On the Subcritical Self-Catalytic Branching Brownian Motions (SBBM)

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Based on joint work with Clayton Barnes (AWS) and Leonid Mytnik (Technion), and joint work with Haojie Hou (BIT)

Introduction

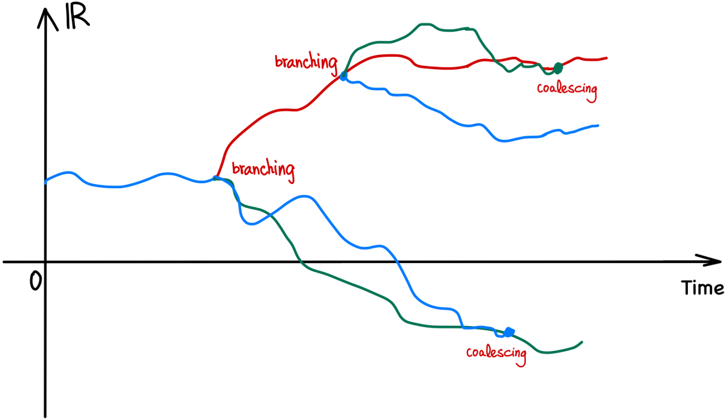
- **Model: Self-Catalytic Branching Brownian Motions (SBBM)**
  A particle system that extends the Branching Brownian Motion by incorporating catalytic branching through pairwise interactions between particles (which will be defined in the next slide).
- **Objectives**:
  - Construct an SBBM that supports infinitely many particles.
  - Characterize the law of the SBBM.
  - Investigate the coming down from infinity (CDI) property of the SBBM.
- **Motivation**:
  - SBBMs offer insights into complex population dynamics, incorporating biological dispersal and intraspecific competition/cooperation.
  - SBBMs serve as moment duals to a class of stochastic reaction-diffusion equations with multiplicative noise.

Model Setup and Parameters

SBBM Dynamics:

- **Initial Configuration**:
  The positions of the initial particles are given by $(x_i)_{i=1}^n \subset \mathbb{R}$.
- **Particle Movement**:
  Each particle performs independent Brownian motion on $\mathbb{R}$.
- **Ordinary Branching**:
  Particles branch independently at rate β_{o}, replaced by k (random) offspring according to the law (p_k).
- **Catalytic Branching**:
  Each pair of particles independently branches at rate β_{c}, based on their intersection local times, replaced by k (random) offspring according to the law (q_k).

An illustration of SBBM



- A graph illustration of SBBM with $p_3 = 1$ and $q_1 = 1$.

Assumptions

- **Catalytic Branching is Subcritical:**

$$\sum kq_k < 2.$$

- **Catalytic Branching is Not Parity-Preserving:**
  There exists an odd k such that $q_k > 0$.
- **A Technical Assumption:**
  There exists $R > 1$ such that

$$\sum R^k p_k < \infty \quad \text{and} \quad \sum R^k q_k < \infty.$$

The Explosion Problem for SBBM

A Priori Consideration:

- The SBBM model is well-defined only up to its explosion time τ_∞.

Definition of the Explosion Time τ_∞:

- Define a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{Z}_+}$ such that:
  - $\tau_0 = 0$.
  - For each $k \geq 0$,

$$\tau_{k+1} := \inf\{t > \tau_k : \text{a branching occurs at time } t\}.$$

- The explosion time is defined as:

$$\tau_\infty := \lim_{k\to\infty} \tau_k.$$

The Explosion Problem:

- Does $\tau_\infty = \infty$?

Non-Explosion Property

Athreya-Tribe (2000, Ann. Probab.)

If the ordinary branching is subcritical, i.e., $\sum kp_k < 1$, then:

$$\mathbb{P}(\tau_\infty = \infty) = 1.$$

Hou-S. (2025, arXiv)

Same result holds without assuming $\sum kp_k < 1$.

Implications:

- The SBBM model remains well-defined for all time.
- For $t \geq 0$, define $Z_t^{(n)}(A) := \#\{\text{particles in set } A \text{ at time } t\}$.
- The process $(Z_t^{(n)})_{t\geq 0}$ is a Markov process taking values in:

$$\mathcal{N} := \{\text{locally finite point measures on } \mathbb{R}\}.$$

- We refer to $Z_\cdot^{(n)}$ as an SBBM (with n initial particles).

The Infinitely Initial Particles Problem

- Consider the scenario where the initial configuration of an SBBM consists of infinitely many particles located at $(x_i)_{i=1}^{\infty} \subset \mathbb{R}$.
- In this case, $\tau_\infty = 0$ almost surely, and the model is a priori not well-defined.

The Infinitely Initial Particles Problem:

- Let $(Z_t^{(n)})_{t\geq 0}$ be an SBBM with initial value $\sum_{i=1}^{n} \delta_{x_i}$.
- What is the limit of the processes $(Z_t^{(n)})_{t\geq 0}$ as $n \to \infty$?

Characterizing the Law:

- How can we characterize the law of $(Z_t^{(n)})_{t\geq 0}$?
- Is the family of processes $\{(Z_t^{(n)})_{t\geq 0} : n \in \mathbb{N}\}$ tight?

The Dual SPDE

- **The dual SPDE of SBBM**:

$$\begin{cases} \partial_t u_t(x) = \frac{\Delta}{2} u_t(x) - \Phi(u_t(x)) + \sqrt{\Psi(u_t(x))}\,\dot{W}_t(x), & t > 0, x \in \mathbb{R}, \\ u_0(x) = f(x), & x \in \mathbb{R}. \end{cases}$$

- **Space-time White Noise**:
$(W_t)_{t\geq 0}$:= a cylindrical Wiener process on $L^2(\mathbb{R})$, such that

$$\mathbb{E}[W_t(\phi) W_s(\psi)] = (t \wedge s)\langle \phi, \psi \rangle.$$

- **Ordinary Branching Mechanism**:

$$\Phi(z) := \beta_{\mathrm{o}} \left(\sum_{k=0}^{\infty} p_k (1-z)^k - (1-z) \right).$$

- **Catalytic Branching Mechanism**:

$$\Psi(z) := \beta_{\mathrm{c}} \left(\sum_{k=0}^{\infty} q_k (1-z)^k - (1-z)^2 \right).$$

The Dual SPDE

- Let $z^* := \inf\{z \in [1,2] : \Psi(z) = 0\}$.
- Under the assumption that the catalytic branching is not parity-preserving, we have $z^* \in [1,2)$.
- Let $C(\mathbb{R},[0,z^*]) := \{\text{Continuous functions from } \mathbb{R} \text{ to } [0,z^*]\}$.
- We say a $C(\mathbb{R},[0,z^*])$-valued continuous process $(u_t)_{t\geq 0}$ is a weak solution, if $\exists$ a space-time white noise $\dot{W}$ such that $\forall \phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}&\langle u_t, \phi\rangle - \langle f, \phi\rangle \\ &= \int_0^t \langle u_s, \frac{\Delta}{2}\phi\rangle ds - \int_0^t \langle \Phi(u_s), \phi\rangle ds + \int_0^t \langle \sqrt{\Psi(u_s)}\phi, dW_s\rangle, \quad \text{a.s.}\end{aligned}$$

Shiga (1994, Can. J. Math.)

For each initial value $f \in C(\mathbb{R},[0,z^*])$, there exists a weak solution to the dual SPDE.

The Duality

- For any $[0, z^*]$-valued function g and point measure μ, define

$$(1-g)^\mu := \prod_{x\in\mathbb{R}} (1-g(x))^{\mu(\{x\})}.$$

Athreya-Tribe (2000, Ann. Probab.)

If the ordinary branching is subcritical, i.e., $\sum kp_k < 1$, then:

$$\mathbb{E}\left[(1-u_t)^{Z_0^{(n)}}\right] = \mathbb{E}\left[(1-u_0)^{Z_t^{(n)}}\right].$$

Hou-S. (2025, arXiv)

The result above holds without assuming $\sum kp_k < 1$.

- **Corollary:** The uniqueness in law holds for the dual SPDE.

SBBM with Infinitely Many Initial Particles

- Let the state space $\mathcal{N}$ be equipped with the vague topology.

Initial Trace (Λ, μ):

- $\Lambda := \{\text{sub-sequential limits of } (x_i)_{i=1}^{\infty}\}$.
- $\mu := \sum_{x_i \notin \Lambda} \delta_{x_i}$.

Hou-S. (2025, arXiv)

There exists an $\mathcal{N}$-valued càdlàg Markov process $(Z_t)_{t>0}$ such that $(Z_t^{(n)})_{t>0}$ converges to $(Z_t)_{t>0}$ as $n \to \infty$ in finite-dimensional distributions. The law of the process $(Z_t)_{t>0}$ is determined by the two branching mechanisms (Φ, Ψ) and the initial trace (Λ, μ).

- We call $(Z_t)_{t>0}$ an SBBM with initial trace (Λ, μ) and branching mechanisms (Φ, Ψ).

Coming Down from Infinity (CDI)

The Local-Time Coalescing Brownian Motions (LCBM):

- If the ordinary branching rate $\beta_{\mathrm{o}} = 0$ and the catalytic branching law satisfies $q_1 = 1$, then the SBBM degenerates into the LCBM.
- In this case, $\Phi = 0$ and $\Psi(z) = z(1-z)$.

Barnes-Mytnik-S. (2024, Ann. Probab.)

Suppose that $(Z_t)_{t>0}$ is an LCBM. Let U be any open interval. Then, almost surely, for every $t > 0$,

$$Z_t(U) < \infty \iff (\Lambda \cup \operatorname{supp}(\mu)) \cap U \text{ is bounded.}$$

Hou-S. (2025, arXiv)

The same result holds for SBBM.

- **The CDI property**: Almost surely, for every $t > 0$,

$$Z_t(\mathbb{R}) < \infty \iff \sup\{|x_i| : i \in \mathbb{N}\} < \infty.$$

The Mean Field Equation (MFE)

From a physic's point of view:

- The MFE for a system of independent Brownian motions is given by the heat equation $\partial_t h = \frac{\Delta}{2} h$. In the sense that

$$\mathbb{E}[\#\{\text{particles in } (x-\frac{1}{2}, x+\frac{1}{2}) \text{ at time } t\}] \approx h_t(x).$$

- The MFE for LCBM is $\partial_t v = \frac{\Delta}{2} v - \frac{\Psi'(0+)}{2} v^2$.

Le Gall (1996, J. Appl. Math. Stochastic Anal.)

There exists a unique non-negative solution $(v_t(x))_{t>0, x\in\mathbb{R}}$ to the PDE

$$\begin{cases} \partial_t v_t(x) = \dfrac{\Delta}{2} v_t(x) - \dfrac{\Psi'(0+)}{2} v_t(x)^2, \quad t>0, x\in\mathbb{R}, \\ \left\{ y \in \mathbb{R} : \forall r>0, \lim\limits_{t\to 0} \displaystyle\int_{y-r}^{y+r} v_t(x)\,dx = \infty \right\} = \Lambda, \\ \lim\limits_{t\to 0} \langle v_t, \phi \rangle = \langle \mu, \phi \rangle, \quad \phi \in C_c(\Lambda^c). \end{cases}$$

Speed of CDI for LCBM

The Speed of CDI Problem

- Assume CDI holds for a process $(N_t)_{t\geq 0}$.
- Can we find a rate function $a(t)$ such that $N_t/a(t) \to 1$ as $t \downarrow 0$?

Barnes-Mytnik-S. (2024, Ann. Probab.)

Suppose that $(Z_t)_{t>0}$ is an LCBM with initial trace (Λ, μ). Let U be an open interval. Suppose that $(\Lambda \cap \mathrm{supp}(\mu)) \cap U$ is bounded and $\Lambda \cap \bar{U} \neq \emptyset$. Then,

$$\left(\int_U v_t(x)\, dx \right)^{-1} Z_t(U) \xrightarrow[t\downarrow 0]{L^1} 1,$$

where $(v_t(x))_{t>0, x\in\mathbb{R}}$ is the solution to the corresponding MFE with initial trace (Λ, μ).

Hou-S. (2025, arXiv)

The same result holds for SBBM.

Comments

Criticality of the Branching:

- It is crucial for our result that the catalytic branching is subcritical, i.e., $\sum kq_k < 2$.
- Barnes-Mytnik-S. (2025, Probab. Theory Related Fields) constructed an SBBM with $p_\infty = 1$ and $q_1 = 1$, and showed that the total population in this model is "reflecting from infinity".
- When the catalytic branching is supercritical, i.e., $\sum kq_k > 2$, we believe that the SBBM will explode in finite time.
- When there is no ordinary branching, i.e., $\beta_{\mathrm{o}} = 0$, and the catalytic branching is critical, i.e., $\sum kq_k = 2$, we believe that the SBBM is non-explosive and rescales to the stochastic heat equation:

$$\partial_t u = \frac{\Delta}{2} u + u\dot{W}.$$

Comments

About the Parity:

- It is crucial for our result that the catalytic branching is not parity-preserving, i.e., there exists an odd number k such that $q_k > 0$.
- Consider an SBBM with no ordinary branching, i.e., $\beta_0 = 0$, and $q_0 = 1$. We call this model the local-time annihilating Brownian motion (LABM).
- LABM is non-explosive and can be defined up to all time, provided there are only finitely many initial particles.
- It can be shown that $\{Z_t^{(n)} : n \in \mathbb{N}\}$ is tight.
- However, the subsequential convergence-in-distribution limit of $\{Z_t^{(n)} : n \in \mathbb{N}\}$ is not unique.
- Hammer-Ortgiese-Völlering (2021, Stochastic Process. Appl.): The entrance laws of the (hard) annihilating Brownian motion are characterized.
- Charaterization of all entrace laws of LABM is still open.

Comments

Examples of Duality:

- We say two Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are dual to each other if there exists a large class of functions $H(x,y)$ such that
$$\mathbb{E}[H(X_t, Y_0)] = \mathbb{E}[H(X_0, Y_t)].$$
- Bachelier (1900, Ann. Sci. École Norm. Sup.): Brownian motion and the heat equation $\partial_t h = \frac{\Delta}{2} h$.
- McKean (1975, Comm. Pure Appl. Math.): Branching Brownian motion and the FKPP equation $\partial_t v = \frac{\Delta}{2} v + v(1-v)$.
- Harris (1978, Ann. Probab.): Coalescing random walk and the voter model.
- Shiga (1986, Math. Appl.): LCBM and the stochastic FKPP equation $\partial_t v = \frac{\Delta}{2} v + \sqrt{v(1-v)}\dot{W}$.
- Tóth-Werner (1998, Probab. Theory Relat. Fields): (Hard) Coalescing Brownian motions and itself.
- **Folklore**: Stochastic heat equation $\partial_t u = \frac{\Delta}{2} u + u\dot{W}$ and itself.
- ...

Comments

Coming Down from Infinity (CDI):

- Feller (1954, Trans. Amer. Math. Soc.): Some diffusion processes in one dimension.
- Aldous (1999, Bernoulli): Kingman's coalescent.
- Schweinsberg (2000, Electron. Comm. Probab.) and Berestycki-Berestycki-Limic (2010, Ann. Probab.): Λ-coalescent.
- Limic-Sturm (2006, Electron. J. Probab.) and Angel-Berestycki-Limic (2012, Probab. Theory Related Fields): Coalescing random walks on graphs.
- Mourrat-Weber (2017, Comm. Math. Phys.): Dynamical Φ^4_3 model (leading to a new construction of the Euclidean Φ^4_3 Field Theory).
- ...

Comments

The Mean Field Equation (MFE)

- The CDI rate of SBBM is characterized by $\partial_t v = \frac{\Delta}{2} v - \frac{\Psi'(0+)}{2} v^2$ despite that the true MFE is $\partial_t \tilde{v} = \frac{\Delta}{2} \tilde{v} + \Phi'(0+)\tilde{v} - \frac{\Psi'(0+)}{2} \tilde{v}^2$.
- This is because $v(s,y) \asymp \tilde{v}(s,y)$ uniformly for $(s,y) \in [0,1] \times \mathbb{R}$.
- The equation $\partial_t v = \frac{\Delta}{2} v - v|v|^\alpha$ with initial trace (Λ, μ) was studied by Marcus-Véron (1999, Comm. Partial Differential Equations) in the PDE literature.
- Watanabe (1968, J. Math. Kyoto Univ.):
  The equation $\partial_t v = \frac{\Delta}{2} v - \frac{\Psi'(0+)}{2} v^2$ is the Laplace dual to the Super-Brownian motion $(X_t)_{t\geq 0}$.
- Le Gall (1996, J. Appl. Math. Stochastic Anal.) used the equation $\partial_t v = \frac{\Delta}{2} v - \frac{\Psi'(0+)}{2} v^2$ to study the Brownian snake, which is related to the super-Brownian motion through a Ray-Knight type theorem.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{The SBBM}} \arrow[rr, leftrightarrow, "\text{Moment dual}"] \arrow[d, leftrightarrow, "\text{Our Results}"'] & & \boxed{\begin{array}{l}\text{The dual SPDE}\\ \partial_t u = \frac{\Delta}{2} u - \Phi(u) + \sqrt{\Psi(u)}\dot{W}\end{array}} \arrow[d, leftrightarrow, "\begin{array}{r}\text{Behave similarly}\\ \text{for small initial data}\\ \text{and small time}\end{array}"'] \\
\boxed{\begin{array}{l}\text{The Mean Field Equation}\\ \partial_t v_t = \frac{\Delta}{2} v - \frac{\Psi'(0+)}{2} v^2\end{array}} \arrow[rr, leftrightarrow, "\text{Laplacian dual}"] & & \boxed{\begin{array}{l}\text{The Super-Brownian motion}\\ \partial_t X = \frac{\Delta}{2} X + \sqrt{\Psi'(0+)X}\dot{W}\end{array}} \\
\boxed{\begin{array}{l}\text{The FKPP equation}\\ \partial_t w = \frac{\Delta}{2} w + \frac{\lambda}{2}(w-1)^2\end{array}} \arrow[u, "\lambda \to \infty"] \arrow[rr, leftrightarrow, "\text{Moment dual}"] & & \boxed{\begin{array}{l}\text{The critical binary BBM}\\ \text{with branching rate } \lambda\end{array}} \arrow[u, "\lambda \to \infty"']
\end{tikzcd}$$

Thanks!