

# EFFECT OF SMALL NOISE ON THE SPEED OF REACTION-DIFFUSION EQUATIONS WITH NON-LIPSCHITZ DRIFT

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ABSTRACT. We consider the  $[0, 1]$ -valued solution  $(u_{t,x} : t \geq 0, x \in \mathbb{R})$  to the one dimensional stochastic reaction diffusion equation with Wright-Fisher noise

$$\partial_t u = \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)} \dot{W}.$$

Here,  $W$  is a space-time white noise,  $\epsilon > 0$  is the noise strength, and  $f$  is a continuous function on  $[0, 1]$  satisfying  $\sup_{z \in [0,1]} |f(z)|/\sqrt{z(1-z)} < \infty$ . We assume the initial data satisfies  $1 - u_{0,-x} = u_{0,x} = 0$  for  $x$  large enough. Recently, it was proved in (Comm. Math. Phys. **384** (2021), no. 2) that the front of  $u_t$  propagates with a finite deterministic speed  $V_{f,\epsilon}$ , and under slightly stronger conditions on  $f$ , the asymptotic behavior of  $V_{f,\epsilon}$  was derived as the noise strength  $\epsilon$  approaches  $\infty$ . In this paper we complement the above result by obtaining the asymptotic behavior of  $V_{f,\epsilon}$  as the noise strength  $\epsilon$  approaches 0: for a given  $p \in [1/2, 1)$ , if  $f(z)$  is non-negative and is comparable to  $z^p$  for sufficiently small  $z$ , then  $V_{f,\epsilon}$  is comparable to  $\epsilon^{-2\frac{1-p}{1+p}}$  for sufficiently small  $\epsilon$ .

## 1. INTRODUCTION

In 1937, Fisher [11] and Kolmogorov, Petrovsky, Piskunov [17] independently studied the wave propagation properties arising from the FKPP equation on  $\mathbb{R}_+ \times \mathbb{R}$

$$(1.1) \quad \partial_t h = \partial_x^2 h + f(h).$$

Fisher was interested in how quickly an advantageous gene (or virus) would propagate through a population living in a linear habitat, such as a shoreline (or train). The solution  $h$  measures the proportion of the population carrying this advantageous gene as the biological system evolves. Under a mild assumption on the Lipschitz function  $f$  with  $f(0) = f(1) = 0$ , for any velocity  $v$  greater or equal to the minimal velocity

$$(1.2) \quad v_{\min} = \sqrt{2f'(0)},$$

there exists a traveling wave solution  $h(t, x) = F_v(x - vt)$  with wave profile denoted by  $F_v$ . With Heaviside initial data  $h(0, x) = \mathbf{1}_{x \leq 0}$ , the shifted solution  $h(t, x + m(t))$  converges uniformly to the wave profile  $F_{v_{\min}}$  with the shift  $m(t)$  having asymptotic speed  $v_{\min}$ . These results have been generalized to include a wider class of initial conditions, a more detailed description of the lower order terms of the wave position, and tail behavior of the wave shape; see [3, 4, 19, 20].

Because the FKPP equation is noiseless, it can be thought to represent a mean field approximation of a microscopic reaction-diffusion process as motivated from a statistical physics perspective [6]. Consequently, there has been recent interest in understanding

analogous questions regarding propagating speed of waves for solutions to the FKPP equation with Wright-Fisher noise, given by the SPDE on  $\mathbb{R}_+ \times \mathbb{R}$ :

$$(1.3) \quad \partial_t u = \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)} \dot{W}$$

where  $f$  is a continuous function on  $[0, 1]$  with  $f(0) = f(1) = 0$  satisfying some regularity conditions, and  $W$  is a space-time white noise on  $\mathbb{R}_+ \times \mathbb{R}$ . Similar to (1.1), the solution  $u$  represents the proportion of the population exhibiting the gene, but now the equation incorporates the random interaction among the populace. The noise term  $\epsilon \sqrt{u(1-u)} \dot{W}$  is motivated by the assumption that these interactions are affected by i.i.d. mean zero random variables independent of time and space, while the variance of the outcome is proportional to the rate of interaction between those with the gene and those without the gene, which is  $u(1-u)$ . The function  $f$  continues to describe the deterministic evolution of the population exhibiting the gene.

The example of  $f(u) = u(1-u)$  was extensively studied in the literature and weak uniqueness, compact interface property, finite speed of the front propagation and other properties were established (see [7, 23, 27, 29]). Moreover there has been a great interest in the asymptotic behavior of the speed of the front propagation,  $V_{f,\epsilon}$ . For a large class of Lipschitz functions  $f$ , including  $f(u) = u(1-u)$ , and also for more general noise coefficients, Mueller, Mytnik, and Quastel, in [21], proved the Brunet-Derrida conjecture (see [5]) on the asymptotic of  $V_{f,\epsilon}$  for small  $\epsilon$ .

Another motivation for the study of the stochastic reaction diffusion equation is its duality relation to the branching-coalescing Brownian motion. Consider a system of particles moving as independent one-dimensional Brownian motions on  $\mathbb{R}$  with generator  $\partial_x^2$ . Assume that each particle independently branches with rate 1 into a random number of particles according to an offspring law  $(q_k)_{k \in \mathbb{Z}_+}$ , and each pair of particles independently coalesce at rate  $\epsilon^2$  according to their intersection local time. Denote by  $(x_i(t) : t \geq 0, i \leq N_t)$  the positions of the particles where  $N_t$  is the number of all particles at time  $t \geq 0$ . In the case of binary branching, i.e.  $q_2 = 1$ , the following duality relation is due to [27] (see also [9]): Let  $u$  be a solution to the SPDE (1.3) with  $f(u) = u(1-u)$ . Assume that  $u$  is independent of the particle system. Then

$$(1.4) \quad \mathbb{E} \left[ \prod_{i=1}^{N_0} (1 - u_{t, x_i(0)}) \right] = \mathbb{E} \left[ \prod_{i=1}^{N_t} (1 - u_{0, x_i(t)}) \right], \quad t \geq 0.$$

This duality relation was first constructed to study the weak uniqueness of the stochastic FKPP equation. It can also be used to study the propagation of the extremal particle in the branching-coalescing Brownian motion. Assume that the particle system starts with a single particle at the position 0. Let  $R(t)$  be the position of the rightmost particle in the system at time  $t$ . From the above duality, by taking  $u_{0,\cdot} = \mathbf{1}_{(-\infty, 0)}$  and using symmetry, one can get  $\mathbb{P}(R(t) > x) = \mathbb{E}[u_{t,x}]$  for every  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ . Then from the existence of the speed of front propagation  $V_{f,\epsilon}$  for  $u$ , one can derive the upper bound on the speed of  $R(t)$ : for any  $\delta > 0$ ,

$$\mathbb{P}(R(t) \leq (V_{f,\epsilon} + \delta)t) \xrightarrow[t \rightarrow \infty]{} 1.$$

One expects that the duality (1.4) holds also for more general drift function  $f$  when it takes the form

$$(1.5) \quad f(z) = 1 - z - g(1 - z), \quad z \in [0, 1],$$

where  $g(z) = \sum_{n=0}^{\infty} z^n q_n$ ,  $z \in [0, 1]$ , is the probability generating function of the offspring law in the branching-coalescing system. This is established for some offspring laws with finite first moment, see [2, Theorem 1]. The case of the offspring law being heavy-tailed, without existence of the first moment, is of particular interest. For example, if one considers the following heavy-tailed offspring law

$$q_0 = q_1 = 0; \quad q_n = \frac{-1}{(n-1)!} \prod_{k=0}^{n-2} (k-p), \quad n \in \mathbb{Z} \cap [2, \infty)$$

where  $p \in (0, 1)$  is a given constant, then, the drift function  $f$  defined via (1.5) takes the form

$$(1.6) \quad f(z) = z^p(1 - z), \quad z \in [0, 1],$$

which is not Lipschitz at 0. As far as we know, the Shiga duality relation (1.4) for such cases has not been proved yet. However, we do conjecture that this duality holds for some Hölder drift functions  $f$  having representation (1.5), and this gives us another motivation to study the SPDE (1.3) with non-Lipschitz drift functions.

In fact, one of such cases has been studied recently in [22], where weak solutions  $u$  to the SPDE (1.3) are investigated under the conditions that

$$(1.7) \quad f \text{ is continuous, and } \sup_{z \in [0,1]} \frac{|f(z)|}{\sqrt{z(1-z)}} < \infty$$

and that the initial value has compact interface, that is,  $u_{0,-x} - 1 = u_{0,x} = 0$  for large enough  $x$ . This includes examples like (1.6) with parameter  $p \in [1/2, 1)$ . Note that in the deterministic case of  $\epsilon = 0$ , the solution to the reaction diffusion equation (1.1), with  $f$  given by (1.6) and with a non-trivial initial value, does exist, but it does not exhibit propagating waves with a finite linear speed as in the case when  $f$  is Lipschitz. Intuitively, this is clear from (1.2) as  $f'(0)$  is infinite. In fact, such solutions  $h$  do not have super-linear speed either, as it follows from a similar argument in [1] that  $\inf_{x \in \mathbb{R}} h(t, x) \rightarrow 1$  as  $t \rightarrow \infty$ . However, this behavior changes drastically when introducing the Wright-Fisher noise. In [22], the authors established the weak uniqueness, compact interface property, and finiteness of front propagation speed for (1.3) when  $\epsilon > 0$ . In particular, they proved that there exists a deterministic  $V_{f,\epsilon} \in \mathbb{R}$ , which only depends on the drift function  $f$  and the noise strength  $\epsilon$ , so that

$$\frac{\sup\{x \in \mathbb{R} : u_{t,x} \neq 0\}}{t} \xrightarrow[t \rightarrow \infty]{} V_{f,\epsilon}, \quad \text{a.s.}$$

It is then natural to study the asymptotics of the speed of this propagation in terms of the strength of the noise. In [22], the authors studied the asymptotic behavior of  $V_{f,\epsilon}$ , when  $\epsilon$  goes to  $\infty$ , under a condition slightly stronger than (1.7). As for the small  $\epsilon$ , the case of Lipschitz  $f$  was treated already in [21], and it was shown there how fast  $V_{f,\epsilon}$

converges to  $V_{f,0}$  as  $\epsilon \downarrow 0$ . In this paper, we complement the above results and consider the asymptotic behavior of  $V_{f,\epsilon}$  when  $\epsilon$  converges to 0 and  $f$  is not necessarily Lipschitz. According to our discussion about the deterministic case of  $\epsilon = 0$ , it is intuitively clear that if, for example,  $f$  is given by (1.6), then  $V_{f,\epsilon}$  should converge to  $\infty$  as  $\epsilon \downarrow 0$ . Our main result shows this, but also answers the much more delicate question: *At what rate does  $V_{f,\epsilon}$  converge to  $\infty$  as  $\epsilon \downarrow 0$ ?*

To state our main result we need to introduce the following conditions on  $f$ .

(1.8)  $f$  is non-negative and there exists  $p \in [1/2, 1)$  such that  $\liminf_{z \downarrow 0} f(z)/z^p > 0$ .

(1.9) There exists  $p \in [1/2, 1)$  such that  $\limsup_{z \downarrow 0} f(z)/z^p < \infty$ .

Note that (1.6) is an example of  $f$  satisfying (1.7)-(1.9).

We now state our main result.

**Theorem 1.1.** *Assume  $f$  satisfies (1.7).*

(a) *If  $f$  satisfies (1.8), then  $\liminf_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f,\epsilon} > 0$ .*

(b) *If  $f$  satisfies (1.9), then  $\limsup_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f,\epsilon} < \infty$ .*

If the drift function  $f$  satisfies (1.7)-(1.9), then Theorem 1.1 implies that there exists  $\epsilon_0 > 0$  and  $c, C > 0$  such that

$$c\epsilon^{-2\frac{1-p}{1+p}} \leq V_{f,\epsilon} \leq C\epsilon^{-2\frac{1-p}{1+p}}, \quad \epsilon \in (0, \epsilon_0).$$

Let us now give some comments on the proof strategy and some intuition on the exponent  $-2\frac{1-p}{1+p}$ . Note that this exponent shows up in both the upper bound and the lower bound, and therefore cannot be improved. This is interesting because our methods for the lower bound and the upper bound are quite different.

For the lower bound, we simply replace the drift  $f$  by some smaller Lipschitz drift  $H \leq f$ . The comparison principle then gives us a lower bound  $V_{H,\epsilon} \leq V_{f,\epsilon}$ . Of course, by choosing different Lipschitz functions  $H$ , one can obtain a family of lower bounds. To obtain the optimal one, we take  $H$  depending on the noise strength  $\epsilon$  in a certain way so that  $H'(0)$  is comparable to  $\epsilon^{-4\frac{1-p}{1+p}}$ . We then use a known result on the propagation speed of stochastic FKPP equation [21] to get the desired lower bound.

For the upper bound, the strategy of replacing the drift by Lipschitz functions is not fruitful because for any Lipschitz function  $H$  greater than a drift function  $f$  satisfying (1.7)-(1.9), it always holds that  $H(0) > 0$ . For a solution  $u$  corresponding to such a drift  $H$ , the state 0 is not locally stable anymore, and typically,  $\sup\{x : u_{t,x} \neq 0\}$  is not even finite. Instead, we use a similar strategy as in [21] to decompose our solution  $u$  as  $v + w$  where  $v$  is a weak solution to the SPDE (1.3) with a moving Dirichlet boundary condition on the line  $\{(t, x) : x = vt\}$ , i.e.

$$\begin{cases} \partial_t v = \partial_x^2 v + f(v) + \epsilon \sqrt{v(1-v)} \dot{W}^v, & x < vt, \\ v = 0, & x \geq vt, \end{cases}$$

where  $W^v$  is a white noise. Here,  $v$ , the speed of the moving boundary, is left to be chosen. It is intuitively clear that if one chooses  $v$  to be larger than  $V_{f,\epsilon}$  then the deviation between  $u$  and  $v$ , which is  $w$ , should be small and not propagate; and if one chooses  $v$  to be smaller

than  $V_{f,\epsilon}$ , then  $w$  will be large and will propagate. Therefore, by searching for a balanced value  $v$  so that  $w$  lies in between those two phases, one can obtain a good estimation on  $V_{f,\epsilon}$ . An insight from [21] suggests that such a value  $v$  can be predicted by finding the solution  $(F, v)$  so that

$$F(x) = 0, \quad x \geq 0; \quad F'(0-) = \epsilon^2; \quad F(-\infty) = 1,$$

and that  $\varrho_{t,x} := F(x - vt)$ ,  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$  satisfies the following non-KPP type PDE:

$$(1.10) \quad \begin{cases} \partial_t \varrho = \partial_x^2 \varrho + f(\varrho), & x < vt, \\ \varrho = 0, & x \geq vt. \end{cases}$$

Replacing the drift  $f$  in (1.10) by some approximating Lipschitz functions, the solution  $(F, v)$  is computable using a similar argument used in [21, Proof of Proposition 2.1], and one can calculate that the balancing value  $v$  should be comparable to  $\epsilon^{-2\frac{1-p}{1+p}}$  for small  $\epsilon$ . (The analytical calculation of solving  $(F, v)$  is omitted, because it is not used in the actual proof of Theorem 1.1.)

To analyze the behavior of  $w$  under this balancing value  $v \sim \epsilon^{-2\frac{1-p}{1+p}}$ , we observe that it satisfies the following equation

$$\partial_t w = \partial_x^2 w + f(u) - f(v) + \epsilon \sqrt{u(1-u)} \dot{W} - \epsilon \sqrt{v(1-v)} \dot{W}^v + \dot{A}_t \delta_{vt}(x)$$

where  $A_t$  is the accumulated mass of  $v$  being “killed” at its boundary before time  $t \geq 0$ . Note that  $f$  is typically not Lipschitz, so unlike in [21] we cannot control the drift term  $f(u) - f(v)$  by  $\|f\|_{\text{Lip}} w$ . To overcome this, we use Dawson’s Girsanov transformation and remove this drift term under a new probability measure. However, similarly to what often happens for finite dimensional diffusion processes, one cannot control the Radon-Nikodym derivative in Dawson’s Girsanov transformation for a long time. So we need to chop off the time into small intervals  $\{[nT, (n+1)T) : n \in \mathbb{Z}_+\}$ , and only perform Dawson’s Girsanov transformation on each of those intervals. By choosing the parameter  $T$  small enough, the transformed  $w$  will then serve as a good approximation of the original  $w$  on each of those intervals. On the other hand, in order to get a reasonably good upper bound for the long time propagation speed, we cannot take  $T$  too small either. So a balanced value has to be chosen for this interval length  $T$ .

Our philosophy of choosing such a value for  $T$  works as follows. Consider the random field  $w$  by standing at the moving boundary  $\{(t, x) : x = vt\}$ , that is to say, consider the random field  $(w_{t,vt+x} : (t, x) \in \mathbb{R}_+ \times \mathbb{R})$ . Say  $L$  is a typical distance for the support of this random field to travel in a time interval of length  $T$ . We want our  $T$  to be chosen so that this  $L$  not only can be explained by the (parabolic) thermal diffusivity, but also does not give excess speed. That is, we want both  $L \sim \sqrt{T}$  and  $L/T \lesssim v$ . Recalling our choice of  $v \sim \epsilon^{-2\frac{1-p}{1+p}}$ , we end up choosing  $T \sim \epsilon^{4\frac{1-p}{1+p}}$ . It turns out that this is also a time span on which Dawson’s Girsanov transformation argument works. We hope this idea, of performing Dawson’s Girsanov transformations on time intervals with a balanced length, can also be useful for finding propagation speed in other spacial stochastic models.

Let us now mention some limitations of our result. Note that we only considered the Wright-Fisher noise  $\sqrt{u(1-u)} \dot{W}$ . It would be interesting to also consider more general

noise  $\sigma(u)\dot{W}$ . We comment here that both in the proofs of our result Theorem 1.1 and of [22, Theorem 1.1], the Wright-Fisher noise is not essential: what has been really used is the property that  $\sqrt{z(1-z)} \sim \sqrt{z}$  for small  $z$ . However, to explore the most general conditions for the noise term  $\sigma$  is out of the scope of the current paper.

Let us also mention here that our result does not give the existence of  $\lim_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f,\epsilon}$ . We conjecture that  $\lim_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f,\epsilon}$  exists provided  $\lim_{z \downarrow 0} f(z)/z^p$  exists. For proving this conjecture, it seems that new ideas and methods will be needed.

The rest of the paper is organized as follows. In Section 2, we recall some preliminary terminology including the solution concept for the SPDE (1.3). In Section 3, we give the proof of Theorem 1.1(a). We give the proof of Theorem 1.1(b) in Section 4 while the proofs of the results used for its proof are given in Sections 5-10. In the Appendix, we gather the proofs of several technical lemmas.

## 2. PRELIMINARY

In this section, we recall some preliminary terminology including the solution concept to the SPDE (1.3). We first give some notation. We say a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfies the usual hypotheses if  $(\Omega, \mathcal{G}, \mathbb{P})$  is a complete probability space with right-continuous filtration  $(\mathcal{F}_t)_{t \geq 0}$  satisfying  $\{A \in \mathcal{G} : \mathbb{P}(A) = 0\} \subset \mathcal{F}_0$ . We impose the usual hypotheses on every filtered probability spaces that will be considered in this paper. Given such a space, denote by  $\mathcal{M}_{\text{loc}}$  the family of adapted continuous local martingales. For any continuous semi-martingale  $M$ , denote by  $\langle M \rangle$  its quadratic variation. Given two continuous semi-martingales  $M, N$ , let  $\langle M, N \rangle$  denote their quadratic covariation. In this paper, we say  $g$  is a random field if it is an  $\mathbb{R}$ -valued stochastic process indexed by  $\mathbb{R}_+ \times \mathbb{R}$ . Denote by  $\mathcal{L}_{\text{loc}}^2$  the family of predictable random fields  $g$  satisfying

$$\int_0^t \int_{\mathbb{R}} g_{s,y}^2 ds dy < \infty, \quad t \geq 0, \quad \text{a.s.}$$

Let  $\mathcal{B}_F(\mathbb{R})$  be the collection of Borel subsets of  $\mathbb{R}$  with finite Lebesgue measure. We say  $W = (W_s(A) : A \in \mathcal{B}_F(\mathbb{R}), s \in \mathbb{R}_+)$  is a white noise if it is an adapted orthogonal martingale measure so that for any  $A, B \in \mathcal{B}_F(\mathbb{R})$  almost surely

$$\langle W.(A), W.(B) \rangle_t = t \cdot \text{Leb}(A \cap B), \quad t \geq 0,$$

where  $\text{Leb}(\cdot)$  is the Lebesgue measure on  $\mathbb{R}$ . Given a white noise  $W$ , Walsh's stochastic integral for  $W$  is a map from  $\mathcal{L}_{\text{loc}}^2$  to  $\mathcal{M}_{\text{loc}}$  which will be denoted by

$$g \mapsto \iint_0^\cdot g_{s,y} W(ds dy).$$

We refer our reader to [14, 30] for more details.

Let us now be precise about the solution concept for the SPDE (1.3). Denote by  $\mathcal{C}_{\text{tem}}$  the space of continuous functions  $g$  on  $\mathbb{R}$  such that

$$\|g\|_{(-n)} := \sup_{x \in \mathbb{R}} |e^{-n|x|} g(x)| < \infty, \quad n > 0.$$



Let  $\mathcal{C}_{\text{tem}}$  be equipped with the topology generated by the norms  $(\|\cdot\|_{(-n)} : n > 0)$ , and set  $\mathcal{C}_{\text{tem}}^+$  as the collection of non-negative elements in  $\mathcal{C}_{\text{tem}}$ . Let  $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}})$  be the space of continuous  $\mathcal{C}_{\text{tem}}$ -valued paths with the topology of uniform convergence on bounded time sets. We say a Borel function  $f$  on  $\mathbb{R}$  satisfies the linear growth condition if  $\sup_{z \in \mathbb{R}} |f(z)|/(1+|z|) < \infty$ . Assume that Borel functions  $f$  and  $\sigma$  on  $\mathbb{R}$  satisfy the linear growth condition. We say that  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, u, W)$  is a weak solution to the SPDE

$$(2.1) \quad \partial_t u = \partial_x^2 u + f(u) + \sigma(u) \dot{W},$$

if  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is a filtered probability space on which a predictable random field  $u$  and a white noise  $W$  are defined so that  $(u_{t,\cdot} : t \geq 0) \in \mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}})$  and

$$(2.2) \quad u_{t,x} = \int_0^t \int_{\mathbb{R}} G_{s,y;t,x} M^u(\text{dsdy}) \quad \text{a.s.} \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

where

$$(2.3) \quad G_{s,y;t,x} := \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} \mathbf{1}_{s < t}, \quad (s, y), (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

and

$$M^u(\text{dsdy}) := u_{0,y} \delta_0(\text{ds}) \text{dy} + f(u_{s,y}) \text{dsdy} + \sigma(u_{s,y}) W(\text{dsdy}).$$

Here, the right hand side of (2.2) is a mixture of the classical integral and Walsh's stochastic integral defined in an obvious way using linearity. With some abuse of notation, we sometimes just use the random field  $u$  to represent the weak solution if there is no risk of confusion. We refer our reader to [28, Theorem 2.1] for an equivalent definition.

Given a subset  $\hat{\mathcal{C}} \subset \mathcal{C}_{\text{tem}}$ , we say a weak solution  $u$  to the SPDE (2.1) is a  $\hat{\mathcal{C}}$ -valued weak solution if  $(u_{t,\cdot} : t \geq 0)$  is a  $\hat{\mathcal{C}}$ -valued process. We say the weak existence of the SPDE (2.1) holds in  $\hat{\mathcal{C}}$  for an initial condition  $g \in \hat{\mathcal{C}}$  if there exists a  $\hat{\mathcal{C}}$ -valued weak solution  $u$  to (2.1) such that  $u_{0,\cdot} = g$ . We say the weak uniqueness of the SPDE (2.1) holds in  $\hat{\mathcal{C}}$  for an initial condition  $g \in \hat{\mathcal{C}}$  if, whenever  $u$  and  $u'$  are two  $\hat{\mathcal{C}}$ -valued weak solutions to the SPDE (2.1) such that  $u_{0,\cdot} = g$  and  $u'_{0,\cdot} = g$ , the  $\mathcal{C}_{\text{tem}}$ -valued processes  $(u_{t,\cdot} : t \geq 0)$  and  $(u'_{t,\cdot} : t \geq 0)$  induce the same law on  $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{\text{tem}})$ .

The weak existence, weak uniqueness, and compact propagation property of the SPDE (1.3) under condition (1.7) and  $\epsilon > 0$  is studied in [22]. Denote by  $\mathcal{C}_{[0,1]}$  the space of continuous functions on  $\mathbb{R}$  taking values in  $[0, 1]$ . Denote by  $\mathcal{C}_I$  the family of functions  $g$  in  $\mathcal{C}_{[0,1]}$  with compact interface, i.e.  $g \in \mathcal{C}_{[0,1]}$  and  $-\infty < L(g) < R(g) < \infty$  where  $L(g) := \inf\{x \in \mathbb{R} : g(x) \neq 1\}$  and  $R(g) := \sup\{x \in \mathbb{R} : g(x) \neq 0\}$ .

**Theorem 2.1** ([22]). *For any  $\epsilon > 0$  and function  $f$  satisfying (1.7), the following holds.*

- (1) *For any initial condition  $g \in \mathcal{C}_I$ , the weak existence and weak uniqueness of (1.3) holds in  $\mathcal{C}_{[0,1]}$ .*
- (2) *For any  $\mathcal{C}_{[0,1]}$ -valued weak solution  $u$  to (1.3) with  $u_{0,\cdot} \in \mathcal{C}_I$ , it holds that*

$$\mathbb{E} \left[ \sup_{s \in [0, t]} |R(u_{s,\cdot}) - L(u_{s,\cdot})| \right] < \infty, \quad t \geq 0.$$

In particular, for any initial condition  $g \in \mathcal{C}_I$ , the weak existence and weak uniqueness of (1.3) holds in  $\mathcal{C}_I$ .

- (3) There exists a deterministic  $V_{f,\epsilon} \in \mathbb{R}$  such that for any  $\mathcal{C}_I$ -valued weak solution  $u$  to (1.3),

$$\lim_{t \rightarrow \infty} \frac{R(u_t)}{t} = V_{f,\epsilon}, \quad a.s.$$

*Remark 2.2.* We refer to  $V_{f,\epsilon}$  as the speed of the traveling front of the SPDE (1.3). We emphasize here that  $V_{f,\epsilon}$  depends only on the drift function  $f$  and the noise strength  $\epsilon$ . To see that it is independent of the initial value  $g$ , we note that for any other  $\tilde{g} \in \mathcal{C}_I$ , there exists a constant  $c \in \mathbb{R}$  such that

$$g(x+c) \leq \tilde{g}(x) \leq g(x-c), \quad x \in \mathbb{R}.$$

Therefore, using the comparison principle and the weak uniqueness, for any weak solutions  $u$  and  $\tilde{u}$  to the SPDE (2.1) with  $u_{0,\cdot} = g$  and  $\tilde{u}_{0,\cdot} = \tilde{g}$  respectively, random field  $\tilde{u}$  will be dominated stochastically by  $(u_{t,x-c} : (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  from above, and by  $(u_{t,x+c} : (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  from below. This indicates that the fronts of  $u$  and  $\tilde{u}$  have the same speed.

Note that in Remark 2.2 we used the comparison principle in the absence of the Lipschitz condition. This is justified by the following lemma whose variants have already appeared in the literature, see [28, Theorem 2.6] and [21, p. 412] for example.

**Lemma 2.3** (Comparison Principle). *Let  $f, \tilde{f}$  and  $\sigma$  be continuous functions on  $\mathbb{R}$  satisfying the linear growth condition. Assume that  $\tilde{f} \leq f$  on  $\mathbb{R}$ . Let  $g, \tilde{g} \in \mathcal{C}_{\text{tem}}$  satisfy  $\tilde{g} \leq g$  on  $\mathbb{R}$ . Then, there exists a weak solution  $u$  to the SPDE (2.1) with  $u_{0,\cdot} = g$ , and a weak solution  $\tilde{u}$  to the SPDE*

$$\partial_t \tilde{u} = \partial_x^2 \tilde{u}^2 + \tilde{f}(\tilde{u}) + \sigma(\tilde{u})\tilde{W}$$

*with  $\tilde{u}_{0,\cdot} = \tilde{g}$ , so that the random field  $\tilde{u}$  is stochastically dominated by the random field  $u$ .*

One can prove the above lemma by following the routine arguments in the proof of [28, Theorem 2.6]. Note that when  $\tilde{f}(0) = \sigma(0) = 0$  and  $\tilde{g} \equiv 0$ , it actually follows directly from [28, Theorem 2.6].

Another general result that will be used very often is the following rescaling lemma of the SPDE (2.1) which can be proved using a similar argument as in [22, Section 4.1].

**Lemma 2.4** (Rescaling). *Suppose that Borel functions  $f$  and  $\sigma$  on  $\mathbb{R}$  satisfies the linear growth condition. Suppose that  $u$  is a weak solution to the SPDE (2.1). Let  $\alpha, \beta > 0$ , and  $v_{t,x} := \beta u_{\alpha^{-4}t, \alpha^{-2}x}$  for each  $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ . Then there exists a white noise  $W^v$  such that  $v$  is a weak solution to the SPDE*

$$\partial_t v = \partial_x^2 v + \alpha^{-4} \beta f(\beta^{-1}v) + \alpha^{-1} \beta \sigma(\beta^{-1}v) \dot{W}^v.$$

We end this section by collecting some notations for function spaces. Given a locally compact separable metric space  $E$ , we denote by  $\mathcal{C}(E)$  the space of continuous functions on  $E$ . We write  $\mathcal{C}_b(E)$ ,  $\mathcal{C}_0(E)$ , and  $\mathcal{C}_c(E)$ , respectively, for the space of continuous



functions on  $E$  that are bounded, vanishing at  $\infty$ , and having compact support, respectively. Use  $\mathcal{C}_\bullet(E)$  to represent one of  $\mathcal{C}(E)$ ,  $\mathcal{C}_b(E)$ ,  $\mathcal{C}_0(E)$  or  $\mathcal{C}_c(E)$ . If  $E \subset \mathbb{R}^d$ , we define  $\mathcal{C}_\bullet^0(E) := \mathcal{C}_\bullet(E)$ , and inductively

$$\mathcal{C}_\bullet^n(E) := \{\phi \in \mathcal{C}_\bullet(E) : \partial_{x_k} \phi \in \mathcal{C}_\bullet^{n-1}(E), \forall k = 1, \dots, d\}, \quad n \in \mathbb{N},$$

and  $\mathcal{C}_\bullet^\infty(E) := \cap_{n=1}^\infty \mathcal{C}_\bullet^n(E)$ . If  $E = \mathbb{T} \times \mathbb{R}$  with the time interval  $\mathbb{T} \subset \mathbb{R}_+$ , we define

$$\mathcal{C}_\bullet^{1,2}(E) := \{\phi \in \mathcal{C}_\bullet(E) : \partial_t \phi, \partial_x^2 \phi \in \mathcal{C}_\bullet(E)\}.$$

### 3. PROOF OF THEOREM 1.1(a)

Note that we only have to prove the result for every  $f \in \{f^{(\delta)} : \delta \in (0, 1)\}$  where

$$f^{(\delta)}(z) := z^p \mathbf{1}_{0 \leq z \leq \delta}, \quad z \in [0, 1].$$

This is because for any general function  $f$  satisfying (1.7) and (1.8), there exists  $c > 0$  and  $0 < \delta < 1$  such that

$$f(z) \geq c f^{(\delta)}(z), \quad z \in [0, 1].$$

Using Lemmas 2.3 and 2.4 we have

$$V_{f,\epsilon} = c^{1/2} V_{c^{-1}f, c^{-1/4}\epsilon} \geq c^{1/2} V_{f^{(\delta)}, c^{-1/4}\epsilon}.$$

Therefore, to show  $\liminf_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f,\epsilon}$  is positive we only have to show  $\liminf_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_{f^{(\delta)},\epsilon}$  is positive. So in the remainder of this section, without loss of generality, let us fix an arbitrary  $\delta \in (0, 1)$  and assume that  $f \equiv f^{(\delta)}$ .

The idea of the proof is to use the comparison principle and the rescaling lemma for the SPDEs to replace our non-Lipschitz drift  $f$  by some continuous tent function

$$H(z; l, h) := \begin{cases} 0, & z \in (-\infty, 0], \\ \frac{2zh}{l}, & z \in (0, l/2), \\ h, & z = l/2, \\ 2h - \frac{2zh}{l}, & z \in (l/2, l), \\ 0, & z \in [l, \infty). \end{cases}$$

Here, the width  $l \in (0, 1]$  and the height  $h > 0$  of this tent function will be chosen more precisely later. This will allow us to analyze the speed of the system using the following result from [21]. For any  $\beta > 0$ , define  $\mathcal{C}_{I,\beta} := \{\beta g : g \in \mathcal{C}_I\}$ .

**Lemma 3.1** ([21]). *There exists a  $\gamma_0 > 0$  so that the following statement holds. Suppose that*

- $\gamma \in (0, \gamma_0)$  and  $\beta > 0$ ;
- $\sigma$  is a non-negative function on  $\mathbb{R}_+$  such that  $\sigma^2$  is Lipschitz and that  $\sigma^2(z) \leq z$  for any  $z \in \mathbb{R}_+$ ;
- for any initial condition  $g \in \mathcal{C}_{I,\beta}$ , the weak existence and the weak uniqueness of the SPDE

$$(3.1) \quad \partial_t v = \partial_x^2 v + H(v; 1, 1/2) + \gamma \sigma(v) \dot{W}$$

holds in  $\mathcal{C}_{I,\beta}$ .

Then for any  $\mathcal{C}_{I,\beta}$ -valued weak solution  $v$  to (3.1), it holds that

$$(3.2) \quad \liminf_{t \rightarrow \infty} \frac{R(v_{t,\cdot})}{t} \geq 2 - \frac{\pi^2}{|\log \gamma^2|^2} - \frac{2\pi^2[11 \log |\log \gamma| - \log(1/2)]}{|\log \gamma^2|^3}, \quad a.s.$$

*Remark 3.2.* Lemma 3.1 is a corollary of [21, Theorem 1.1] except that now the function  $\sigma$  is not required to satisfy [21, (1.5)] and the parameter  $\gamma_0$  is universal. We justify this by observing that condition [21, (1.5)] is actually not needed in the proof of the lower bound of [21, Theorem 1.1], and that the parameter  $\gamma_0$ , chosen as the  $\epsilon_0$  from [21, Lemma 4.1], is only related to the drift function  $f$ , which, in our case, is the fixed tent function  $H(\cdot; 1, 1/2)$ .

*Proof of Theorem 1.1(a). Step 1.* Let us fix the value  $\gamma := \min\{\gamma_0, e^{-4}\}$  where  $\gamma_0$  is given as in Lemma 3.1. One can easily check now the right hand side of (3.2) is larger than 1.

*Step 2.* Let  $q := 4/(1+p)$  and define  $\epsilon_0 > 0$  so that  $2(\epsilon_0/(\sqrt{2}\gamma))^q = \delta$ . Fix an arbitrary  $\epsilon \in (0, \epsilon_0)$  and define  $\varepsilon := \epsilon/(\sqrt{2}\gamma)$ . Observe that  $2\varepsilon^q \leq \delta$ . As a consequence, we have that the tent function  $H(\cdot; 2\varepsilon^q, \varepsilon^{qp}) \leq f(\cdot)$ .

*Step 3.* Let  $u$  be a  $\mathcal{C}_I$ -valued weak solution to the SPDE (1.3). Now, from Step 2 and the comparison principle, we can construct a  $\mathcal{C}_I$ -valued weak solution  $\underline{u}$  to the SPDE

$$(3.3) \quad \partial_t \underline{u} = \partial_x^2 \underline{u} + H(\underline{u}; 2\varepsilon^q, \varepsilon^{qp}) + \epsilon \sqrt{\underline{u}(1-\underline{u})} \dot{W}^{\underline{u}}$$

where  $W^{\underline{u}}$  is a white noise, so that the random field  $\underline{u}$  is stochastically dominated by  $u$ .

*Step 4.* Define the random field

$$v_{t,x} := \beta \underline{u}_{\alpha^{-4}t, \alpha^{-2}x}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

where  $\alpha := \varepsilon^{-(1-p)q/4}$  and  $\beta := 1/(2\varepsilon^q)$ . Then, we can easily verify from Lemma 2.4 that there exists a white noise  $W^v$  such that  $v$  is a  $\mathcal{C}_{I,\beta}$ -valued weak solution to the SPDE

$$(3.4) \quad \partial_t v = \partial_x^2 v + H(v; 1, 1/2) + \gamma \sqrt{v(1-2\varepsilon^q v)} \dot{W}^v.$$

*Step 5.* We now verify from Lemma 3.1 and Step 1 that  $\liminf_{t \rightarrow \infty} \frac{R(v_{t,\cdot})}{t} \geq 1$ . Note that the weak uniqueness of the SPDE (3.4) in  $\mathcal{C}_{I,\beta}$  is inherited from the weak uniqueness of the SPDE (3.3) in  $\mathcal{C}_I$ , which is justified by Theorem 2.1(2).

*Final Step.* Note that for any  $t \geq 0$ ,

$$\begin{aligned} R(\underline{u}_{t,\cdot}) &= \sup\{x \in \mathbb{R} : \underline{u}_{t,x} \neq 0\} = \sup\{\alpha^{-2}x \in \mathbb{R} : \beta \underline{u}_{\alpha^{-4}(t), \alpha^{-2}x} \neq 0\} \\ &= \alpha^{-2} \sup\{x \in \mathbb{R} : v_{\alpha^4 t, x} \neq 0\} = \alpha^{-2} R(v_{\alpha^4 t, \cdot}). \end{aligned}$$

Therefore,

$$\begin{aligned} V_{f,\epsilon} &= \lim_{t \rightarrow \infty} \frac{R(u_{t,\cdot})}{t} \stackrel{\text{Step 3}}{\geq} \liminf_{t \rightarrow \infty} \frac{R(\underline{u}_{t,\cdot})}{t} = \liminf_{t \rightarrow \infty} \frac{\alpha^{-2} R(v_{\alpha^4 t, \cdot})}{t} \\ &= \liminf_{t \rightarrow \infty} \frac{\alpha^2 R(v_{\alpha^4 t, \cdot})}{\alpha^4 t} \stackrel{\text{Step 5}}{\geq} \alpha^2 = \varepsilon^{-2 \frac{1-p}{1+p}} = (\sqrt{2}\gamma)^{2 \frac{1-p}{1+p}} \cdot \epsilon^{-2 \frac{1-p}{1+p}}. \end{aligned}$$

Finally, noticing that  $\epsilon$  is arbitrarily chosen from  $(0, \epsilon_0)$ , and that  $\gamma = \min\{\gamma_0, e^{-4}\}$  is independent of the choice of this  $\epsilon$ , by taking  $\epsilon \downarrow 0$ , we get

$$\liminf_{\epsilon \downarrow 0} \epsilon^{2 \frac{1-p}{1+p}} V_{f,\epsilon} \geq (\sqrt{2}\gamma)^{2 \frac{1-p}{1+p}} > 0. \quad \square$$

## 4. PROOF OF THEOREM 1.1(b)

We first assume without loss of generality that  $f = \tilde{f}$  where

$$\tilde{f}(z) := z^p \wedge \sqrt{1-z}, \quad z \in [0, 1].$$

We can do this because for any general  $f$  satisfying (1.7) and (1.9), it holds that

$$K := \sup_{z \in [0, 1]} f(z)/\tilde{f}(z) < \infty.$$

By using Lemmas 2.3 and 2.4, we can then verify that

$$\epsilon^{2\frac{1-p}{1+p}} V_{f, \epsilon} = \epsilon^{2\frac{1-p}{1+p}} \sqrt{K} V_{f/K, \epsilon/K^{1/4}} \leq K^{\frac{1}{1+p}} \tilde{\epsilon}^{2\frac{1-p}{1+p}} V_{\tilde{f}, \tilde{\epsilon}}$$

where  $\tilde{\epsilon} = \epsilon/K^{1/4}$ . From here, it is clear that if Theorem 1.1(b) holds for  $f = \tilde{f}$ , then it also holds for every  $f$  satisfying (1.7) and (1.9).

To get an upper bound for the speed, we will construct a sequence of updating frontiers using a non-negative deterministic function  $F : \mathbb{R} \rightarrow \mathbb{R}$ , and control the propagation of  $u$  using an updating procedure. More precisely, the  $n$ -th updating frontier will be defined as

$$F^{(n)}(x) := F(x - ndvT), \quad x \in \mathbb{R}, \quad n \in \mathbb{Z}_+$$

where  $d, v, T > 0$  are parameters that will be specified shortly. Define  $\xi_0 = 0$ . For each  $n \in \mathbb{Z}_+$ , we inductively run a process  $\tilde{u}$ , satisfying SPDE (1.3), starting from time  $\xi_n$  and configuration data  $1 \wedge F^{(n)}$ , stopped right before the stopping time

$$\xi_{n+1} := (\xi_n + T) \wedge \inf\{t \geq \xi_n : \tilde{u}_{t,x} > 1 \wedge F^{(n+1)}(x) \text{ for some } x \in \mathbb{R}\}.$$

By the comparison principle,  $\tilde{u}$  will travel faster than the original process. This allows us to get an upper bound by calculating the speed of the new process  $\tilde{u}$ . However, in order to get a reasonably good upper bound, we need to choose  $F$ ,  $d$ ,  $v$  and  $T$ , parameters in this updating procedure, very carefully. So, for the sake of precision, let us first give our choice of  $F, d, v$  and  $T$  here, along with several other quantities that will be used throughout the rest of the paper.

(4.1) Let us fix a constant  $\theta \in (1/2, 1)$  and define

$$\kappa := (p^{\frac{p}{1-p}} - p^{\frac{1}{1-p}}) \theta^{\frac{p}{p-1}} (1 - \theta)^{\frac{1}{p-1}}.$$

Since  $p \in [1/2, 1)$  we can verify that  $\kappa > 0$  and  $\kappa^{p-1} \leq 1$ .

(4.2) Fix a  $\mathcal{K} > 0$  large enough so that

$$2^5 \sum_{n=1}^{\infty} \exp(-2^{-22} e^{-2\theta(2-\theta)} \mathcal{K} e^{(2\theta-1)n}) \leq 1/8.$$

(4.3) Let us fix a constant  $\gamma > 0$  small enough so that

$$2^7 \sqrt{\gamma} \exp\{3\gamma\nu\} \leq 1/8; \quad \nu^p \leq \nu/4; \quad \mathcal{K}/\gamma \geq 2.$$

where  $\nu := 2^4 + (2^5 \mathcal{K} + 2^{14} \mathcal{K}^{1/2}) \gamma^{-1}$ .

(4.4) Define  $k := \mathcal{K}/\gamma$  and  $d := k + \nu + 1$ .

(4.5) For each  $\epsilon > 0$  let us define  $\varepsilon$ ,  $v$ ,  $T$  and  $L$  so that the following hold:

$$\varepsilon = \gamma\epsilon^2, \quad \varepsilon = \kappa v^{\frac{p+1}{p-1}}, \quad T = v^{-2}, \quad L = v^{-1}.$$

(4.6) Let us fix an  $\epsilon_0 > 0$  small enough so that for any  $\epsilon \in (0, \epsilon_0)$ ,

$$v\varepsilon L \leq 1/4 \quad \text{and} \quad 2^{13}\sqrt{\epsilon^2 L} \leq 1/4.$$

(4.7) For each  $\epsilon > 0$  define

$$F(x) := \frac{\varepsilon}{\theta_V}(e^{-\theta_V x} - 1)\mathbf{1}_{x \leq 0}, \quad x \in \mathbb{R}.$$

With the choice of the above quantities, we can verify the following proposition whose proof is postponed to Section 5.

**Proposition 4.1.** *For any  $\epsilon \in (0, \epsilon_0)$  and any  $\mathcal{C}_I$ -valued weak solution  $u$  to the SPDE (1.3) on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  with  $u_{0,\cdot} = 1 \wedge F$ , it holds that*

$$\mathbb{P}(\forall (t, x) \in [0, T] \times \mathbb{R}, u_{t,x} \leq 1 \wedge F(x - dvT)) \geq 1/2.$$

Below we show that this proposition is sufficient for Theorem 1.1(b).

*Proof of Theorem 1.1(b).* Fix an arbitrary  $\epsilon \in (0, \epsilon_0)$ , and let  $u$  be a  $\mathcal{C}_I$ -valued weak solution to the SPDE (1.3) with  $u_{0,\cdot} = 1 \wedge F$ . Let the process  $\tilde{u}$  be constructed using the updating procedure described at the beginning of this section with parameters  $F$ ,  $d$ ,  $v$  and  $T$  given as in (4.1)–(4.7). The corresponding updating times are denoted by  $(\xi_n)_{n \geq 0}$ . By the comparison principle, without loss of generality, we assume that  $\tilde{u}$  and  $u$  are constructed on the same filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  such that  $\tilde{u}_{t,x} \geq u_{t,x}$  for each  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ . Note, from the strong Markov property, that  $(\xi_{n+1} - \xi_n)_{n \in \mathbb{Z}_+}$  is a sequence of i.i.d. random variables. Also note from Proposition 4.1 that

$$\mathbb{P}(\xi_1 = T) \geq \mathbb{P}(\forall (t, x) \in [0, T] \times \mathbb{R}, u_{t,x} \leq 1 \wedge F(x - dvT)) \geq 1/2.$$

So by the strong law of large numbers, we have almost surely

$$\lim_{n \rightarrow \infty} \frac{\xi_n}{n} = \mathbb{E}[\xi_1] \geq T \cdot \mathbb{P}(\xi_1 = T) \geq T/2.$$

Also observe that from the way  $\tilde{u}$  is constructed, we always have

$$R(\tilde{u}_{\xi_n, \cdot}) = R(1 \wedge F^{(n)}) = ndvT, \quad n \in \mathbb{N}.$$

Now we can verify that

$$\begin{aligned} V_{f,\epsilon} &:= \lim_{t \rightarrow \infty} \frac{R(u_t)}{t} \leq \liminf_{t \rightarrow \infty} \frac{R(\tilde{u}_t)}{t} \leq \liminf_{n \rightarrow \infty} \frac{R(\tilde{u}_{\xi_n})}{\xi_n} \\ &\leq 2dv = 2d(\kappa^{-1}\gamma\epsilon^2)^{-\frac{1-p}{1+p}}. \end{aligned}$$

Finally, note that  $d$ ,  $\kappa$  and  $\gamma$  are independent of the choice of  $\epsilon \in (0, \epsilon_0)$ , and hence we are done.  $\square$

## 5. PROOF OF PROPOSITION 4.1

Let us fix an arbitrary  $\epsilon \in (0, \epsilon_0)$ . Let  $\varepsilon, v, L, T, F, k$  and  $\nu$  be given as in (4.1)-(4.7). The function  $F$  plays an important role in the updating procedure described in Section 4. The main reason we choose  $F$  as in (4.7) is given by the following analytical lemma. Let us define

$$(5.1) \quad \varrho_{t,x} := F(x - vt), \quad t \geq 0, x \in \mathbb{R}; \quad \bar{f}(z) := (1 - \theta)\theta v^2 z + (1 - \theta)v\varepsilon, \quad z \in \mathbb{R}_+.$$

**Lemma 5.1.** *For every  $z \in [0, 1]$ , it holds that  $\bar{f}(z) \geq f(z)$ . Moreover,  $(\varrho_{t,x} : t \geq 0, x \in \mathbb{R})$  is the solution to the PDE*

$$(5.2) \quad \begin{cases} \partial_t \varrho = \partial_x^2 \varrho + \bar{f}(\varrho), & x < vt, \\ \varrho = 0, & x \geq vt, \end{cases}$$

with initial condition  $\varrho_{0,\cdot} = F$ .

*Proof. Step 1.* It can be verified directly that  $\varrho$  satisfies (5.2).

*Step 2.* To finish the proof, we show that  $\bar{f}(z) \geq z^p$  for all  $z \geq 0$ . Note that

- $\bar{f}$  is a linear function with slope  $(1 - \theta)\theta v^2$ ; and
- $z \mapsto z^p$  is a concave function on  $[0, \infty)$ .

So we only have to show that  $\bar{f}(z_0) \geq z_0^p$  where  $z_0 > 0$  solves  $\partial_z z^p|_{z=z_0} = (1 - \theta)\theta v^2$ . Actually, it is easy to calculate that  $z_0 = (p^{-1}(1 - \theta)\theta v^2)^{\frac{1}{p-1}}$ . From this, and how  $\kappa$  and  $\varepsilon$  are defined in (4.1) and (4.5), we can verify that  $\bar{f}(z_0) - z_0^p = 0$ .  $\square$

In the rest of this paper, let us assume  $\sigma(z) = \sqrt{z(1 - z)}$  for  $z \in [0, 1]$ . To build a connection between  $\varrho$  and  $u$  we use the following two SPDEs:

$$(5.3) \quad \begin{cases} \partial_t v = \partial_x^2 v + f(v) + \epsilon \sigma(v) \dot{W}^v, & x < vt, \\ v = 0, & x \geq vt; \end{cases}$$

and

$$(5.4) \quad \begin{cases} \partial_t \bar{v} = \partial_x^2 \bar{v} + \bar{f}(\bar{v}) + \epsilon \sigma(\bar{v}) \dot{W}^{\bar{v}}, & x < vt, \\ \bar{v} = 0, & x \geq vt. \end{cases}$$

Let us be precise about the solution concept of (5.3) and (5.4) by first introducing a kernel  $G^{(v)}$ . For each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ , let  $B = (B_t)_{t \geq s}$  be a one dimensional Brownian motion with generator  $\partial_x^2$  initiated at time  $s$  and position  $y$  defined on a filtered probability space with probability measure denoted as  $\Pi_{s,y}$ . Let us define

$$(5.5) \quad \rho := \inf\{t : B_t \geq vt\}.$$

Denote by  $b\mathcal{B}(\mathbb{R})$  the space of all bounded Borel functions on  $\mathbb{R}$ . It can be verified that for each  $0 \leq s < t < \infty$  and  $y < vs$  there exists a unique continuous map  $x \mapsto G_{s,y;t,x}^{(v)}$  from  $(-\infty, vt)$  to  $(0, \infty)$  such that

$$\int_{-\infty}^{vt} G_{s,y;t,x}^{(v)} \varphi(x) dx = \Pi_{s,y}[\varphi(B_t); t < \rho], \quad \varphi \in b\mathcal{B}(\mathbb{R}).$$

The precise expression of  $G^{(v)}$  can be calculated using the reflection principle and the Girsanov transformation for the Brownian motion (see [21, Proof of Lemma 6.2]). We define  $G_{s,y;t,x}^{(v)} = 0$  on  $\{(s, y; t, x) : 0 \leq s < t, y < vs, x < vt\}^c$  for convention.

We say  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, P, v, W^v)$  is a weak solution to the SPDE (5.3), if  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, P)$  is a filtered probability space on which a predictable random field  $v$  and a white noise  $W^v$  are defined so that  $(v_{t,\cdot} : t \geq 0)$  is a  $\mathcal{C}_{\text{tem}}$ -valued continuous process satisfying

$$v_{t,x} = \int_0^t \int_0^x G_{s,y;t,x}^{(v)} M^v(ds dy), \quad \text{a.s.} \quad t > 0, x \in \mathbb{R},$$

where

$$(5.6) \quad M^v(ds dy) := v_{0,y} \delta_0(ds) dy + f(v_{s,y}) ds dy + \sigma(v_{s,y}) W^v(ds dy).$$

With some abuse of notation, we sometimes only use the random field  $v$  to represent a weak solution to the SPDE (5.3) if there is no risk of confusion. Given a subset  $\hat{\mathcal{C}} \subset \mathcal{C}_{\text{tem}}$ , we say a weak solution  $v$  to the SPDE (5.3) is a  $\hat{\mathcal{C}}$ -valued weak solution if  $(v_{t,\cdot} : t \geq 0)$  is a  $\hat{\mathcal{C}}$ -valued process. The concept of weak solution to the SPDE (5.4) is given in a similar way.

The main idea behind the proof of Proposition 4.1 is that  $v$  can be shown to satisfy the property which is similar to that desired for  $u$  in Proposition 4.1; and if  $u$  and  $v$  have the same initial value  $1 \wedge F$ , then they can be coupled in such a way that they don't deviate from each other "too much" before time  $T$ . This coupling is described in the following proposition whose proof are postponed to Section 6. The difference between  $u$  and  $v$  in the coupling will be controlled by a random field  $w$ .

**Proposition 5.2.** *There exists  $(v, W^v; \bar{v}, W^{\bar{v}}; w, W^w; u, W^u)$  defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, P)$  such that the followings holds.*

- (1)  $W^v, W^{\bar{v}}, W^w$  and  $W^u$  are white noises where  $W^v$  and  $W^w$  are independent of each other.
- (2)  $v$  is a  $\mathcal{C}_{[0,1]}$ -valued weak solution to the SPDE (5.3) with  $v_{0,\cdot} = 1 \wedge F$ .
- (3)  $\bar{v}$  is a  $\mathcal{C}_{\text{tem}}^+$ -valued weak solution to the SPDE (5.4) with  $\bar{v}_{0,\cdot} = F$ .
- (4) Almost surely  $\bar{v} \geq v$  on  $\mathbb{R}_+ \times \mathbb{R}$ .
- (5)  $u$  is a  $\mathcal{C}_I$ -valued weak solution to the SPDE (1.3) with  $W = W^u$ ,  $u_{0,\cdot} = 1 \wedge F$ .
- (6)  $w$  is a non-negative predictable random field so that  $(w_{t,\cdot} : t \geq 0)$  is a  $\mathcal{C}_{\text{tem}}$ -valued continuous process, and that, for each  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$  and  $t \geq 0$  almost surely

$$(5.7) \quad \begin{aligned} \int \phi_{t,x} w_{t,x} dx &= \int_0^t \int_0^x w_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) ds dy + \int_0^t \phi_{s,vs} dA_s \\ &\quad + \int_0^t \int_0^x \phi_{s,y} (f_{s,y}^w ds dy + \epsilon \sigma_{s,y}^w W^w(ds dy)). \end{aligned}$$

Here,  $f^w$  and  $\sigma^w$  are random fields given so that for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\begin{aligned} f_{s,y}^w &:= |f(v_{s,y} + w_{s,y}) - f(v_{s,y})| \mathbf{1}_{y \in [-L, vT+L], v_{s,y} + w_{s,y} \leq v \varepsilon L}, \\ \sigma_{s,y}^w &:= \sqrt{|\sigma(v_{s,y} + w_{s,y})^2 - \sigma(v_{s,y})^2| \vee \frac{w_{s,y}}{2}}; \end{aligned}$$



and  $(A_t)_{t \geq 0}$  is a real-valued adapted non-decreasing continuous process so that for each  $t \geq 0$  almost surely

$$A_t = \int_0^t \int_0^s \Pi_{s,y}(\rho \leq t) M^v(ds dy).$$

(7) It holds almost surely that  $u = v + w$  on  $[0, \tau] \times \mathbb{R}$ . Here the optional time

$$\tau := \min\{T, \tau_1, \tau_2\}$$

is defined using

$$\tau_1 := \inf \left\{ t \in [0, T] : \int_0^t ds \int_{[-L, \nu T + L]^c} w_{s,y} dy > 0 \right\},$$

$$\tau_2 := \inf \left\{ t \in [0, T] : v_{t,x} + w_{t,x} \geq \nu \varepsilon L \text{ for some } x \in [-L, \nu T + L] \right\},$$

with the convention that the infimum of the empty set is infinite.

We will show that  $v$  satisfies a similar property which we desired for  $u$ . This is done in the following proposition whose proof is postponed to Section 9.

**Proposition 5.3.** *Let  $v$  be given by Proposition 5.2. Then  $P(\tau_3 < T) < 1/8$  where*

$$\tau_3 := \inf \{ t \in [0, T] : v_{t,x} \geq F(x - \nu t) + k\varepsilon L e^{-\theta \nu(x - \nu t)} \mathbf{1}_{x \leq \nu t} \text{ for some } x \in (-\infty, \nu t] \}.$$

From Proposition 5.2 (7), the difference between  $u$  and  $v$  can be controlled by the process  $w$  up to the stopping time  $\tau$ . We use the following two propositions to control this stopping time. Their proofs are postponed later to Sections 7 and 10 respectively.

**Proposition 5.4.** *Let  $\tau_1$  be given by Proposition 5.2. Then it holds that  $P(\tau_1 < T) < 1/8$ .*

**Proposition 5.5.** *Let  $\tau_1$  and  $\tau_2$  be given by Proposition 5.2. Let  $\tau_3$  be given by Proposition 5.3. Then it holds that  $P(\tau_2 < T, \tau_3 \geq T, \tau_1 \geq T) < 1/8$ .*

We are now ready to give the proof of Proposition 4.1 using Propositions 5.2-5.5.

*Proof of Proposition 4.1.* Thanks to the weak uniqueness, we only have to prove the desired result for a specific  $\mathcal{C}_I$ -valued weak solutions with initial value  $1 \wedge F$ . So, let us take the weak solution  $u$  to the SPDE 1.3 given as in Proposition 5.2. Let also  $v, w, \tau_1, \tau_2$  be as in Proposition 5.2, and  $\tau_3$  as in Proposition 5.3. To get the desired result we only have to verify that

$$(5.8) \quad \bigcap_{i=1,2,3} \{ \tau_i \geq T \} \subset \{ \forall (t, x) \in [0, T] \times \mathbb{R}, u_{t,x} \leq 1 \wedge F(x - \nu t) \},$$

since by Propositions 5.3-5.5,

$$\begin{aligned} P\left( \bigcap_{i=1,2,3} \{ \tau_i \geq T \} \right) &= 1 - P(\{ \tau_1 < T \} \cup \{ \tau_2 < T, \tau_1 \geq T, \tau_3 \geq T \} \cup \{ \tau_3 < T \}) \\ &\geq 1 - (P(\tau_1 < T) + P(\tau_2 < T, \tau_1 \geq T, \tau_3 \geq T) + P(\tau_3 < T)) \geq 1/2. \end{aligned}$$

In the rest of the proof, we verify (5.8). First note that for any  $x \in \mathbb{R}$  and  $l > 0$ ,

$$F(x - l) - F(x) = \frac{\varepsilon}{\theta \nu} (e^{-\theta \nu(x-l)} - 1) \mathbf{1}_{x-l \leq 0} - \frac{\varepsilon}{\theta \nu} (e^{-\theta \nu x} - 1) \mathbf{1}_{x \leq 0}$$

$$(5.9) \quad \geq \frac{\varepsilon}{\theta v} e^{-\theta v x} (e^{\theta v l} - 1) \mathbf{1}_{x \leq 0} \geq \varepsilon l e^{-\theta v x} \mathbf{1}_{x \leq 0} \geq \varepsilon l \mathbf{1}_{x \leq 0}.$$

Then notice that almost surely on the event  $\cap_{i=1,2,3} \{\tau_i \geq T\}$ , we have

$$\begin{aligned} u_{t,x} &= v_{t,x} + w_{t,x}, & t \in [0, T], x \in \mathbb{R}; \\ w_{t,x} &\leq \nu \varepsilon L \mathbf{1}_{x \in [-L, \nu T + L]}, & t \in [0, T], x \in \mathbb{R}; \\ v_{t,x} &\leq F(x - \nu t) + k \varepsilon L e^{-\theta v(x - \nu t)} \mathbf{1}_{x \leq \nu t}, & t \in [0, T], x \in \mathbb{R}. \end{aligned}$$

Therefore, almost surely on event  $\cap_{i=1,2,3} \{\tau_i \geq T\}$ , we have that for any  $(t, x) \in [0, T] \times \mathbb{R}$ ,

$$\begin{aligned} u_{t,x} &= v_{t,x} + w_{t,x} \leq F(x - \nu t - kL) + \nu \varepsilon L \mathbf{1}_{x \in [-L, \nu T + L]} \\ &\leq F(x - \nu T - kL) + \nu \varepsilon L \mathbf{1}_{x - \nu T - kL \leq 0} \leq F(x - \nu T - (k + \nu)L). \end{aligned}$$

In the second inequity above, we used the fact that  $F$  is non-increasing and that  $k \geq 1$ . The third inequity follows easily by (5.9).

Finally, noticing that  $u_{t,x} \leq 1$  and according to (4.5) that  $L = \nu T$ , (5.8) follows.  $\square$

## 6. PROOF OF PROPOSITION 5.2

We will need the following lemma to identify the killing of  $v$  at its boundary. Its proof is given in the Appendix.

**Lemma 6.1.** *Suppose that  $v$  is a  $\mathcal{C}_{[0,1]}$ -valued weak solution to the SPDE (5.3) with  $v_{0,\cdot} = 1 \wedge F$ . Then*

(1) *for each  $\phi \in \mathcal{C}_c^{1,2}(\mathbb{R}_+ \times \mathbb{R})$  and  $t \geq 0$  it holds almost surely that,*

$$(6.1) \quad \begin{aligned} \int \phi_{t,x} v_{t,x} dx &= \int \int_0^t \phi_{s,y} M^v(ds dy) + \int \int_0^t v_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) ds dy \\ &\quad - \int \int_0^t \Pi_{s,y}[\phi_{\rho, B_\rho}; t \geq \rho] M^v(ds dy) \end{aligned}$$

where  $M^v$  is given by (5.6) and  $\rho$  is defined in (5.5);

(2) *there exists an adapted, real-valued, almost surely non-decreasing process  $(A_t)_{t \geq 0}$  satisfying that for each  $t \geq 0$  and bounded Borel measurable function  $\psi$  on  $\mathbb{R}_+$ ,*

$$(6.2) \quad \int_0^t \psi_s dA_s = \int \int_0^t \Pi_{s,y}[\psi_\rho; t \geq \rho] M^v(ds dy), \quad \text{a.s.}$$

In what follows, we refer to  $(A_t)_{t \geq 0}$  as the killing process of  $v$  at its boundary.

*Proof of Proposition 5.2. Step 1.* Using a strategy similar to the proof of [21, Proposition 5.1], we can verify that there exists a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and stochastic elements  $(v, W^v; \bar{v}, W^{\bar{v}}; w, W^w)$  on it such that

- $W^v, W^{\bar{v}}$  and  $W^w$  are white noises where  $W^w$  is independent of  $W^v$ ; and
- (2), (3), (4) and (6) of Proposition 5.2 hold.

Lemma 6.1 is used here to justify that the second term on the right hand side of (5.7) is well-defined.

*Step 2.* Define optional time  $\tau$  as in Proposition 5.2 (7) using  $v$  and  $w$  constructed in Step 1. Extending the space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  if necessary, we can construct a pair  $(u, \widetilde{W})$  so that

- $\widetilde{W}$  is a white noise independent of  $(v, W^v; \bar{v}, W^{\bar{v}}; w, W^w)$ ;
- $u$  is a  $\mathcal{C}_{[0,1]}$ -valued weak solution to the SPDE

$$\begin{cases} u = v + w, & \text{on } [0, \tau] \times \mathbb{R}, \\ \partial_t u = \partial_x^2 u + f(u) + \epsilon \sigma(u) \dot{\widetilde{W}}, & \text{on } [\tau, \infty) \times \mathbb{R}. \end{cases}$$

The existence of such  $u$  after the optional time  $\tau$  is due to [28, Theorem 2.6].

*Step 3.* We will show that almost surely

$$(6.3) \quad \sigma_{s,y}^w = \sqrt{\sigma(v_{s,y} + w_{s,y})^2 - \sigma(v_{s,y})^2}, \quad (s, y) \in [0, \tau] \times \mathbb{R}$$

and

$$f_{s,y}^w = f(v_{s,y} + w_{s,y}) - f(v_{s,y}), \quad (s, y) \in [0, \tau] \times \mathbb{R}.$$

This is obvious for  $(s, y) \in [0, \tau] \times [-L, \nu T + L]^c$  since in this case  $w_{s,y} = 0$ . Let us now consider the case  $(s, y) \in [0, \tau] \times [-L, \nu T + L]$ . Note that in this case, from the definition of  $\tau$  and (4.6) we have  $v_{s,y} + w_{s,y} \leq \nu \varepsilon L \leq 1/4$ . From the properties of functions  $f$  and  $\sigma$ , we observe that for any  $\mathbf{v}, \mathbf{w} \in [0, 1]$ , we have

$$\sqrt{|\sigma(\mathbf{v} + \mathbf{w})^2 - \sigma(\mathbf{v})^2| \vee \frac{\mathbf{w}}{2}} = \sqrt{\sigma(\mathbf{v} + \mathbf{w})^2 - \sigma(\mathbf{v})^2}$$

and

$$|f(\mathbf{v} + \mathbf{w}) - f(\mathbf{v})| = f(\mathbf{v} + \mathbf{w}) - f(\mathbf{v})$$

provided  $\mathbf{v} + \mathbf{w} \leq \nu \varepsilon L \leq 1/4$ . Thus, the desired result in this step follows.

*Step 4.* We can verify that there exists a white noise  $W^u$  so that for any  $g \in \mathcal{L}_{\text{loc}}^2$ ,

$$(6.4) \quad \begin{aligned} \iint_0^t g_{s,y} W^u(\text{d} s \text{d} y) &= \iint_0^t \frac{g_{s,y} \mathbf{1}_{B_{s,y}}}{\sigma(v_{s,y} + w_{s,y})} (\sigma(v_{s,y}) W^v(\text{d} s \text{d} y) + \sigma_{s,y}^w W^w(\text{d} s \text{d} y)) \\ &\quad + \iint_0^t g_{s,y} \mathbf{1}_{B_{s,y}^c} \widetilde{W}(\text{d} s \text{d} y), \quad t \geq 0, \text{ a.s.}, \end{aligned}$$

where for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$  the event  $B_{s,y} := \{s \leq \tau, \sigma(v_{s,y} + w_{s,y}) > 0\}$ . To see this, one only have to calculate the quadratic variation of the right hand side of (6.4) using (6.3) and the fact that  $W^w, W^v$  and  $\widetilde{W}$  are mutually independent.

*Final step.* Observe from Step 4 that for any  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\iint_0^t G_{s,y;t,x} \sigma(u_{s,y}) W^u(\text{d} s \text{d} y) = \iint_0^t G_{s,y;t,x} (\sigma(v_{s,y}) W^v(\text{d} s \text{d} y) + \sigma_{s,y}^w W^w(\text{d} s \text{d} y))$$

holds almost surely on the event  $\{t \leq \tau\}$ ; also

$$\iint_\tau^t G_{s,y;t,x} \sigma(u_{s,y}) W^u(\text{d} s \text{d} y) = \iint_\tau^t G_{s,y;t,x} \sigma(u_{s,y}) \widetilde{W}(\text{d} s \text{d} y)$$

holds almost surely on the event  $\{t > \tau\}$ . We can then verify that  $u$  is a  $\mathcal{C}_{[0,1]}$ -valued weak solution to the SPDE (1.3) with  $W = W^u$ ,  $u_{0,\cdot} = 1 \wedge F$ . Thus, Proposition 5.2 (5) follows from Theorem 2.1 (2).  $\square$

## 7. PROOF OF PROPOSITION 5.4

Let us write (5.7) in the following short form:

$$\partial_t w = \partial_x^2 w + f^w + \sigma^w \dot{W}^w + \delta_{vt}(x) \dot{A}_t.$$

The first step of the proof is to remove the drift term  $f^w$  using Dawson's Girsanov transformation. We summarize this transformation in the following lemma. We refer the reader to [8, Section 10.2.1] for its proof. Note that the definition of the stochastic integrals depends on the underlying probability measure  $P$ . So sometimes we write “ $P$ - $\int\int$ ” for the stochastic integrals to emphasize this dependence.

**Lemma 7.1.** *Suppose that  $W$  is a white noise defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, P)$ . Suppose that  $h$  is a real-valued predictable random field satisfying*

$$\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^\infty \int_0^\infty h_{s,y}^2 ds dy \right\} \right] < \infty.$$

*Then under the probability measure  $Q$  given by*

$$dQ := \exp \left\{ \int_0^\infty \int_0^\infty h_{s,y} W(ds dy) - \frac{1}{2} \int_0^\infty \int_0^\infty h_{s,y}^2 ds dy \right\} dP,$$

*there exists a white noise  $\tilde{W}$  satisfying that for each  $g \in \mathcal{L}_{\text{loc}}^2$  almost surely*

$$Q\text{-} \int_0^t \int_0^t g_{s,y} \tilde{W}(ds dy) = P\text{-} \int_0^t \int_0^t g_{s,y} W(ds dy) - \int_0^t \int_0^t h_{s,y} g_{s,y} ds dy.$$

*Remark 7.2.* Let  $Q$  and  $P$  be the probability measure in Lemma 7.1. One can verify that  $Q$  and  $P$  are mutually absolute continuous. In other word,  $A \subset \Omega$  is a  $Q$ -null set if and only if  $A$  is a  $P$ -null set. Therefore, the filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, Q)$  also satisfies the usual hypotheses; and there is no need to distinguish between “ $P$ -a.s.” and “ $Q$ -a.s.”.

Later in the proof of Proposition 5.4, we will construct a new probability measure  $Q$ , using Lemma 7.1, under which  $w$  will satisfy

$$\partial_t w = \partial_x^2 w + \sigma^w \tilde{W}^w + \delta_{vt}(x) \dot{A}_t, \quad t \in [0, T], x \in \mathbb{R}$$

where  $\tilde{W}^w$  is a white noise under  $Q$ . In order to study the support of  $w$  under this new probability, we will need the following proposition whose proof is postponed later to Section 8. In what follows, we say that a random measure  $\mu$  on a Polish space  $S$  defined in a probability space  $(\Omega, \mathcal{G}, P)$  is with finite mean if its mean measure

$$(P\mu)(\cdot) := \int_\Omega \mu(\omega, \cdot) P(d\omega)$$

is a finite measure on  $S$ . For more on random measures see [16].

**Proposition 7.3.** *Let  $\tilde{T} > 0$  be arbitrary. Suppose that  $\tilde{w}$  is an adapted non-negative continuous random field, defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F})_{t \geq 0}, \mathbb{Q})$ , such that  $(\tilde{w}_{t,\cdot} : t \geq 0)$  is a  $\mathcal{C}_{\text{tem}}$ -valued continuous process, and for each  $t \in [0, \tilde{T}]$  and  $\phi \in C_c^\infty([0, \tilde{T}] \times \mathbb{R})$ ,*

$$(7.1) \quad \int \phi_{t,x} \tilde{w}_{t,x} dx = \int \int_0^t \tilde{w}_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) ds dy + \int \int_0^t \phi_{s,y} (\tilde{\sigma}_{s,y} W(ds dy) + \mu(ds dy)), \text{ a.s.}$$

Here  $\tilde{\sigma}$  is a predictable random field,  $W$  is a white noise, and  $\mu$  is a random measure on  $\mathbb{R}_+ \times \mathbb{R}$  with finite mean. Suppose that there exist deterministic  $\tilde{\vartheta} \geq \vartheta > 0$  satisfying that almost surely  $\tilde{\vartheta} \sqrt{\tilde{w}} \geq \tilde{\sigma} \geq \vartheta \sqrt{\tilde{w}}$  on  $\mathbb{R}_+ \times \mathbb{R}$ . Then for each  $-\infty \leq a < b \leq \infty$  it holds that

$$\mathbb{Q} \left( \int_0^{\tilde{T}} ds \int_{[a,b]^c} \tilde{w}_{s,y} dy > 0 \right) \leq \mathbb{E}^{\mathbb{Q}} \left[ \int \int_0^{\tilde{T}} (\zeta_{\tilde{T}-s,b-y}^{\vartheta} + \zeta_{\tilde{T}-s,y-a}^{\vartheta}) \mu(ds dy) \right]$$

where

$$(7.2) \quad \zeta_{s,y}^{\vartheta} := \begin{cases} 0, & s \geq 0, y = \infty; \\ \frac{2^8 \sqrt{s}}{\vartheta^2 y^3} e^{-\frac{y^2}{24s}}, & s \geq 0, y > 0; \\ \infty, & s \geq 0, y \leq 0. \end{cases}$$

In order to control the support of  $w$  using the above proposition, we will investigate the expectation of  $A_t$  under the new probability measure  $\mathbb{Q}$  which is absolutely continuous with respect to the original probability measure. Recall that  $A_t$  is given in Lemma 6.1(2) and can be considered as the amount of mass of  $v$  killed at the line  $\{(s, y) \in \mathbb{R}_+ \times \mathbb{R} : y = vs, s \leq t\}$ . We will show that under the new probability  $\mathbb{Q}$ ,  $v$  is still a weak solution to the SPDE (5.3), and, in fact, for any such weak solution, we can derive the upper bound on the expectation of  $A_t$  using the following lemma whose proof is postponed in the Appendix.

**Lemma 7.4.** *Suppose that  $v$  is a  $\mathcal{C}_{[0,1]}$ -valued weak solution to the SPDE (5.3) defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  with  $v_{0,\cdot} = 1 \wedge F$ . Let  $(A_t)_{t \geq 0}$  be given as in Lemma 6.1 (2). Then,*

$$\mathbb{E}^{\mathbb{Q}}[A_t - A_r] \leq \varepsilon(t - r), \quad 0 \leq r \leq t < \infty.$$

As for showing that  $v$  is a weak solution to the SPDE (5.3), under the new probability  $\mathbb{Q}$ , this will be done with the help of the following lemma whose proof is standard and therefore is omitted (one can replicate the analogous classical proof for Brownian motions).

**Lemma 7.5.** *Suppose the conditions of Lemma 7.1 hold. Suppose that under probability  $\mathbb{P}$ ,  $W'$  is a white noise which is independent of  $W$ . Then under  $\mathbb{Q}$ ,  $W'$  is still a white noise. Moreover, for each  $t \geq 0$  and  $g \in \mathcal{L}_{\text{loc}}^2$ , it holds that*

$$\mathbb{Q}\text{-} \int \int_0^t g_{s,y} W'(ds dy) = \mathbb{P}\text{-} \int \int_0^t g_{s,y} W'(ds dy) \quad \text{a.s.}$$

We are now ready to give the proof of Proposition 5.4.

*Proof of Proposition 5.4. Step 1.* Noticing from (4.6) that  $\nu\varepsilon L \leq 1/4$ , and the fact that for any  $x, y \in [0, 1/4]$ ,

$$|f(y) - f(x)| = |y^p - x^p| = \left| \int_x^y pz^{p-1}dz \right| \leq \int_0^{|y-x|} pz^{p-1}dz = |y - x|^p,$$

we have almost surely for each  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\begin{aligned} h_{t,x} &:= \frac{f_{t,x}^w}{\epsilon \sigma_{t,x}^w} \mathbf{1}_{\sigma_{t,x}^w > 0, t \leq T} \\ &\leq \frac{|f(v_{t,x} + w_{t,x}) - f(v_{t,x})| \mathbf{1}_{x \in (-L, \nu T + L), v_{t,x} + w_{t,x} \leq \nu\varepsilon L} \mathbf{1}_{w_{t,x} > 0, t \leq T}}{\epsilon \sqrt{w_{t,x}/2}} \\ &\leq \sqrt{2}\epsilon^{-1} w_{t,x}^{p-\frac{1}{2}} \mathbf{1}_{x \in (-L, \nu T + L), w_{t,x} \leq \nu\varepsilon L, t \leq T} \leq \sqrt{2}\epsilon^{-1} (\nu\varepsilon L)^{p-\frac{1}{2}} \mathbf{1}_{x \in (-L, \nu T + L), t \leq T}. \end{aligned}$$

*Step 2.* We construct a probability measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{G})$  such that

$$(7.3) \quad d\mathbb{Q} = \exp \left\{ - \iint_0^\infty h_{s,y} W^w(dsdy) - \frac{1}{2} \iint_0^\infty h_{s,y}^2 dsdy \right\} d\mathbb{P}.$$

We can do this thanks to Step 1 since

$$\mathbb{E}^{\mathbb{P}} \left[ \exp \left\{ \frac{1}{2} \iint_0^\infty h_{s,y}^2 dsdy \right\} \right] < \infty.$$

*Step 3.* We verify that for any  $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$  and  $t \in [0, T]$ , almost surely

$$\begin{aligned} \int \phi_{t,x} w_{t,x} dx &= \iint_0^t w_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) dsdy + \int_0^t \phi_{s,\nu s} dA_s + \\ &\quad \mathbb{Q}\text{-} \iint_0^t \phi_{s,y} \epsilon \sigma_{s,y}^w \tilde{W}^w(dsdy) \end{aligned}$$

where  $\tilde{W}^w$  is a white noise on the filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  given as in Lemma 7.1 so that

$$\mathbb{Q}\text{-} \iint_0^\cdot g_{s,y} \tilde{W}^w(dsdy) = \mathbb{P}\text{-} \iint_0^\cdot g_{s,y} W^w(dsdy) - \iint_0^\cdot h_{s,y} g_{s,y} dsdy, \quad \text{a.s. } g \in \mathcal{L}_{\text{loc}}^2.$$

*Step 4.* We will show that for each  $t \geq 0$  and non-negative continuous function  $\psi$  on  $\mathbb{R}_+$  the following holds:

$$(7.4) \quad \mathbb{E}^{\mathbb{Q}} \left[ \int_0^t \psi_s dA_s \right] \leq \varepsilon \int_0^t \psi_s ds.$$

To see this, we verify from Lemma 7.5 that with respect to the filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ :

- $W^v$  is still a white noise;
- $v$  is still a weak solution to the SPDE (5.3) with  $v_{0,\cdot} = 1 \wedge F$ ;
- $(A_t)_{t \geq 0}$  is still the killing process of  $v$ ; see Lemma 6.1 (2).



Therefore from Lemma 7.4, we have  $\mathbb{E}^Q[A_t - A_r] \leq \varepsilon(t - r)$  for each  $0 \leq r \leq t < \infty$ . From this we can verify that (7.4) holds for each  $t \geq 0$  and each non-negative  $\psi \in \mathcal{K}$  where

$$(7.5) \quad \mathcal{K} := \left\{ \sum_{k \in \mathbb{N}} n_k \mathbf{1}_{(t_k, t_{k+1}]} : (n_k)_{k \in \mathbb{N}} \subset \mathbb{R} \text{ is bounded, } (t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+ \text{ is unbounded and strictly increasing} \right\}.$$

Now the desired result in this step follows from monotone convergence theorem and the fact that for any non-negative continuous function  $\psi$  on  $\mathbb{R}_+$  there exists a non-negative sequence  $(\psi^{(n)})_{n \in \mathbb{N}} \subset \mathcal{K}$  such that  $\psi^{(n)} \uparrow \psi$  pointwise as  $n \uparrow \infty$ .

*Step 5.* We will show that  $Q(\tau_1 < T) \leq 2^{14}\gamma$ . Note that almost surely

$$\epsilon \sqrt{w_{t,x}} \geq \epsilon \sigma_{t,x}^w \geq \frac{\epsilon}{\sqrt{2}} \sqrt{w_{t,x}}, \quad t \geq 0, x \in \mathbb{R}.$$

So from Proposition 7.3, Steps 3 and 4 we get that

$$\begin{aligned} Q(\tau_1 < T) &\leq \mathbb{E}^Q \left[ \int_0^T \int_0^T (\zeta_{T-s, (vT+L)-x}^{\epsilon/\sqrt{2}} + \zeta_{T-s, x-(-L)}^{\epsilon/\sqrt{2}}) \delta_{vs}(dx) dA_s \right] \\ &\leq \varepsilon \int_0^T (\zeta_{T-s, (vT+L)-vs}^{\epsilon/\sqrt{2}} + \zeta_{T-s, vs-(-L)}^{\epsilon/\sqrt{2}}) ds \leq 2\varepsilon \int_0^T \zeta_{T-s, L}^{\epsilon/\sqrt{2}} ds. \end{aligned}$$

Here in the last inequality, we used the fact that for any given  $s \geq 0$ , map  $x \mapsto \zeta_{s,x}^{\epsilon/\sqrt{2}}$  is non-increasing on  $\mathbb{R}$ . Now we have

$$\begin{aligned} Q(\tau_1 < T) &\leq 2\varepsilon \int_0^T \frac{2^9(T-s)^{1/2}}{\epsilon^2 L^3} e^{-\frac{L^2}{2^4(T-s)}} ds = \frac{2^{10}\gamma}{L^3} \int_0^T s^{1/2} e^{-\frac{L^2}{8s}} ds \\ &\leq \frac{2^{10}\gamma T^{5/2}}{L^3} \frac{2^4}{L^2} \int_{s=0}^{s=T} e^{-\frac{L^2}{2^4 s}} d\left(-\frac{L^2}{2^4 s}\right) \leq \frac{2^{14}\gamma T^{5/2}}{L^5} e^{-\frac{L^2}{2^4 T}} \leq 2^{14}\gamma. \end{aligned}$$

*Final Step.* Noticing that  $\tilde{W}^w$  is a white noise under  $Q$ , we can verify that for each  $q \in [1, \infty)$  the expectation of

$$m^{(q)} := \exp \left\{ q \iint_0^\infty h_{s,y} \tilde{W}^w(ds dy) - \frac{q^2}{2} \iint_0^\infty h_{s,y}^2 ds dy \right\}$$

under  $Q$  equals to 1. Also note from (7.3) and Lemma 7.1 we have that

$$\begin{aligned} \frac{dP}{dQ} &= \exp \left\{ \iint_0^\infty h_{s,y} W^w(ds dy) + \frac{1}{2} \iint_0^\infty h_{s,y}^2 ds dy \right\} \\ &= \exp \left\{ \iint_0^\infty h_{s,y} \tilde{W}^w(ds dy) - \frac{1}{2} \iint_0^\infty h_{s,y}^2 ds dy \right\} = m^{(1)}. \end{aligned}$$

Now we can verify using Cauchy–Schwartz inequality that

$$\begin{aligned} P(\tau_1 < T) &= \mathbb{E}^Q[\mathbf{1}_{\{\tau_1 < T\}} m^{(1)}] \leq Q(\tau_1 < T)^{\frac{1}{2}} \mathbb{E}^Q[(m^{(1)})^2]^{\frac{1}{2}} \\ &= Q(\tau_1 < T)^{\frac{1}{2}} \mathbb{E}^Q \left[ m^{(2)} \exp \left\{ \iint_0^\infty h_{s,y}^2 ds dy \right\} \right]^{\frac{1}{2}}. \end{aligned}$$

Finally, using (4.1), (4.3), (4.5) and Steps 1, 5 we have that

$$\begin{aligned} P(\tau_1 < T) &\leq Q(\tau_1 < T)^{\frac{1}{2}} \exp\{(vT + 2L)T\epsilon^{-2}(\nu\epsilon L)^{2p-1}\} \\ &= Q(\tau_1 < T)^{\frac{1}{2}} \exp\{3\gamma\kappa^{2p-2}\nu^{2p-1}\} \leq 2^7\sqrt{\gamma}\exp\{3\gamma\nu\} \leq 1/8. \end{aligned} \quad \square$$

### 8. PROOF OF PROPOSITION 7.3

In this section we will give the proof of Proposition 7.3 following a strategy similar to that used in [29, Proof of Proposition 3.2]. Next lemma deals with properties of the solutions to the so-called log-Laplace equations which play very important role in studying properties of superprocesses (see e.g. [12]). Its proof is standard and thus is moved to the Appendix.

**Lemma 8.1.** *Let  $\tilde{T} > 0, \vartheta > 0$  and  $\psi \in \mathcal{C}_0^2(\mathbb{R})$  be non-negative. There exists a unique non-negative  $\phi \in \mathcal{C}_b^{1,2}([0, \tilde{T}] \times \mathbb{R})$  such that*

$$(8.1) \quad \begin{cases} \partial_t \phi_{t,x} = -\partial_x^2 \phi_{t,x} + \frac{1}{2}(\vartheta \phi_{t,x})^2 - \psi_x, & (t, x) \in [0, \tilde{T}] \times \mathbb{R}; \\ \phi_{\tilde{T},x} = 0, & x \in \mathbb{R}. \end{cases}$$

Furthermore, if  $b \in \mathbb{R}$  and  $\psi = 0$  on  $(-\infty, b]$ , then

$$(8.2) \quad \phi_{t,x} \leq \zeta_{\tilde{T}-t, b-x}^{\vartheta}, \quad (t, x) \in [0, \tilde{T}] \times \mathbb{R}$$

where  $\zeta^{\vartheta}$  is given by (7.2).

In order to study the property of  $\tilde{w}$  using the above testing function  $\phi$ , we need the following lemma whose prove is postponed later in this section.

**Lemma 8.2.** *Under the conditions of Proposition 7.3, it holds that*

$$\sup_{t \in [0, \tilde{T}]} \mathbb{E}^Q \left[ \int \tilde{w}_{t,x} dx \right] < \infty.$$

Furthermore, (7.1) holds almost surely for each  $t \in [0, \tilde{T}]$  and  $\phi \in \mathcal{C}_b^{1,2}([0, \tilde{T}] \times \mathbb{R})$ .

We are now ready to give the proof of Proposition 7.3.

*Proof of Proposition 7.3. Step 1.* We only need to prove the desired result for the case  $-\infty = a < b < \infty$ . In fact, in the case of  $a = -\infty, b = \infty$ , nothing needs to be proved. And if the desired result holds for the case  $-\infty = a < b < \infty$ , then by symmetry, it also holds for the case  $-\infty < a < b = \infty$ . For the only left case  $-\infty < a < b < \infty$ , we use

$$Q\left(\int_0^{\tilde{T}} ds \int_{(a,b)^c} \tilde{w}_{s,y} dy > 0\right) \leq Q\left(\int_0^{\tilde{T}} ds \int_{-\infty}^a \tilde{w}_{s,y} dy > 0\right) + Q\left(\int_0^{\tilde{T}} ds \int_b^{\infty} \tilde{w}_{s,y} dy > 0\right).$$

*Step 2.* Fix  $b \in \mathbb{R}$  and a non-negative  $\psi \in \mathcal{C}_0^2(\mathbb{R})$  with support  $\{x \in \mathbb{R} : \psi_x > 0\} = (b, \infty)$ . For each  $n > 0$ , let  $\phi^{(n)} \in \mathcal{C}_b^{1,2}([0, \tilde{T}] \times \mathbb{R})$  be given by Lemma 8.1 with  $\psi$  replaced by  $n\psi$  and  $\vartheta$  from Proposition 7.3. For any  $n > 0$ , define process

$$M_t^{(n)} := n \iint_0^t \tilde{w}_{s,y} \psi_y ds dy + \int \tilde{w}_{t,x} \phi_{t,x}^{(n)} dx, \quad t \in [0, \tilde{T}].$$

*Step 3.* We note that

$$\begin{aligned} \mathbb{Q}\left(\int_0^{\tilde{T}} ds \int_b^\infty \tilde{w}_{s,y} dy > 0\right) &= \mathbb{Q}\left(\iint_0^{\tilde{T}} \tilde{w}_{s,y} \psi_y ds dy > 0\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}\left(1 - \exp\left\{-n \iint_0^{\tilde{T}} \tilde{w}_{s,y} \psi_y ds dy\right\}\right) = \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}(1 - e^{-M_{\tilde{T}}^{(n)}}). \end{aligned}$$

*Step 4.* We will verify that

$$\mathbb{E}^{\mathbb{Q}}(1 - e^{-M_{\tilde{T}}^{(n)}}) \leq \mathbb{E}^{\mathbb{Q}}\left[\iint_0^{\tilde{T}} \phi_{s,y}^{(n)} \mu(ds dy)\right], \quad n > 0.$$

In fact, from Lemma 8.2 we have for each  $t \in [0, \tilde{T}]$  almost surely

$$\begin{aligned} M_t^{(n)} &\stackrel{(7.1)}{=} n \iint_0^t \tilde{w}_{s,y} \psi_y ds dy + \iint_0^t \tilde{w}_{s,y} (\partial_y^2 \phi_{s,y}^{(n)} + \partial_s \phi_{s,y}^{(n)}) ds dy \\ &\quad + \iint_0^t \tilde{\sigma}_{s,y} \phi_{s,y}^{(n)} W(ds dy) + \iint_0^t \phi_{s,y}^{(n)} \mu(ds dy). \end{aligned}$$

Therefore, we have almost surely

$$\langle M^{(n)} \rangle_t = \iint_0^t (\tilde{\sigma}_{s,y} \phi_{s,y}^{(n)})^2 ds dy, \quad t \in [0, \tilde{T}].$$

Now, we use Itô's formula and get that for any  $t \in [0, \tilde{T}]$  almost surely,

$$\begin{aligned} e^{-M_t^{(n)}} - 1 &= \int_0^t (-e^{-M_s^{(n)}}) dM_s^{(n)} + \frac{1}{2} \int_0^t e^{-M_s^{(n)}} d\langle M^{(n)} \rangle_s \\ &= \iint_0^t (-e^{-M_s^{(n)}}) (n\tilde{w}_{s,y} \psi_y + \tilde{w}_{s,y} (\partial_y^2 \phi_{s,y}^{(n)} + \partial_s \phi_{s,y}^{(n)})) ds dy \\ &\quad + \iint_0^t (-e^{-M_s^{(n)}} \phi_{s,y}^{(n)}) (\tilde{\sigma}_{s,y} W(ds dy) + \mu(ds dy)) + \frac{1}{2} \iint_0^t e^{-M_s^{(n)}} (\tilde{\sigma}_{s,y} \phi_{s,y}^{(n)})^2 ds dy \\ &\stackrel{(8.1)}{=} \frac{1}{2} \iint_0^t e^{-M_s^{(n)}} (\phi_{s,y}^{(n)})^2 (\tilde{\sigma}_{s,y}^2 - \vartheta^2 \tilde{w}_{s,y}) ds dy \\ (8.3) \quad &+ \iint_0^t (-e^{-M_s^{(n)}} \phi_{s,y}^{(n)}) \tilde{\sigma}_{s,y} W(ds dy) + \iint_0^t (-e^{-M_s^{(n)}} \phi_{s,y}^{(n)}) \mu(ds dy). \end{aligned}$$

Note that the second integral on the right hand side of (8.3) is a  $L^2$ -bounded martingale on  $[0, \tilde{T}]$  since from Lemma 8.2,

$$\mathbb{E}^{\mathbb{Q}}\left[\iint_0^{\tilde{T}} (-e^{-M_s^{(n)}} \phi_{s,y}^{(n)})^2 (\tilde{\sigma}_{s,y})^2 ds dy\right] \leq \|\phi^{(n)}\|_\infty^2 \tilde{\vartheta}^2 \mathbb{E}^{\mathbb{Q}}\left[\iint_0^{\tilde{T}} \tilde{w}_{s,y} ds dy\right] < \infty.$$

Noticing that  $\tilde{\sigma}^2 \geq \vartheta^2 \tilde{w}$  on  $\mathbb{R}_+ \times \mathbb{R}$ , we can take expectation on (8.3) and get that

$$\mathbb{E}^{\mathbb{Q}}[1 - e^{-M_{\tilde{T}}^{(n)}}]$$

$$\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{2} \int_0^{\tilde{T}} \int_0^{\tilde{T}} e^{-M_s^{(n)}} (\phi_{s,y}^{(n)})^2 (\vartheta^2 \tilde{w}_{s,y} - (\tilde{\sigma}_{s,y})^2) \mathrm{d}s \mathrm{d}y + \int_0^{\tilde{T}} \int_0^{\tilde{T}} e^{-M_s^{(n)}} \phi_{s,y}^{(n)} \mu(\mathrm{d}s \mathrm{d}y) \right] \\
&\leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{T}} \int_0^{\tilde{T}} e^{-M_s^{(n)}} \phi_{s,y}^{(n)} \mu(\mathrm{d}s \mathrm{d}y) \right] \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{T}} \int_0^{\tilde{T}} \phi_{s,y}^{(n)} \mu(\mathrm{d}s \mathrm{d}y) \right].
\end{aligned}$$

*Final step.* The desired result now follows from Steps 3, 4 and Lemma 8.1.  $\square$

Let us now give the proof of Lemma 8.2.

*Proof of Lemma 8.2. Step 1.* It is routine (c.f. [28, Theorem 2.1]) to verify that

$$\tilde{w}_{t,x} = \int_0^t \int_0^{\tilde{T}} G_{s,y;t,x} (\tilde{\sigma}_{s,y} W(\mathrm{d}s \mathrm{d}y) + \mu(\mathrm{d}s \mathrm{d}y)), \quad \text{a.s.} \quad (t, x) \in [0, \tilde{T}] \times \mathbb{R}.$$

*Step 2.* For an arbitrary fixed  $(t, x) \in [0, \tilde{T}] \times \mathbb{R}$ , we will show that

$$\mathbb{E}^{\mathbb{Q}}[\tilde{w}_{t,x}] \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^t \int_0^{\tilde{T}} G_{s,y;t,x} \mu(\mathrm{d}s \mathrm{d}y) \right].$$

To do this, for each  $u \geq 0$ , define  $I_u := II_u + III_u$  where

$$II_u := \int_0^u \int_0^{\tilde{T}} G_{s,y;t,x} \mu(\mathrm{d}s \mathrm{d}y); \quad III_u := \int_0^u \int_0^{\tilde{T}} G_{s,y;t,x} \tilde{\sigma}_{s,y} W(\mathrm{d}s \mathrm{d}y).$$

We can verify that if  $u \geq t$ , then  $I_u = \tilde{w}_{t,x}$ , and if  $u \in [0, t]$ , then from stochastic Fubini theorem we get

$$I_u = \int G_{t,x;u,z} \tilde{w}_{u,z} \mathrm{d}z, \quad \text{a.s.}$$

In particular,  $(I_u)_{u \geq 0}$  is a non-negative process. Note that  $(III_u)_{u \geq 0}$  is a local martingale. So there exists a sequence of stopping time  $(\rho_n)_{n \in \mathbb{N}}$  such that for each  $n \in \mathbb{N}$ ,  $(III_{u \wedge \rho_n})_{u \geq 0}$  is a martingale, and  $\rho_n \uparrow \infty$  almost surely as  $n \uparrow \infty$ . Now for any fixed  $u \geq 0$  we can verify from Fatou's lemma that  $\mathbb{Q}[I_u] \leq \liminf_{n \rightarrow \infty} \mathbb{Q}[I_{u \wedge \rho_n}] \leq \mathbb{Q}[II_u]$ . In particular  $\mathbb{Q}[\tilde{w}_{t,x}] = \mathbb{Q}[I_t] \leq \mathbb{Q}[II_t]$  as desired.

*Step 3.* From Fubini's theorem we can verify from Step 2 that for each  $t \in [0, \tilde{T}]$ ,

$$\mathbb{E}^{\mathbb{Q}} \left[ \int \tilde{w}_{t,x} \mathrm{d}x \right] \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^t \int_0^{\tilde{T}} \mu(\mathrm{d}s \mathrm{d}y) \int G_{s,y;t,x} \mathrm{d}x \right] \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{T}} \int_0^{\tilde{T}} \mu(\mathrm{d}s \mathrm{d}y) \right] < \infty.$$

This proves the first part of the lemma.

*Step 4.* Let  $g$  and sequence  $(g_n)_{n \in \mathbb{N}}$  be  $\mathbb{R}$ -valued Borel functions on a Polish space  $S$ . We say  $(g_n)_{n \in \mathbb{N}}$  converges to  $g$  bounded pointwise if  $(g_n)_{n \in \mathbb{N}}$  converges to  $g$  pointwise, and  $\sup_{n \in \mathbb{N}, s \in S} |g_n(s)| < \infty$ . Fix any  $\phi \in \mathcal{C}_b^{1,2}([0, \tilde{T}] \times \mathbb{R})$ . Then it is easy to get that there exists a sequence of  $(\phi^{(n)} : n \in \mathbb{N})$  in  $\mathcal{C}_c^\infty([0, \tilde{T}] \times \mathbb{R})$  such that,  $(\phi^{(n)})_{n \in \mathbb{N}}$ ,  $(\partial_t \phi^{(n)})_{n \in \mathbb{N}}$ ,  $(\partial_x \phi^{(n)})_{n \in \mathbb{N}}$  and  $(\partial_x^2 \phi^{(n)})_{n \in \mathbb{N}}$  converges bounded pointwise to  $\phi$ ,  $\partial_t \phi$ ,  $\partial_x \phi$  and  $\partial_x^2 \phi$ , respectively.

*Final Step.* From Steps 3, 4, bounded convergence theorem, [15, Proposition 17.6] and the fact that  $\tilde{\sigma}^2 \leq \tilde{\vartheta} \tilde{w}$  on  $[0, \tilde{T}] \times \mathbb{R}$ , we can verify that (7.1) holds almost surely for each  $t \in [0, \tilde{T}]$  and  $\phi \in \mathcal{C}_b^{1,2}([0, \tilde{T}] \times \mathbb{R})$ .  $\square$

## 9. PROOF OF PROPOSITION 5.3

We first need the following lemma to control the small time fluctuation of certain random fields. This lemma is modified from [21, Lemma 6.1] in order to incorporate the small time intervals. Its proof follows the lines of the proof of [21, Lemma 6.1] and therefore is omitted.

**Lemma 9.1.** *Suppose that*

- (1)  $\tilde{T} > 0$ ,  $\tilde{L} > 0$ ,  $a \in \mathbb{R}$  are arbitrary and  $\mathbf{H} := [0, \tilde{T}] \times [a, a + \tilde{L}]$ ;
- (2)  $(g_{s,y;t,x} : (s,y), (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  and  $(\eta_{t,x} : (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  are deterministic non-negative functions satisfying

$$B := \sup_{(t',x'),(t,x) \in \mathbf{H}} \frac{\int_0^\infty (g_{s,y;t',x'} - g_{s,y;t,x})^2 \eta_{s,y} ds dy}{\left| \frac{x'-x}{\tilde{L}} \right| + \left| \frac{t'-t}{\tilde{T}} \right|^{1/2}} < \infty;$$

- (3)  $W$  is a white noise defined on a filtered probability space  $(\Omega, \mathcal{G}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ;
- (4)  $\tilde{\sigma}$  is a predictable random field on  $\Omega$  such that almost surely  $\tilde{\sigma}^2 \leq \eta$  on  $\mathbb{R}_+ \times \mathbb{R}$ ;
- (5)  $Z$  is a continuous random field on  $\Omega$  such that for all  $(t,x) \in \mathbf{H}$ ,

$$Z_{t,x} = \int_0^\infty \int_{\mathbb{R}} g_{s,y;t,x} \tilde{\sigma}_{s,y} W(ds dy) \quad a.s.$$

Then for each  $z \geq 0$ ,

$$\mathbb{P} \left( \sup_{(t,x),(t',x') \in \mathbf{H}} |Z_{t',x'} - Z_{t,x}| > z\sqrt{B} \right) \leq 2^5 e^{-z^2/2^{12}}.$$

Next result is a simple corollary of the above lemma.

**Corollary 9.2.** *Lemma 9.1 still holds if its conditions (1) and (2) are replaced by:*

- (1')  $\tilde{v} > 0$  and  $a \in \mathbb{R}$  are arbitrary and

$$\mathbf{H} := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : t \in [0, \tilde{v}^{-2}], x - \tilde{v}t \in [a, a + \tilde{v}^{-1}]\};$$

- (2')  $(g_{s,y;t,x} : (s,y), (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  and  $(\eta_{t,x} : (t,x) \in \mathbb{R}_+ \times \mathbb{R})$  are (deterministic) non-negative functions satisfying

$$B := \sup_{(t',x'),(t,x) \in \mathbf{H}} \frac{\tilde{v}^{-1} \int_0^\infty (g_{s,y;t',x'} - g_{s,y;t,x})^2 \eta_{s,y} ds dy}{|(x' - \tilde{v}t') - (x - \tilde{v}t)| + |t' - t|^{1/2}} < \infty.$$

In order to control the quantity  $B$  in Lemma 9.1 and Corollary 9.2 we will be using the following analytical lemma whose proof is given in the Appendix.

**Lemma 9.3.** *For any  $\tilde{v} > 0$  and  $(t,x), (t',x') \in \mathbb{R}_+ \times \mathbb{R}$  satisfying*

$$(9.1) \quad t, t' \in [0, \tilde{v}^{-2}]; \quad x - \tilde{v}t, x' - \tilde{v}t' \in (-\infty, 0]; \quad |(x' - \tilde{v}t') - (x - \tilde{v}t)| \leq \tilde{v}^{-1},$$

*it holds that*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} (G_{s,y;t',x'}^{(\tilde{v})} - G_{s,y;t,x}^{(\tilde{v})})^2 e^{-\tilde{v}(y-\tilde{v}s)} ds dy \\ & \leq 2^9 e^{-\tilde{v}(x-\tilde{v}t)} (|(x' - \tilde{v}t') - (x - \tilde{v}t)| + |t' - t|^{1/2}). \end{aligned}$$

We are now ready to give the proof of Proposition 5.3.

*Proof of Proposition 5.3. Step 1.* Define

$$I_{t,x} := \varepsilon k L e^{-\theta v(x-vt)} \mathbf{1}_{x \leq vt}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Let  $\bar{v}$  be as in Proposition 5.2. Then by part (4) of that proposition we have that  $\bar{v} \geq v$  on  $\mathbb{R}_+ \times \mathbb{R}$  almost surely. Therefore, in order to prove Proposition 5.3, we only have to show that  $P(\tilde{\tau}_3 < T) < 1/8$  holds with

$$\tilde{\tau}_3 := \inf\{t \in [0, T] : \bar{v}_{t,x} \geq F(x - vt) + I_{t,x} \text{ for some } x \in (-\infty, vt]\}.$$

*Step 2.* Define  $\tilde{Z}_{t,x} := e^{-\alpha t}(\bar{v}_{t,x} - \varrho_{t,x})$  for each  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$  where  $\varrho$  is given in (5.1) and  $\alpha := \theta(1 - \theta)v^2$ . Then it can be verified from Lemma 5.1 that

$$\tilde{Z}_{t,x} = \int_0^t \int_0^x G_{s,y;t,x}^{(v)} \varepsilon e^{-\alpha s} \sigma(\bar{v}_{s,y}) W^{\bar{v}}(ds dy), \quad \text{a.s.} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

From this we immediately get that

$$\tilde{\tau}_3 = \inf\{t \in [0, T] : \tilde{Z}_{t,x} \geq e^{-\alpha t} I_{t,x} \text{ for some } x \in (-\infty, vt]\}.$$

*Step 3.* We show that almost surely

$$\varepsilon e^{-\alpha t} \sigma(\bar{v}_{t,x}) \leq \varepsilon \sigma(\bar{v}_{t,x}) \leq \sqrt{\eta_{t,x}}, \quad (t, x) \in [0, \tilde{\tau}_3] \times \mathbb{R}$$

where

$$\eta_{t,x} := 2kL\varepsilon^2 e^{-\theta v(x-vt)} \mathbf{1}_{x \leq vt}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

In fact, almost surely for each  $(t, x) \in [0, \tilde{\tau}_3] \times \mathbb{R}$ ,

$$\begin{aligned} \sigma(\bar{v}_{t,x})^2 &\leq \bar{v}_{t,x} \leq F(x - vt) + I_{t,x} \\ &\leq \frac{\varepsilon}{\theta v} (e^{-\theta v(x-vt)} - 1) \mathbf{1}_{x \leq vt} + \varepsilon k L e^{-\theta v(x-vt)} \mathbf{1}_{x \leq vt} \\ &\leq (2 + k) L \varepsilon e^{-\theta v(x-vt)} \mathbf{1}_{x \leq vt}. \end{aligned}$$

Note from (4.3) and (4.4) that  $k \geq 2$ . The desired result in this step follows.

*Step 4.* From Step 3 we can verify that almost surely  $Z = \tilde{Z}$  on  $[0, \tilde{\tau}_3] \times \mathbb{R}$  where  $Z$  is a continuous random field so that

$$Z_{t,x} = \int_0^t \int_0^x G_{s,y;t,x}^{(v)} \left( \sqrt{\eta_{s,y}} \wedge (\varepsilon e^{-\alpha s} \sigma(\bar{v}_{s,y})) \right) W^{\bar{v}}(ds dy) \quad \text{a.s.} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Thus from Step 2, we get that

$$\tilde{\tau}_3 = \inf\{t \in [0, T] : Z_{t,x} \geq e^{-\alpha t} I_{t,x} \text{ for some } x \leq vt\}, \quad \text{a.s.}$$

*Step 5.* Define

$$\Gamma_n := \{(t, x) \in [0, T] \times \mathbb{R} : x - vt \in (-nL, -(n-1)L)\}, \quad n \in \mathbb{N}.$$

We can verify from (4.5) that for each  $n \in \mathbb{N}$  and  $(t, x) \in \Gamma_n$ ,

$$e^{-\alpha t} I_{t,x} \geq e^{-\alpha T} \varepsilon k L e^{\theta v(n-1)L} = C_\theta \varepsilon k L e^{\theta n} =: \Pi_n$$

where  $C_\theta := e^{-\theta(2-\theta)}$ .



*Step 6.* For each  $n \in \mathbb{N}$ , we can get from Lemma 9.3 and (4.5) that

$$B_n = \sup_{(t,x),(t',x') \in \Gamma_n} \frac{v^{-1} \int_0^\infty (G_{s,y;t,x}^{(v)} - G_{s,y;t',x'}^{(v)})^2 \eta_{s,y} ds dy}{|(x' - vt') - (x - vt)| + |t' - t|^{1/2}} \leq 2^{10} k L^2 \epsilon^2 \varepsilon e^n.$$

In fact, for  $(t, x), (t', x') \in \Gamma_n$ , since (9.1) holds, we have from Lemma 9.3 that

$$\begin{aligned} \int_0^\infty (G_{s,y;t,x}^{(v)} - G_{s,y;t',x'}^{(v)})^2 \eta_{s,y} ds dy &\leq 2^2 k L \epsilon^2 \varepsilon \int_0^\infty (G_{s,y;t,x}^{(v)} - G_{s,y;t',x'}^{(v)})^2 e^{-v(y-vs)} ds dy \\ &\leq 2 k L \epsilon^2 \varepsilon 2^9 e^{-v(x-vt)} (|(x' - vt') - (x - vt)| + |t' - t|^{1/2}) \\ &\leq 2^{10} k L \epsilon^2 \varepsilon e^{nvL} (|(x' - vt') - (x - vt)| + |t' - t|^{1/2}). \end{aligned}$$

Noting from (4.5) that  $vL = 1$ , the desired result in this step follows.

*Step 7.* From Step 6, (4.5) that  $\varepsilon = \gamma \epsilon^2$ , (4.4) that  $\gamma k = \mathcal{K}$ , and Corollary 9.2 we can obtain

$$\begin{aligned} \mathbb{P}\left(\sup_{\Gamma_n} Z \geq \Pi_n\right) &\leq \mathbb{P}\left(\sup_{\Gamma_n} Z \geq 2^{-5} C_\theta \sqrt{\varepsilon/\epsilon^2} \sqrt{k} e^{(\theta-1/2)n} \sqrt{B_n}\right) \\ &\leq \mathbb{P}\left(\sup_{\Gamma_n} Z \geq 2^{-5} C_\theta \sqrt{\mathcal{K}} e^{(\theta-1/2)n} \sqrt{B_n}\right) \\ &\leq 2^5 \exp(-2^{-22} C_\theta^2 \mathcal{K} e^{(2\theta-1)n}). \end{aligned}$$

*Final Step.* Using Steps 4, 5 and 7, we can verify that

$$\begin{aligned} \mathbb{P}(\tilde{\tau}_3 < T) &\leq \mathbb{P}\left(\exists(t, x) \in \bigcup_{n=1}^\infty \Gamma_n : Z_{t,x} \geq e^{-\alpha t} \mathbf{I}_{t,x}\right) \\ &\leq \sum_{n=1}^\infty \mathbb{P}\left(\sup_{\Gamma_n} Z \geq \Pi_n\right) \leq 2^5 \sum_{n=1}^\infty \exp(-2^{-22} C_\theta^2 \mathcal{K} e^{(2\theta-1)n}) \leq 1/8 \end{aligned}$$

where we used (4.2) in the last inequality.  $\square$

## 10. PROOF OF PROPOSITION 5.5

We will need the following analytical lemma whose proof can be found in the Appendix.

**Lemma 10.1.** *For any  $\tilde{v} > 0$  and  $(t, x), (t', x') \in \mathbb{R}_+ \times \mathbb{R}$  satisfying*

$$t, t' \in [0, \tilde{v}^{-2}]; \quad x, x' \in [-2\tilde{v}^{-1}, 2\tilde{v}^{-1}]$$

*it holds that*

$$\int_0^\infty \int_0^\infty (G_{s,y;t',x'} - G_{s,y;t,x})^2 e^{-\tilde{v}y} ds dy \leq 2^7 (|x' - x| + |t' - t|^{1/2}).$$

Note that the upper bound in the above lemma is uniform in  $\tilde{v}$ . Let us now give the proof of Proposition 5.5.

*Proof of Proposition 5.5. Step 1.* it is easy to see that on the event  $\{\tau_1 \geq T, \tau_3 \geq T\}$ , the following holds almost surely: for each  $(s, y) \in [0, \tau_2 \wedge T] \times \mathbb{R}$ ,

$$v_{s,y} \leq F(s - vy) + \varepsilon k L e^{-\theta v(x-vt)} \mathbf{1}_{x \leq vt} \leq 2k\varepsilon L e^{-\theta v(y-vs)},$$

$$\begin{aligned} f_{s,y}^w &\leq w_{s,y}^p \mathbf{1}_{y \in [-L, \nu T + L], w_{s,y} \leq \nu \varepsilon L} \leq (\nu \varepsilon L)^p, \\ \sigma_{s,y}^w &= \sigma(v_{s,y} + w_{s,y})^2 - \sigma(v_{s,y})^2 \leq w_{s,y} \leq \nu \varepsilon L \mathbf{1}_{y \in [-L, \nu T + L]}. \end{aligned}$$

*Step 2.* Note that  $v + w$  admits the following mild form (c.f. [28, Theorem 2.1]):

$$\begin{aligned} v_{t,x} + w_{t,x} &= \int_0^t G_{s,y;t,x} \left( v_{0,y} dy \delta_0(ds) + (f(v_{s,y}) + f_{s,y}^w) ds dy + \right. \\ &\quad \left. \varepsilon \sigma(v_{s,y}) W^v(ds dy) + \varepsilon \sigma_{s,y}^w W^w(ds dy) \right), \quad \text{a.s.} \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}. \end{aligned}$$

Therefore, almost surely on the event  $\{\tau_1 \geq T, \tau_3 \geq T\}$ , we have  $v + w = \tilde{u}$  on  $[0, \tau_2 \wedge T] \times \mathbb{R}$ . Here,  $\tilde{u} := \sum_{i=1}^5 Z^{(i)}$  where  $\{Z^{(i)} : i = 1, \dots, 5\}$  is a list of continuous random fields defined so that for each  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\begin{aligned} Z_{t,x}^{(1)} &= \mathbf{1}_{t=0} (1 \wedge F(x)) + \mathbf{1}_{t>0} \int G_{0,y;t,x} (1 \wedge F(y)) dy, \\ Z_{t,x}^{(2)} &= \int_0^t \int_0^s G_{s,y;t,x} f(v_{s,y} \wedge (2k\varepsilon L e^{-\theta v(y-\nu s)})) ds dy, \quad \text{a.s.} \\ Z_{t,x}^{(3)} &= \int_0^t \int_0^s G_{s,y;t,x} (f_{s,y}^w \wedge ((\nu \varepsilon L)^p \mathbf{1}_{y \in [-L, \nu T + L]})) ds dy, \quad \text{a.s.} \\ Z_{t,x}^{(4)} &= \varepsilon \int_0^t \int_0^s G_{s,y;t,x} \sigma(v_{s,x} \wedge (2k\varepsilon L e^{-\theta v(y-\nu s)})) W^v(ds dx), \quad \text{a.s.} \\ Z_{t,x}^{(5)} &= \varepsilon \int_0^t \int_0^s G_{s,y;t,x} (\sigma_{s,y}^w \wedge (\nu \varepsilon L \mathbf{1}_{y \in [-L, \nu T + L]})) W^w(ds dx), \quad \text{a.s.} \end{aligned}$$

*Step 3.* Clearly,  $\tau_2 = \tilde{\tau}_2$  holds almost surely on the event  $\{\tau_1 \geq T, \tau_3 \geq T\}$  where  $\tilde{\tau}_2 := \inf\{t \in [0, T] : \tilde{u}_{t,x} \geq \nu \varepsilon L \text{ for some } x \in [-L, \nu T + L]\}$ .

*Step 4.* We will show that

$$\sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(1)} \leq \nu_1 \varepsilon L.$$

where  $\mathbf{H} := [0, T] \times [-L, \nu T + L]$  and  $\nu_1 := 2^3$ . Note that from (4.7), (2.3) and (4.5), for any  $(t, x) \in \mathbf{H}$ , we have

$$\begin{aligned} Z_{t,x}^{(1)} &\leq \int G_{0,y;t,x} \frac{\varepsilon}{\theta \nu} e^{-\theta \nu y} dy = \frac{\varepsilon}{\theta \nu} e^{-\theta \nu x} \int \frac{e^{-\frac{y^2}{4t} + \theta \nu y}}{\sqrt{4\pi t}} dy = \frac{\varepsilon}{\theta \nu} e^{-\theta \nu x} e^{\theta^2 \nu^2 t} \\ &\leq \theta^{-1} e^{\theta^2 + \theta} \varepsilon L \leq \nu_1 \varepsilon L. \end{aligned}$$

*Step 5.* We will show that

$$\sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(2)} \leq \nu_2 \varepsilon L, \quad \text{a.s.}$$

where  $\nu_2 := 2^4 k$ . Note that from (4.1) that  $\kappa^{p-1} \leq 1$ , (2.3) and (4.5) we can verify that,

$$Z_{t,x}^{(2)} \leq \int_0^t \int_0^s G_{s,y;t,x} (2k\varepsilon L e^{-\theta v(y-\nu s)})^p ds dy \leq (2k\varepsilon L)^p \int_0^t e^{p\theta \nu^2 s} ds \int G_{s,y;t,x} e^{-p\theta \nu y} dy$$

$$\begin{aligned}
&= (2k\varepsilon L)^p e^{-p\theta vx} e^{p^2\theta^2 v^2 t} \int_0^t e^{(p\theta v^2 - p^2\theta^2 v^2)s} ds \leq (2k\varepsilon L)^p e^{-p\theta vx} t e^{p\theta v^2 t} \\
&\leq (2k\varepsilon L)^p e^{p\theta vL} T e^{p\theta v^2 T} = 2^p k^p e^{2p\theta} \kappa^{p-1} \varepsilon L \leq 2^4 k\varepsilon L = \nu_2 \varepsilon L, \quad \forall (t, x) \in \mathbf{H}, \quad \text{a.s.}
\end{aligned}$$

*Step 6.* We will show that

$$\sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(3)} \leq \nu_3 \varepsilon L, \quad \text{a.s.}$$

where  $\nu_3 := \nu^p$ . In fact, from (2.3) and (4.5) we can verify that

$$\begin{aligned}
Z_{t,x}^{(3)} &\leq \int_0^t \int G_{s,y;t,x} (\nu \varepsilon L)^p ds dy = (\nu \varepsilon L)^p \int_0^t ds \int G_{s,y;t,x} dy \\
&\leq T (\nu \varepsilon L)^p = \nu^p \kappa^{p-1} \varepsilon L \leq \nu_3 \varepsilon L, \quad \forall (t, x) \in \mathbf{H}, \quad \text{a.s.}
\end{aligned}$$

*Step 7.* We will show that

$$\mathbb{P} \left( \sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(4)} > \nu_4 \varepsilon L \right) \leq 2^{-4}$$

where  $\nu_4 := 2^{13} \mathcal{K}^{1/2} \gamma^{-1}$ . First note that almost surely for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\varepsilon^2 \sigma(v_{s,y} \wedge (2k\varepsilon L e^{-\theta v(y-v s)}))^2 \leq 2\varepsilon^2 k\varepsilon L e^{-\theta v(y-v s)} \mathbf{1}_{y \leq v s} =: \eta_{s,y}^{(4)}.$$

Then note that for each  $(t, x), (t', x') \in \mathbf{H}$ , using Lemma 10.1,

$$\begin{aligned}
&\frac{1}{2\varepsilon^2 k\varepsilon L} \int_0^\infty \int (G_{s,y;t',x'} - G_{s,y;t,x})^2 \eta_{s,y}^{(4)} ds dy \\
&\leq \int_0^\infty \int (G_{s,y;t',x'} - G_{s,y;t,x})^2 e^{-v(y-v s)} ds dy \leq 2^7 (|x' - x| + |t' - t|^{1/2}).
\end{aligned}$$

Therefore,

$$B^{(4)} := \sup_{(t,x),(t',x') \in \mathbf{H}} \frac{\int_0^\infty \int (G_{s,y;t',x'} - G_{s,y;t,x})^2 \eta_{s,y}^{(4)} ds dy}{\left| \frac{x'-x}{3L} \right| + \left| \frac{t'-t}{T} \right|^{1/2}} \leq 2^{10} \varepsilon^2 k\varepsilon L^2 =: \tilde{B}^{(4)}.$$

Taking  $z = 2^8$ , we get from Lemma 9.1 that

$$\mathbb{P} \left( \sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(4)} > z \sqrt{\tilde{B}^{(4)}} \right) \leq \mathbb{P} \left( \sup_{(t,x),(t',x') \in \mathbf{H}} |H_{t',x'}^{(4)} - H_{t,x}^{(4)}| > z \sqrt{B^{(4)}} \right) \leq 2^5 e^{-z^2/2^{12}} \leq 2^{-4}.$$

To finish this step we note that

$$z \sqrt{\tilde{B}^{(4)}} = 2^{13} \sqrt{\varepsilon^2 k\varepsilon L^2} = \nu_4 \varepsilon L.$$

*Step 8.* We will show that

$$\mathbb{P} \left( \sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(5)} > \nu_5 \varepsilon L \right) \leq 2^{-4}$$

with  $\nu_5 = \nu/4$ . First note that almost surely for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\varepsilon^2 (\sigma_{s,y}^w \wedge (\nu \varepsilon L \mathbf{1}_{y \in [-L, vT+L]}))^2 \leq \varepsilon^2 \nu^2 \varepsilon^2 L^2 =: \eta_{s,y}^{(5)}.$$

Then note that for each  $(t, x), (t', x') \in \mathbf{H}$ , using [28, Lemma 6.2(1)] (c.f. Lemma 10.1),

$$\frac{1}{\epsilon^2 \nu^2 \epsilon^2 L^2} \iint_0^\infty (G_{s,y;t',x'} - G_{s,y;t,x})^2 \eta_{s,y}^{(5)} ds dy \leq 2^7 (|x' - x| + |t' - t|^{1/2}).$$

Therefore,

$$B^{(5)} := \sup_{(t,x),(t',x') \in \mathbf{H}} \frac{\iint_0^\infty (G_{s,y;t',x'} - G_{s,y;t,x})^2 \eta_{s,y}^{(5)} ds dy}{\left| \frac{x' - x}{3L} \right| + \left| \frac{t' - t}{T} \right|^{1/2}} \leq 2^9 \epsilon^2 \nu^2 \epsilon^2 L^3 =: \tilde{B}^{(5)}.$$

Taking  $z = 2^8$ , we get from Lemma 9.1 that

$$\mathbb{P}\left(\sup_{(t,x) \in \mathbf{H}} Z_{t,x}^{(5)} > z \sqrt{\tilde{B}^{(5)}}\right) \leq \mathbb{P}\left(\sup_{(t,x),(t',x') \in \mathbf{H}} |Z_{t',x'}^{(5)} - Z_{t,x}^{(5)}| > z \sqrt{\tilde{B}^{(5)}}\right) \leq 2^5 e^{-z^2/2^{12}} \leq 2^{-4}.$$

To finish this step we note from (4.6) that

$$z \sqrt{\tilde{B}^{(5)}} = 2^{13} \sqrt{\epsilon^2 \nu^2 \epsilon^2 L^3} = 2^{13} \sqrt{\epsilon^2 L} \nu \epsilon L \leq \nu_5 \epsilon L.$$

*Final step.* We note from (4.3) and (4.4) that

$$\sum_{i=1}^5 \nu_i = 2^3 + 2^4 k + \nu^p + 2^{13} \mathcal{K}^{1/2} \gamma^{-1} + \nu/4 \leq \nu.$$

Also note from Steps 2 and 8 that

$$\begin{aligned} \mathbb{P}(\tau_2 < T, \tau_1 \geq T, \tau_3 \geq T) &= \mathbb{P}(\tilde{\tau}_2 < T, \tau_1 \geq T, \tau_3 \geq T) \leq \mathbb{P}(\tilde{\tau}_2 < T) \\ &= \mathbb{P}(\{\tilde{u} \leq \nu \epsilon L \text{ on } [0, T] \times [-L, \nu T + L]\}^c) \\ &\leq \mathbb{P}\left(\bigcup_{i=1}^5 \{Z^{(i)} \leq \nu_i \epsilon L \text{ on } [0, T] \times [-L, \nu T + L]\}^c\right). \end{aligned}$$

Now from Steps 3-7, we have

$$\begin{aligned} &\mathbb{P}(\tau_2 < T, \tau_1 \geq T, \tau_3 \geq T) \\ &\leq \sum_{i=1}^5 \mathbb{P}(\{Z^{(i)} \leq \nu_i \epsilon L \text{ on } [0, T] \times [-L, \nu T + L]\}^c) \leq 2^{-3}. \end{aligned} \quad \square$$

## APPENDIX A. PROOF OF LEMMA 6.1

*Proof of Lemma 6.1 (1). Step 1.* Using the stochastic Fubini theorem (cf. [14, Lemma 2.4] for example) we can verify that for all  $t \geq 0$

$$\int v_{t,x} \phi_{t,x} dx = \int dx \iint_0^t \phi_{t,x} G_{s,y;t,x}^{(\nu)} M^v(ds dy) = \iint_0^t M^v(ds dy) \int \phi_{t,x} G_{s,y;t,x}^{(\nu)} dx, \quad \text{a.s.}$$

*Step 2.* Using the stochastic Fubini theorem again we can verify that for all  $t \geq 0$

$$\iint_0^t (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) v_{r,x} dr dx = \iint_0^t dr dx \iint_0^r (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) G_{s,y;r,x}^{(\nu)} M^v(ds dy)$$

$$= \iint_0^t M^v(dsdy) \iint_s^t G_{s,y;r,x}^{(v)} (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) dr dx, \quad \text{a.s.}$$

*Step 3.* We show that for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ ,

$$\int G_{s,y;t,x}^{(v)} \phi_{t,x} dx + \Pi_{s,y}[\phi_{\rho,B_\rho}; t \geq \rho] = \phi_{s,y} + \iint_s^t G_{s,y;r,x}^{(v)} (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) dr dx.$$

In fact, according to Ito's formula (see [24, p. 147] for example), we know that under probability  $\Pi_{s,y}$ ,

$$\phi_{t,B_t} - \phi_{s,y} - \int_s^t (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x})|_{x=B_r} dr = \int_s^t \partial_x \phi_{r,x}|_{x=B_r} dB_r, \quad t \geq s,$$

is a zero-mean  $L^2$ -martingale. Then, according to optional sampling theorem (see [15, Theorem 7.29] for example) we have

$$\Pi_{s,y}[\phi_{t \wedge \rho, B_{t \wedge \rho}}] = \phi_{s,y} + \int_s^t \Pi_{s,y}[(\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x})|_{x=B_r}; r < \rho] dr.$$

*Step 4.* We note from the fact  $\phi \in \mathcal{C}_c^{1,2}(\mathbb{R}_+ \times \mathbb{R})$  and that  $v, f, \sigma$  take values in  $[0, 1]$ , the following stochastic integral

$$\iint_0^t \phi_{s,y} M^v(dsdy) = \int \phi_{0,y} v_{0,y} dy + \iint_0^t \phi_{s,y} f(v_{s,y}) ds dy + \iint_0^t \phi_{s,y} \sigma(v_{s,y}) W^v(dsdy)$$

is well-defined.

*Final Step.* We verify that almost surely,

$$\begin{aligned} \int v_{t,x} \phi_{t,x} dx &\stackrel{\text{Step 1}}{=} \iint_0^t M^v(dsdy) \int G_{s,y;t,x}^{(v)} \phi_{t,x} dx \\ &\stackrel{\text{Step 3}}{=} \iint_0^t M^v(dsdy) \left( \phi_{s,y} + \iint_s^t G_{s,y;r,x}^{(v)} (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) dr dx - \Pi_{s,y}[\phi_{\rho,B_\rho}; t \geq \rho] \right) \\ &\stackrel{\text{Steps 2 and 4}}{=} \iint_0^t \phi_{s,y} M^v(dsdy) + \iint_0^t (\partial_r \phi_{r,x} + \partial_x^2 \phi_{r,x}) v_{r,x} dr dx \\ &\quad - \iint_0^t \Pi_{s,y}[\phi_{\rho,B_\rho}; t \geq \rho] M^v(dsdy) \end{aligned}$$

as desired.  $\square$

*Proof of Lemma 6.1 (2).* For each  $t \geq 0$ , choose a  $\phi \in \mathcal{C}_c^{1,2}(\mathbb{R}_+ \times \mathbb{R})$  such that  $\phi_{s,vs} = 1$  for every  $s \in [0, t]$ . Use this  $\phi$  in (6.1) to get that for each  $t \geq 0$  the following random variable is well defined:

$$\begin{aligned} \tilde{A}_t &:= \iint_0^t \Pi_{s,y}[t \geq \rho] M^v(dsdy) \\ &= - \int \phi_{t,x} v_{t,x} dx + \iint_0^t \phi_{s,y} M^v(dsdy) + \iint_0^t v_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) ds dy. \end{aligned}$$

It's easy to see that  $(\tilde{A}_t)_{t \geq 0}$  has a continuous modification which will be denoted by  $(A_t)_{t \geq 0}$ . To see that  $(A_t)_{t \geq 0}$  is almost surely non-decreasing, define

$$\phi_{t,x}^{(m)} := \varphi_{(x-vt)m}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, m \in \mathbb{N}$$

where

$$\varphi_x := \begin{cases} 0, & x \in [1, \infty), \\ (18x^2 + 6x + 1)(1 - x)^3, & x \in [0, 1], \\ (x + 1)^3, & x \in [-1, 0], \\ 0, & x \in (-\infty, -1]. \end{cases}$$

Use  $\phi^{(m)}$  instead of  $\phi$  in (6.1) to get that, for each  $m \in \mathbb{N}$  and  $t \geq 0$ ,

$$A_t = \mathbf{I}_t^{(m)} + \mathbf{II}_t^{(m)}, \quad \text{a.s.}$$

where

$$\mathbf{I}_t^{(m)} := \int \int_0^t \phi_{s,y}^{(m)} M^v(\mathrm{d}s \mathrm{d}y) - \int \phi_{t,x}^{(m)} v_{t,x} \mathrm{d}x$$

and

$$\mathbf{II}_t^{(m)} := \int \int_0^t v_{s,y} (\partial_s \phi_{s,y}^{(m)} + \partial_y^2 \phi_{s,y}^{(m)}) \mathrm{d}s \mathrm{d}y.$$

Observe that  $\phi_{s,y}^{(m)} \downarrow 0$  as  $m \uparrow \infty$  on  $\{(s, y) \in \mathbb{R}_+ \times \mathbb{R} : y < vs\}$ . This allows us to use the monotone convergence theorem and [15, Proposition 17.6] to get that for each  $t \geq 0$ ,  $\mathbf{I}_t^{(m)}$  converges to 0 in probability as  $m \rightarrow \infty$ . Fix arbitrary  $r < t$  in  $\mathbb{R}_+$ . [15, Lemma 4.2] allows us to choose an unbounded  $\mathbf{N} \subset \mathbb{N}$  so that  $\mathbf{I}_t^{(m)} - \mathbf{I}_r^{(m)}$  convergence to 0 almost surely as  $m \rightarrow \infty, m \in \mathbf{N}$ . Now we have almost surely

(A.1)

$$\begin{aligned} A_t - A_r &= \lim_{m \rightarrow \infty, m \in \mathbf{N}} (\mathbf{II}_t^{(m)} - \mathbf{II}_r^{(m)}) \\ &= \lim_{m \rightarrow \infty, m \in \mathbf{N}} \int_r^t \mathrm{d}s \int_{vs - \frac{1}{m}}^{vs} v_{s,y} \cdot 3(1 + (y - vs)m) \left( -vm(1 + (y - vs)m) + 2m^2 \right) \mathrm{d}y \\ &\geq \lim_{m \rightarrow \infty, m \in \mathbf{N}} \int_r^t \mathrm{d}s \int_{vs - \frac{1}{m}}^{vs} v_{s,y} \cdot 3(1 + (y - vs)m) (-vm + 2m^2) \mathrm{d}y \geq 0. \end{aligned}$$

From this and the fact that  $(A_t)_{t \geq 0}$  has continuous sample path, we have that  $t \mapsto A_t$  is non-decreasing almost surely.

Denote by  $\mathrm{b}\mathcal{B}(\mathbb{R}_+)$  the space of bounded Borel functions on  $\mathbb{R}_+$ . Fix a time  $t \geq 0$  and define  $\mathcal{H} := \{\psi \in \mathrm{b}\mathcal{B}(\mathbb{R}_+) : (6.2) \text{ holds for } \psi\}$ . From the definition of  $(A_t)_{t \geq 0}$  and the fact that it has non-decreasing sample path almost surely, we can verify that  $\mathcal{K} \subset \mathcal{H}$  where  $\mathcal{K}$  is given by (7.5). One can verify from monotone convergence theorem and [15, Proposition 17.6] that  $\mathcal{H}$  is a monotone vector space in the sense of [26, p. 364]. Also observe that  $\mathcal{H}$  is closed under multiplication. Therefore using monotone class theorem ([26, Theorem A0.6]) we get  $\mathrm{b}\mathcal{B}(\mathbb{R}_+) = \sigma(\mathcal{K}) \subset \mathcal{H}$ .  $\square$



## APPENDIX B. PROOF OF LEMMA 7.4

*Proof of Lemma 7.4. Step 1.* Let  $\varrho$  be given as in (5.1). Note that from Lemma 5.1,  $\varrho$  is a solution to PDE (5.2). We define

$$A_t^\varrho := \int_0^t \int_0^s \Pi_{s,y}[\rho \leq t] M^\varrho(ds dy), \quad t \geq 0,$$

the killing process of  $\varrho$  at its boundary, where  $M^\varrho(ds dy) := \varrho_{0,y} \delta_0(ds) dy + \bar{f}(\varrho_{s,y}) ds dy$ . Similar to Lemma 6.1, we can verify that  $t \mapsto A_t^\varrho$  is a real-valued non-decreasing continuous function on  $\mathbb{R}_+$ , and for each  $t \geq 0$  and  $\phi \in \mathcal{C}_c^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ , it holds that

$$(B.1) \quad \int \phi_{t,x} \varrho_{t,x} dx = \int_0^t \int_0^s \phi_{s,y} M^\varrho(ds dy) + \int_0^t \int_0^s \varrho_{s,y} (\partial_s \phi_{s,y} + \partial_y^2 \phi_{s,y}) ds dy - \int_0^t \phi_{s,vs} dA_s^\varrho.$$

*Step 2.* We show that  $A_t^\varrho = \varepsilon t$  for each  $t \in \mathbb{R}_+$ . To do this, we use an argument similar to the one we used for (A.1), and obtain from (B.1) that

$$A_t^\varrho = \lim_{m \rightarrow \infty} \int_0^t ds \int_{vs - \frac{1}{m}}^{vs} \varrho_{s,y} \cdot 3(1 + (y - vs)m) \left( -vm(1 + (y - vs)m) + 2m^2 \right) dy.$$

Now we can verify from bounded convergence theorem that

$$\begin{aligned} A_t^\varrho &= \lim_{m \rightarrow \infty} \int_0^t ds \int_{-1}^0 F(u/m) \cdot 3(1 + u) (-v(1 + u) + 2m) du \\ &= \int_0^t ds \int_{-1}^0 F'(0-) \cdot 6(1 + u) u du = \varepsilon t. \end{aligned}$$

For the following Steps 3-5, we fix an arbitrary  $t \in \mathbb{R}_+$  and  $x \in (-\infty, vt]$ .

*Step 3.* It holds that  $\mathbb{E}^\mathbb{Q}[v_{t,x}] \leq \mathbb{E}^\mathbb{Q}[I]$  where

$$I := \int_0^t \int_0^s G_{s,y;t,x}^v (v_{0,y} \delta_0(ds) dy + f(v_{s,y}) ds dy).$$

In fact note that almost surely  $v_{t,x} = I + II_t$  where

$$II_u := \epsilon \int_0^u G_{s,y;t,x}^v \sigma(v_{s,y}) W^v(ds dy), \quad u \geq 0$$

is a local martingale. Therefore, we can choose a sequence of stopping time  $(\rho_n)_{n \in \mathbb{N}}$  so that for each  $n \in \mathbb{N}$ ,  $(II_{u \wedge \rho_n})_{u \geq 0}$  is a martingale; and almost surely  $\rho_n \uparrow \infty$  when  $n \uparrow \infty$ . Now from the fact that  $v_{t,x}$  is non-negative, we can verify from Fatou's lemma that  $\mathbb{E}^\mathbb{Q}[v_{t,x}] \leq \liminf_{n \rightarrow \infty} \mathbb{E}^\mathbb{Q}[I + II_{t \wedge \rho_n}] = \mathbb{E}^\mathbb{Q}[I]$ .

*Step 4.* We show that  $\mathbb{E}^\mathbb{Q}[v_{t,x}] \leq \tilde{v}_{t,x}$  where

$$\tilde{v}_{t,x} := \int_0^t \int_0^s G_{s,y;t,x}^v (v_{0,y} \delta_0(ds) dy + \bar{f}(\mathbb{E}^\mathbb{Q}[v_{s,y}]) ds dy).$$

In fact, noticing from Lemma 5.1 that  $\bar{f} \geq f$ , we have

$$\tilde{v}_{t,x} = \mathbb{E}^\mathbb{Q} \left[ \int_0^t \int_0^s G_{s,y;t,x}^v (v_{0,y} \delta_0(ds) dy + \bar{f}(v_{s,y}) ds dy) \right] \geq \mathbb{E}^\mathbb{Q}[I]$$

where  $I$  is given as in Step 3. Now the desired result in this step follows from Step 3.

*Step 5.* It holds that  $\mathbb{E}^Q[v_{t,x}] \leq \varrho_{t,x}$ . To see this, we first observe from Lemma 5.1 that  $\varrho$  admits the following mild form

$$\varrho_{t,x} = \iint_0^t G_{s,y;t,x}^{(v)} (\varrho_{0,y} \delta_0(ds) dy + (\alpha \varrho_{s,y} + \beta) ds dy)$$

where  $\alpha := \theta(1-\theta)v^2$  and  $\beta := (1-\theta)v\varepsilon$ . Using Feynman-Kac formula (c.f. [10, Lemma 1.5. on p. 1211]) we have that

$$\varrho_{t,x} = e^{\alpha t} \iint_0^t G_{s,y;t,x}^{(v)} e^{-\alpha s} (\varrho_{0,y} \delta_0(ds) dy + \beta ds dy).$$

Similarly, using Feynman-Kac formula for  $\tilde{v}$ , we get

$$\tilde{v}_{t,x} := e^{\alpha t} \iint_0^t G_{s,y;t,x}^{(v)} e^{-\alpha s} (v_{0,y} \delta_0(ds) dy + (-\alpha \tilde{v}_{s,y} + \alpha \mathbb{E}^Q[v_{s,y}] + \beta) ds dy).$$

Observing from the above two equations and Step 4, we have that  $\tilde{v}_{t,x} \leq \varrho_{t,x}$ . Using Step 4 again, we get the desired result in this step.

*Step 6.* We show that for any  $0 \leq r < t < \infty$ , it holds that  $\mathbb{E}^Q[A_t - A_r] \leq A_t^e - A_r^e$ . To do this, note that almost surely  $0 \leq A_t - A_r = \text{III} + \text{IV}_t$  where

$$\begin{aligned} \text{III} &:= \iint_r^t \Pi_{s,y}[\rho \leq t] (v_{0,y} \delta_0(ds) dy + f(v_{s,y}) ds dy); \\ \text{IV}_u &:= \iint_r^{u \wedge t} \Pi_{s,y}[\rho \leq t] \sigma(v_{s,y}) W^v(ds dy), \quad u \geq r. \end{aligned}$$

Since  $(\text{IV}_u)_{u \geq r}$  is a local martingale, we can choose a sequence of stopping time  $(\tilde{\rho}_n)_{n \in \mathbb{N}}$  so that for each  $n \in \mathbb{N}$ ,  $(\text{IV}_{u \wedge \tilde{\rho}_n})_{u \geq r}$  is a martingale; and almost surely  $\tilde{\rho}_n \uparrow \infty$  when  $n \uparrow \infty$ . From Fatou's Lemma we have  $\mathbb{E}^Q[A_t - A_r] \leq \liminf_{n \rightarrow \infty} \mathbb{E}^Q[\text{III} + \text{IV}_{t \wedge \tilde{\rho}_n}] = \mathbb{E}^Q[\text{III}]$ . From Lemma 5.1 that  $\bar{f} \geq f$ , Steps 1 and 5, we can verify that

$$\mathbb{E}^Q[\text{III}] \leq \iint_r^t \Pi_{s,y}[\rho \leq t] (\varrho_{0,y} \delta_0(ds) dy + \bar{f}(\varrho_{s,y}) ds dy) = A_t^e - A_r^e.$$

The desired result in this step then follows.

*Final Step.* The desired result in this lemma follows from Steps 2 and 6.  $\square$

## APPENDIX C. PROOF OF LEMMA 8.1

*Proof of Lemma 8.1.* The existence and uniqueness for (8.1) is given in [12]. The proof of the upper bound (8.2) is also pretty standard, see e.g. derivation of (5) in the proof of Proposition 3.2 in [29], or the relevant steps in the proof of Lemma 2.6 in [22].

We give the upper bound for  $\phi$  in (8.2) provided  $\psi = 0$  on  $(-\infty, b]$  for an arbitrary  $b \in \mathbb{R}$ . First, using the connection between solutions to (8.2) and super-Brownian motion and due to [13, Theorem 1] we can derive

$$(C.1) \quad \frac{1}{2} \vartheta^2 \phi_{s,y} \leq \frac{9}{(y-b)^2}, \quad (s, y) \in [0, T] \times (-\infty, b).$$

Now, let  $(B_t)_{t \geq s}$  be a one-dimensional Brownian motion with generator  $\partial_x^2$  initiated at time  $s$  and position  $y$ ; it induces the probability measure  $\Pi_{s,y}$  on the canonical path space. Then, from the fact that  $\phi \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{R})$ , we can use Ito's formula and the optional sampling theorem to get

$$(C.2) \quad \phi_{s,y} = \Pi_{s,y} \left[ \phi_{\tilde{\rho}, B_{\tilde{\rho}}} - \int_s^{\tilde{\rho}} \left( \frac{1}{2} \vartheta \phi_{r, B_r}^2 - \psi_{B_r} \right) dr \right].$$

that for each optional time  $\tilde{\rho} \in [s, T]$ , defined on the probability space where  $B$  is defined. Choose an arbitrary  $z \in (y, b)$ . Denote by  $\rho_z$  the first time for the Brownian motion  $B$  hitting  $\{z\}$ . Replacing  $\tilde{\rho}$  in (C.2) by  $T \wedge \rho_z$ , we get from (C.1) that

$$(C.3) \quad \phi_{s,y} \leq \Pi_{s,y} [\phi_{T \wedge \rho_z, B_{T \wedge \rho_z}}] \leq \frac{18}{\vartheta^2 (z - b)^2} \Pi_{s,y} (\rho_z < T).$$

From the reflecting principle we have

$$(C.4) \quad \begin{aligned} \Pi_{s,y} (\rho_z < T) &= 2\Pi_{0,0} [B_{T-s} \geq z - y] = 2 \int_{z-y}^{\infty} \frac{1}{\sqrt{4\pi(T-s)}} e^{-\frac{u^2}{4(T-s)}} du \\ &\leq 2 \int_{z-y}^{\infty} \frac{1}{\sqrt{4\pi(T-s)}} \frac{u}{z-y} e^{-\frac{u^2}{4(T-s)}} du \leq \frac{2}{\sqrt{\pi}} \frac{\sqrt{T-s}}{z-y} e^{-\frac{(z-y)^2}{4(T-s)}}. \end{aligned}$$

Note that  $z \in (y, b)$  is chosen arbitrarily. So taking  $z = \frac{y+b}{2}$  in (C.3) and (C.4), we get

$$\phi_{s,y} \leq \frac{18}{\vartheta^2 (z - b)^2} \frac{2}{\sqrt{\pi}} \frac{\sqrt{T-s}}{z-y} e^{-\frac{(z-y)^2}{4(T-s)}} \leq \frac{2^8}{\vartheta^2} \frac{\sqrt{T-s}}{(b-y)^3} e^{-\frac{(b-y)^2}{2^4(T-s)}}. \quad \square$$

#### APPENDIX D. PROOF OF LEMMA 9.3

*Proof of Lemma 9.3.* Let us fix an arbitrary  $\tilde{v} > 0$  and arbitrary  $(t, x), (t', x') \in \mathbb{R}_+ \times \mathbb{R}$  satisfying (9.1). Define  $z := x - \tilde{v}t$  and  $z' := x' - \tilde{v}t'$ . By the symmetry between  $(t, x)$  and  $(t', x')$ , we can assume without loss of generality that  $\frac{\tilde{v}}{2}(z' - z) + \frac{\tilde{v}^2}{4}(t' - t) \geq 0$ .

*Step 1.* Note that one can give the precise expression of  $G^{(\tilde{v})}$  using the reflection principle and Girsanov transformation for the Brownian motion (see [21, Proof of Lemma 6.2]). In fact, for each  $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ , we have

$$G_{s,y;t,x}^{(\tilde{v})} = \rho_{s,y-\tilde{v}s;t,z}^{(1)} - \rho_{s,y-\tilde{v}s;t,z}^{(-1)}$$

where

$$\rho_{s,y;t,z}^{(i)} := e^{-\frac{\tilde{v}}{2}(z-y) - \frac{\tilde{v}^2}{4}(t-s)} G_{s,y;t,z}^{(i)} \mathbf{1}_{y,z \leq 0}, \quad i \in \{1, -1\}.$$

Now from the fact that the squares of the sum of two numbers is bounded by twice the sum of the squares of those two numbers, we have

$$(G_{s,y;t',x'}^{(\tilde{v})} - G_{s,y;t,x}^{(\tilde{v})})^2 \leq 2 \sum_{i=1,-1} (\rho_{s,y-\tilde{v}s;t',z'}^{(i)} - \rho_{s,y-\tilde{v}s;t,z}^{(i)})^2, \quad (s, y) \in \mathbb{R}_+ \times \mathbb{R}.$$

*Step 2.* We show that for each  $i \in \{-1, 1\}$  we have

$$I_i := \int_0^\infty \int_0^\infty (\rho_{s,y;t',z'}^{(i)} - \rho_{s,y;t,z}^{(i)})^2 e^{-\tilde{v}(y-z)} ds dy \leq 4(II_i + III_i)$$

where

$$\text{II}_i := \iint_0^\infty ((\gamma - 1)G_{s,y;t',iz'})^2 ds dy; \quad \text{III}_i := \iint_0^\infty (G_{s,y;t',iz'} - G_{s,y;t,iz})^2 ds dy$$

and  $\gamma := e^{-\frac{\tilde{v}}{2}(z'-z) - \frac{\tilde{v}^2}{4}(t'-t)}$ . In fact we can verify that

$$\begin{aligned} \text{I}_i &= \iint_0^\infty e^{-\tilde{v}(z-y) - \frac{\tilde{v}^2}{2}(t-s)} \mathbf{1}_{y \leq 0} (\gamma G_{s,y;t',iz'} - G_{s,y;t,iz})^2 e^{-\tilde{v}(y-z)} ds dy \\ &\leq e^{\frac{\tilde{v}^2}{2}|t'-t|} \iint_0^\infty (\gamma G_{s,y;t',iz'} - G_{s,y;t,iz})^2 ds dy. \end{aligned}$$

The desired result in this step then follows from (9.1) that  $\tilde{v}^2|t - t'| \leq 1$ .

*Step 3.* We show that for each  $i \in \{-1, 1\}$  we have

$$\text{II}_i \leq (|z' - z| + |t' - t|^{1/2})/4$$

where  $\text{II}_i$  is given in Step 2. In fact,

$$\begin{aligned} \text{II}_i &= (\gamma - 1)^2 \iint_0^{t'} \frac{e^{-\frac{(iz'-y)^2}{2(t'-s)}}}{4\pi(t'-s)} ds dy = (\gamma - 1)^2 \frac{\sqrt{t'}}{\sqrt{2\pi}} \\ &\leq \left( \frac{\tilde{v}}{2}(z' - z) + \frac{\tilde{v}^2}{4}(t' - t) \right)^2 \frac{\tilde{v}^{-1}}{\sqrt{2\pi}} \leq 2 \left( \frac{\tilde{v}}{2}(z' - z) \right)^2 \frac{\tilde{v}^{-1}}{\sqrt{2\pi}} + 2 \left( \frac{\tilde{v}^2}{4}(t' - t) \right)^2 \frac{\tilde{v}^{-1}}{\sqrt{2\pi}} \\ &= \frac{\tilde{v}|z' - z|}{2\sqrt{2\pi}}|z' - z| + \frac{\tilde{v}^3|t' - t|^{3/2}}{8\sqrt{2\pi}}|t' - t|^{1/2}. \end{aligned}$$

Here, in the first inequality, we used the fact that  $\frac{\tilde{v}}{2}(z' - z) + \frac{\tilde{v}^2}{4}(t' - t) \geq 0$ . The desired result in this step then follows from (9.1) that  $\tilde{v}|z - z'| \leq 1$  and  $\tilde{v}^2|t - t'| \leq 1$ .

*Step 4.* We note from [28, Lemma 6.2(1)] that there exists a universal constant  $\tilde{C} > 0$ , independent of our choice of  $(t, x)$ ,  $(t', x')$  and  $\tilde{v}$ , such that  $\text{III}_i \leq \tilde{C}(|z' - z| + |t' - t|^{1/2})$  for each  $i \in \{-1, 1\}$ . In fact, one can take  $\tilde{C} = 2^7$  (c.f. Lemma 10.1).

*Final Step.* From Step 1, we know that

$$\begin{aligned} &\iint_0^\infty (G_{s,y;t',x'}^{(\tilde{v})} - G_{s,y;t,x}^{(\tilde{v})})^2 \frac{e^{-\tilde{v}(y-\tilde{v}s)}}{e^{-\tilde{v}(x-\tilde{v}t)}} ds dy \\ &\leq \sum_{i \in \{-1, 1\}} \iint_0^\infty (\rho_{s,y-\tilde{v}s;t',z'}^{(i)} - \rho_{s,y-\tilde{v}s;t,z}^{(i)})^2 \frac{e^{-\tilde{v}(y-\tilde{v}s)}}{e^{-\tilde{v}z}} ds dy = \sum_{i \in \{-1, 1\}} \text{I}_i. \end{aligned}$$

The desired result in this Lemma then follows from Steps 2, 3 and 4.  $\square$

## APPENDIX E. PROOF OF LEMMA 10.1

*Proof of Lemma 10.1.* Let us fix an arbitrary  $\tilde{v} > 0$ . First note that

$$\begin{aligned} &\iint_0^\infty (G_{s,y;t',x'} - G_{s,y;t,x})^2 e^{-\tilde{v}y} ds dy \\ &\leq 2 \iint_0^\infty (G_{s,y;t,x'} - G_{s,y;t,x})^2 e^{-\tilde{v}y} ds dy + 2 \iint_0^\infty (G_{s,y;t',x'} - G_{s,y;t,x'})^2 e^{-\tilde{v}y} ds dy \end{aligned}$$

$$=: 2\text{I} + 2\text{II}, \quad (t, x), (t', x') \in \mathbb{R}_+ \times \mathbb{R}.$$

To finish the proof it is sufficient to show that

$$(E.1) \quad \text{I} \leq 2^6 |x' - x|, \quad t \in [0, \tilde{v}^{-2}], \quad x, x' \in [-2\tilde{v}^{-1}, 2\tilde{v}^{-1}],$$

$$(E.2) \quad \text{II} \leq 2^6 |t' - t|^{1/2}, \quad t, t' \in [0, \tilde{v}^{-2}], \quad x' \in [-2\tilde{v}^{-1}, 2\tilde{v}^{-1}].$$

We will prove only (E.1), and leave the proof of (E.2), which is tedious but not much different, to the reader.

To prove (E.1) we assume without loss of generality that  $z := x' - x \geq 0$ . Note that

$$2^{-3}\text{I} \leq e^{\tilde{v}x}\text{I} = \iint (G_{s,y;t,x'} - G_{s,y;t,x})^2 e^{-\tilde{v}(y-x)} ds dy = \iint (G_{s,y;t,z} - G_{s,y;t,0})^2 e^{-\tilde{v}y} ds dy.$$

From the expression of  $G$  in (2.3), we have

$$\begin{aligned} 2^{-3}\text{I} &\leq \int_0^t \int_0^t \frac{1}{4\pi s} (e^{-\frac{(y-z)^2}{4s}} - e^{-\frac{y^2}{4s}})^2 e^{-\tilde{v}y} ds dy \\ &= \int_0^t \int_0^t \frac{1}{4\pi s} (e^{-\frac{y^2}{2s}} - 2e^{-\frac{y^2+(y-z)^2}{4s}} + e^{-\frac{(y-z)^2}{2s}}) e^{-\tilde{v}y} ds dy \\ &= \int_0^t \frac{ds}{4\pi s} \int (e^{-\frac{y^2}{2s} - \tilde{v}y} - 2e^{-\frac{y^2}{2s} + (\frac{z}{2s} - \tilde{v})y - \frac{z^2}{4s}} + e^{-\frac{y^2}{2s} + (\frac{z}{s} - \tilde{v})y - \frac{z^2}{2s}}) dy. \end{aligned}$$

From the fact that

$$\int e^{-ay^2 + by} dy = \int e^{-a(y - \frac{b}{2a})^2 + \frac{b^2}{4a}} dy = e^{\frac{b^2}{4a}} \int e^{-ay^2} dy = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}, \quad a > 0, b \in \mathbb{R},$$

we can get

$$\begin{aligned} 2^{-3}\text{I} &\leq \int_0^t \frac{ds}{4\pi s} \left( \sqrt{2s\pi} e^{\frac{s}{2}\tilde{v}^2} - 2e^{-\frac{z^2}{4s}} \sqrt{2s\pi} e^{\frac{s}{2}(\frac{z}{2s} - \tilde{v})^2} + e^{-\frac{z^2}{2s}} \sqrt{2s\pi} e^{\frac{s}{2}(\frac{z}{s} - \tilde{v})^2} \right) \\ &\leq \frac{1}{2} \int_0^t \frac{e^{\frac{\tilde{v}^2}{2}s}}{\sqrt{2\pi}} (1 - 2e^{-\frac{z^2}{8s} - \frac{z\tilde{v}}{2}} + e^{-z\tilde{v}}) \frac{ds}{\sqrt{s}} \\ &\leq \frac{1}{2} \int_0^{z^2 \wedge t} (1 + e^{-z\tilde{v}}) \frac{ds}{\sqrt{s}} + \frac{1}{2} \int_{z^2 \wedge t}^t (2|1 - e^{-\frac{z^2}{8s} - \frac{z\tilde{v}}{2}}| + |e^{-z\tilde{v}} - 1|) \frac{ds}{\sqrt{s}}. \end{aligned}$$

Now using the fact that  $|1 - e^{-z}| \leq z$  for  $z \in \mathbb{R}_+$ , we have

$$\begin{aligned} 2^{-3}\text{I} &\leq (1 + e^{-z\tilde{v}})z + \mathbf{1}_{z^2 \leq t} \int_{z^2}^t \left( \frac{z^2}{8s} + \tilde{v}z \right) \frac{ds}{\sqrt{s}} \\ &\leq 2z + \int_{z^2}^\infty \frac{z^2}{8s} \frac{ds}{\sqrt{s}} + \int_0^t \tilde{v}z \frac{ds}{\sqrt{s}} = (2 + \frac{1}{4} + 2\tilde{v}\sqrt{t})z \leq 2^3 z. \end{aligned}$$

This gives us (E.1). As we have mentioned we omit the proof of (E.2) and thus we are done.  $\square$

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