1} \text{ for some } R \geq 1.$$

Examples of Hölder drift

Consider the Wright-Fisher SHE with Hölder continuous drift:

$$(2) \qquad \begin{cases} \partial_t u = \frac{\Delta}{2} u + u^{\alpha}(1-u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

According to	uniqueness in law holds for (2) provided
Shiga (1988) or Athreya-Tribe (2000)	$\alpha = 1$
Mueller-Mytnik-Ryzhik (2021)	$\alpha \in [\frac{1}{2}, 1]$ and f has compact interface
Barnes-Mytnik-S. (2025+)	$\alpha \in (0, 1]$

This is expected, since the uniqueness in law holds for the SDE

$$\mathrm{d}X_t = X_t^{\alpha}(1 - X_t)\mathrm{d}t + \sqrt{X_t(1 - X_t)}\mathrm{d}B_t; \quad X_0 = x \in [0, 1].$$

What if "$\alpha = 0$"? Pay attention that $u^{\alpha}(1-u)$ converges to the discontinuous drift $(1-u) - \mathbf{1}_{\{0\}}(u)$ when $\alpha \downarrow 0$.

Weak existence is non-trivial for discontinous drifts

- Denote by $u^{(\alpha)}$ the solution to the SPDE (2) with parameter α.
- It is standard to verify that the family of random elements $\{u^{(\alpha)} : \alpha \in (0,1]\}$ is tight in the path space $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}, [0,1]))$.
- By Skorohod's embedding, we can assume WLOG that there exists a sequence $\alpha_n \downarrow 0$ such that almost surely $u := \lim_{n\to\infty} u^{(\alpha_n)}$ exists in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}(\mathbb{R}, [0,1]))$.
- However, $z_n \to z$ in $[0,1]$ does not necessarily imply that $z_n^{\alpha_n} \to z^0 = \mathbf{1}_{(0,1]}(z)$. For a counter example, consider $z_n := \exp(-\frac{\log 2}{\alpha_n}) \to 0 =: z$, but $z_n^{\alpha_n} = \frac{1}{2} \not\to \mathbf{1}_{(0,1]}(z) = 0$.
- So, from the standard "martingale problem argument", it is not clear if u solve the SPDE

$$\begin{cases} \partial_t u = \frac{\Delta}{2} u + (1-u) - \mathbf{1}_{\{0\}}(u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

Examples for discontinuous drifts

Nevertheless, our result implies the following:

Barnes-Mytnik-S. (2025+, to appear in Probab. Theory Related Fields)

For each $\delta \in [-1, 1]$, the weak existence and uniqueness in law hold for the SPDE

$$(3) \begin{cases} \partial_t u = \frac{\Delta}{2} u + (1-u) + \delta \mathbf{1}_{\{0\}}(u) + \sqrt{u(1-u)}\dot{W}, & x \in \mathbb{R}, t \geq 0, \\ u_0 = f, & x \in \mathbb{R}. \end{cases}$$

We can also show that δ is a relevant parameter!

Barnes-Mytnik-S. (2025+, to appear in Probab. Theory Related Fields)

Suppose that $f \not\equiv 1$. The distributions of the solution to the SPDE (3) are different for each $\delta \in [-1, 1]$.

Examples for discontinuous drifts

This is drastically different from the SDE

$$(4) \quad \mathrm{d}X_t = [(1 - X_t) + \delta \mathbf{1}_{\{0\}}(X_t)]\mathrm{d}t + \sqrt{X_t(1 - X_t)}\mathrm{d}B_t.$$

where δ is basically irrelevant.

Simple fact

When $\delta = -1$, the uniquess in law does not hold for the SDE (4). When $\delta \in (-1.1]$, the uniquess in law does hold, but the distributions of the solution to the SDE (4) are the same for different $\delta \in (-1, 1]$.

Insight: The Wright-Fisher noise has a very different regularizing effect in the SPDE setting compared to the SDE setting!

Overview

Here is an overview of what we know about the 1-D SHE

$$\partial_t u_t = \frac{\Delta}{2} u_t + b(u) + \sigma(u)\dot{W}.$$

$\sigma(u)$ \ $b(u)$	0	Lipschitz	Hölder	Discontinuous	Measurable
deterministic: 0	well-posed	well-posed	non-uniqueness for some drift	non-uniqueness for some drift	non-uniqueness for some drift
additive: 1	well-posed	well-posed	well-posed	well-posed	well-posed
Lipschitz & non-degenerate	well-posed	well-posed	well-posed	well-posed	well-posed
3/4-Hölder & non-degenerate	well-posed	weakly well-posed	weakly well-posed	?	?
Feller noise: $\sqrt{u}$	weakly well-posed	weakly well-posed for some drift	?	?	?
Wright-Fisher noise: $\sqrt{u(1-u)}$	weakly well-posed	weakly well-posed	weakly well-posed for some drift	weakly well-posed for some drift	?

Duality method

- We say two Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are dual to each other if there exists a large class of functions $H(x, y)$ such that

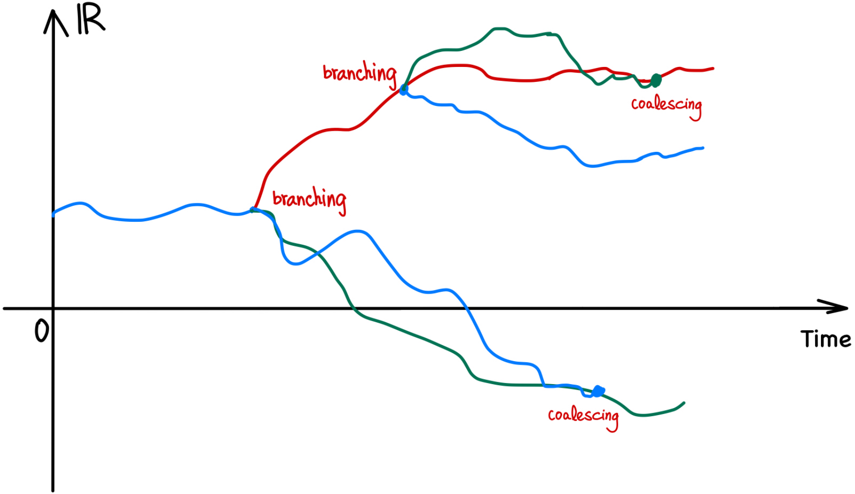
$$\mathbb{E}[H(X_t, Y_0)] = \mathbb{E}[H(X_0, Y_t)].$$

- Bachelier (1900, Ann. Sci. École Norm. Sup.):
  Brownian motion and the heat equation $\partial_t h = \frac{\Delta}{2} h$.
- McKean (1975, Comm. Pure Appl. Math.): Branching Brownian motion and the FKPP equation $\partial_t v = \frac{\Delta}{2} v + v(1-v)$.
- Harris (1978, Ann. Probab.):
  Coalescing random walk and the voter model.
- Shiga (1986, Math. Appl.): LCBM and the stochastic FKPP equation $\partial_t v = \frac{\Delta}{2} v + \sqrt{v(1-v)}\dot{W}$.
- Tóth-Werner (1998, Probab. Theory Relat. Fields):
  (Hard) Coalescing Brownian motions and itself.
- **Folklore**: Stochastic heat equation $\partial_t u = \frac{\Delta}{2} u + u\dot{W}$ and itself.
- ...

The Dual of the Wright-Fisher SHEs

- The dual of Wright-Fisher SHEs are coalescing-branching Brownian motions (CBBMs).
- Two parameters:
  - *Branching rate* $\mu > 0$.
  - *Offspring distribution* $(p_k)_{k \in \{0,\infty\} \cup \mathbb{N}}$.
- Three dynamics:
  - *Spatial movement:* Particle move as independent Brownian motions.
  - *Branching:* Each particle branches into a random number of particles with the rate μ. The offspring number is sampled according to the distribution $(p_k)_{k \in \{0,\infty\} \cup \mathbb{N}}$.
  - *Coalescing:* Each pair of particles coalesces as one particle with rate $1/2$ according to their intersection local time.

An illustration of the dual particle system



Explosion in CBBM

- To build a duality relation between CBBMs and the Wright-Fisher SHEs, we take

$$\mu := \sum_{k \in \{0,\infty\} \cup \mathbb{N} \setminus \{1\}}^{\infty} |b_k|$$

and $p_1 := 0$, $p_k := |b_k|/\mu$ for $k \in \{0, \infty\} \cup \mathbb{N} \setminus \{1\}$.

- The dynamic is well-defined up to the explosion time

$$\tau_\infty := \lim_{n \to \infty} \inf\{t \geq 0 : \#\text{particles} \geq n\}.$$

- (b_k) satisfies AT's condition $\implies$ $p_\infty = 0$ and (p_k) has exponential moment $\implies$ $\tau_\infty = \infty$ a.s.
- If AT's condition does not hold (especially when $p_\infty = |b_\infty|/\mu > 0$) the explosion might happen in finite time.
- The definition of the particle system needs more justification!

Coming down from infinity

- A coalescing Brownian motion (CBM) is CBBM with $p_1 = 1$.
- Define CBM with infinitely many initial particles as the limit of a sequence of CBMs with finite initial particles.
- Denote by $Z_t(A)$ the number of particles in a domain A at time t of a CBM with infinitely many initial particles, i.e. $Z_0(\mathbb{R}) = \infty$.

Barnes-Mytnik-S. (2024, Ann. Probab.)

The total population $Z_t(\mathbb{R}) < \infty$ for every $t > 0$
$\iff$ $Z_0(\cdot)$ is compactly supported.
Moreover, in this case

$$\left(\int v_t(x)\mathrm{d}x\right)^{-1} Z_t(\mathbb{R}) \xrightarrow{L^1} 1, \quad t \downarrow 0$$

where $(v_t(x))_{t\geq 0, x\in\mathbb{R}}$ is the unique non-negative solution to the 1d PDE $\partial_t v_t = \frac{\Delta}{2} v_t - v_t^2/2$ with initial value $v_0 = Z_0$.

Reflecting from infinity

- Similarly, we can justify the definition of the CBBM for arbitrary offspring distribution (allowing $p_\infty > 0$).
- It is defined as the limit of a sequence of CBBMs with truncated offspring distributions.
- Denote by $X_t(\mathbb{R})$ the total population of a CBBM with arbitrary branching rate and arbitrary offspring distribution.

Barnes-Mytnik-S. (2025+, to appear in Probab. Theory Related Fields)

If $X_0(\mathbb{R}) < \infty$, then $X_t(\mathbb{R})$ is "reflecting" from ∞.

- This "reflecting from infinity" property of the dual particle system is the key to the well-posedness of the corersponding SPDE.

Thanks!