Effect of small noise on the speed of reaction-diffusion equation with non-Lipschitz drift

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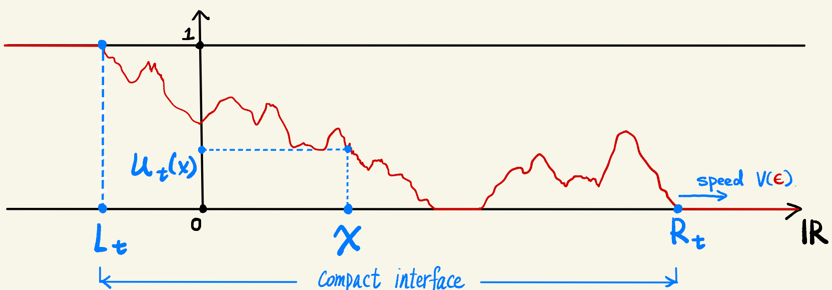
Stochastic reaction-diffusion equation

$$\text{①} \quad \underbrace{\partial_t u = \tfrac{1}{2}\partial_x^2 u}_{\text{heat equation}} + \underbrace{u(1-u)}_{\text{FKPP drift}} + \underbrace{\epsilon}_{\text{noise strength}} \underbrace{\sqrt{u(1-u)}\,\dot{W}}_{\text{Wright-Fisher white noise}} \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}\ ; \quad \epsilon > 0.$$

Shiga (1988), Mueller & Sowers (1995)

The weak existence and weak uniqueness of SPDE ① hold in

$C_I := \{$ Continuous $[0,1]$-valued functions on $\mathbb{R}$ with compact interface $\}$.



Four equivalent solution concepts:

- Mild form
- Schwartz distribution
- Martingale problem

Walsh (1984), Iwata (1987)

- Dual process of branching-coalescing Brownian motion.

Shiga (1988)

Mueller & Sowers (1995), Conlon & Doering (2005)

$\forall \epsilon > 0$, $\exists$ a deterministic $V(\epsilon)\in\mathbb{R}$ s.t. $\dfrac{R_t}{t} \xrightarrow[t\to\infty]{} V(\epsilon)$ a.s.

N-branching random walk

Bérard & Gouéré (2010).

Branching-coalescing Brownian motion

Mueller, Mytnik & Quastel (2011).

N-branching Brownian motion

Maillard (2012). . . . .

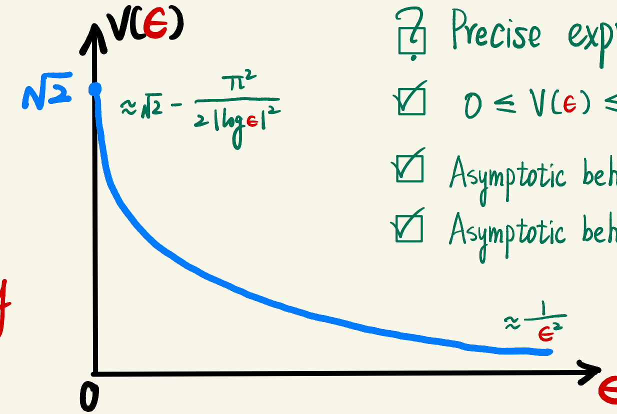
Branching Brownian motion in a strip

Berestycki, Berestycki & Schweinsberg (2013).

L-branching Brownian motion

Pain (2016).

universality



? Precise expression.

☑ $0 \le V(\epsilon) \le \sqrt{2}$.

☑ Asymptotic behavior for small ϵ.

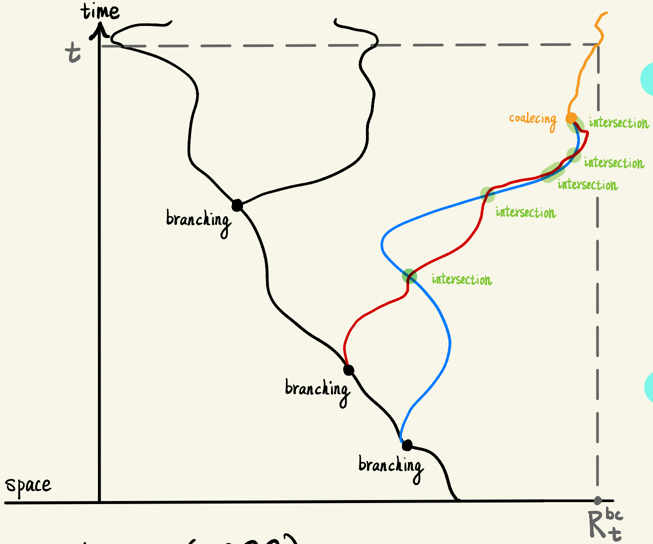
☑ Asymptotic behavior for large ϵ.

Brunet & Derrida conjecture (1997), Mueller, Mytnik & Quastel (2011)

$$V(\epsilon) = \sqrt{2} - \frac{\pi^2}{2|\log \epsilon|^2} + O\left(\frac{\log|\log \epsilon|}{|\log \epsilon|^3}\right), \qquad \epsilon \to 0.$$

Conlon & Doering (2005), Mueller, Mytnik & Ryzhik (2021)

$$\lim_{\epsilon \to +\infty} \epsilon^2 V(\epsilon) = 1.$$



Branching-coalescing Brownian motion

Independently
- $\forall$ particle moves as Brownian motion.
- $\forall$ particle branches into 2 particles with rate 1.
- $\forall$ pair of particles coalesce into 1 particle with rate ϵ^2 according to their intersection local time.

Notation:

$I_t = \{$ particles alive at time $t\}$.

$X_t^i =$ Position of particle $i \in I_t$ at time t.

Shiga (1988)

If the branching-coalescing Brownian motion is independent of the weak solution $u_t(x)$,

then $\mathbb{E} \prod_{i\in I_0} [1-u_t(X_0^i)] = \mathbb{E} \prod_{i \in I_t} [1-u_0(X_t^i)]$.

$\xrightarrow{\text{LHS}}$ finite moments of u_t $\xrightarrow{\text{determine}}$ distribution of u_t $\longrightarrow$ weak uniqueness.

$\xrightarrow{\text{taking } u_0 = 1_{(-\infty,0)}}$ $\mathbb{P}(R_t^{bc} > x) = \mathbb{E}\, u_t(x)$ $\longrightarrow$ propagation of branching-coalescing Brownian motion.

Benguria & Depassier (1996): In application, people are interested in reaction-diffusion equation $\partial_t u = \frac{1}{2}\partial_x^2 u + f(u)$ with more general reaction term.

And in many cases when f is lipschitz, the system has finite propagation speed.

Aguirre & Escobedo (1986): If $f(u) = u^p(1-u)$ with $0<p<1$, then the system doesn't have finite propagation.
non-Lipschitz

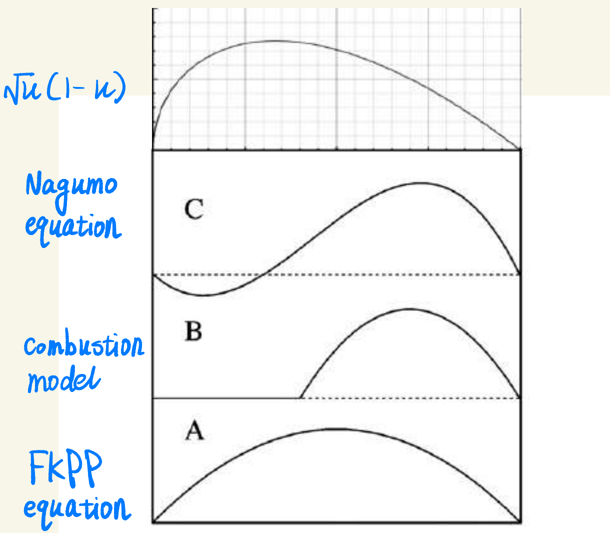


FIG. 1. The three basic types of reaction terms that arise in different applications.

Image of $f(u)$ © Benguria & Depassier (1996)

Athreya & Tribe (2000):

A large class of stochastic reaction-diffusion equations

$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \sigma(u)\dot{W}$$

With Lipschitz f and σ^2, have duality relation to some branching-coalescing type particle system.

Conjecture

Shiga's duality holds between $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\,\dot{W}$ and branching-coalescing Brownian motion whose offspring law has generating function $u - (1-u)^p u$. $(0<p<1)$

Examples: types A, B, C & $f(u)=u^p(1-u)$ with $\frac{1}{2}\le p\le 1$.

(2) $\partial_t u=\frac{1}{2}\partial_x^2 u+f(u)+\epsilon\sqrt{u(1-u)}\,\dot{W}$ where $f\in C([0,1])$ and $\sup_{u\in(0,1)}\frac{|f(u)|}{\sqrt{u(1-u)}}<+\infty$

Mueller, Mytnik & Ryzhik (2021)

- $\forall\,\epsilon>0$, the weak existence and weak uniqueness of SPDE (2) hold in $C_I:=\{$ Continuous $[0,1]$-valued functions on $\mathbb{R}$ with compact interface $\}$.
- $\forall\,\epsilon>0$, $\exists$ a deterministic $V_f(\epsilon)\in\mathbb{R}$ s.t. $\frac{R_t}{t}\xrightarrow[t\to\infty]{}V_f(\epsilon)$ a.s.

Idea: $\partial_t u=\frac{1}{2}\partial_x^2 u+\epsilon\sqrt{u(1-u)}\,\dot{W}$.

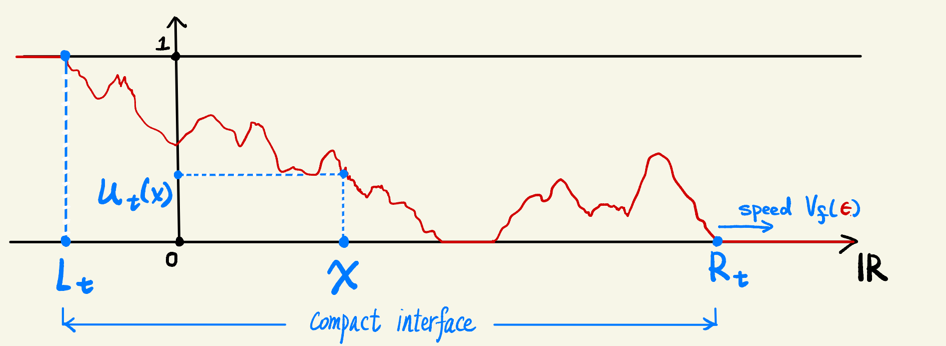
(2) Girsanov transformation

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}=\exp\left(M_t-\frac{1}{2}\langle M\rangle_t\right)$$

$$M_t=\int_{L_t}^{R_t}\int_0^t\frac{f(u)}{\sqrt{u(1-u)}}\,dW$$

bounded

Finite propagation $\xleftarrow{\text{due to}}$ Noise $\epsilon>0$



Mueller, Mytnik & Ryzhik (2021)

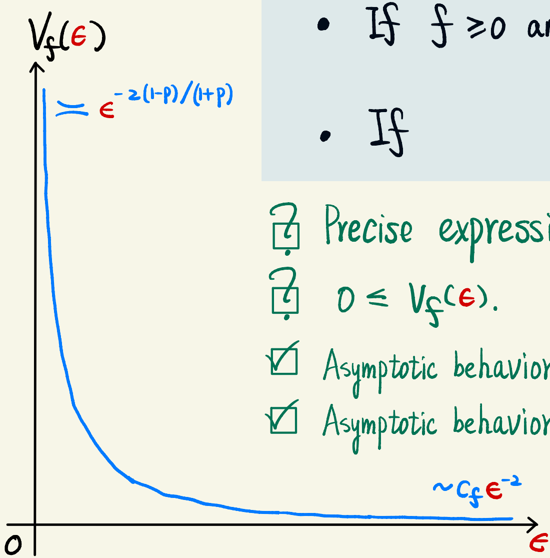
$\forall\ f \in C([0,1])$ satisfying a "slightly stronger" condition than $\sup_{u\in(0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$,

$\exists$ a deterministic $C_f \in \mathbb{R}$ s.t. $\lim_{\epsilon\to+\infty} \epsilon^2 V_f(\epsilon) = C_f$.

Mytnik, Barnes & S. (2021+)

$\forall\ f \in C([0,1])$ s.t. $\sup_{u\in(0,1)} \frac{|f(u)|}{\sqrt{u(1-u)}} < +\infty$:

- If $f \geq 0$ and $\exists\ \frac{1}{2} \leq p \leq 1$ s.t. $\liminf_{u\downarrow 0} u^{-p} f(u) > 0$, then $\liminf_{\epsilon\downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon) > 0$;
- If $\exists\ \frac{1}{2} \leq p \leq 1$ s.t. $\limsup_{u\downarrow 0} u^{-p} f(u) < +\infty$, then $\limsup_{\epsilon\downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon) < +\infty$.



- ☐ Precise expression.
- ☐ $0 \leq V_f(\epsilon)$.
- ☑ Asymptotic behavior for small ϵ.
- ☑ Asymptotic behavior for large ϵ.

For example, if $f(u) = u^p(1-u)$ with $\frac{1}{2} \leq p \leq 1$, then

$\exists\ c, C > 0\ \&\ \epsilon_0 > 0$ s.t.

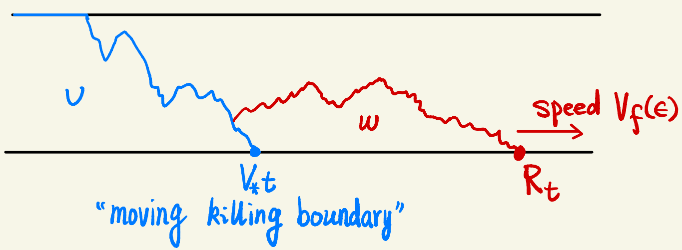
$$c\,\epsilon^{-2\frac{1-p}{1+p}} \leq V_f(\epsilon) \leq C\,\epsilon^{-2\frac{1-p}{1+p}}, \qquad 0 < \epsilon \leq \epsilon_0$$

Same exponent, can not be improved.

Step 1: Choose a killing boundary.

Decompose $u = v + w$ where
$$\begin{cases} \partial_t v = \frac{1}{2}\partial_x^2 v + f(v) + \epsilon\sqrt{v(1-v)}\,\dot{W}', & x < V_* t \\ v = 0, & x \geq V_* t \end{cases}$$

left to be chosen



If we choose:

- $V_* \gg V_f(\epsilon) \xrightarrow{\text{then}} w$ will be small,
- $V_* \ll V_f(\epsilon) \xrightarrow{\text{then}} w$ will be large.

Idea: If $\exists\, V_*$ s.t. w is neither too small nor too large,

Then $V_* \approx V_f(\epsilon)$.

Criticality

An insight: The balanced value for V_* can be predicted by finding the solution (F, V_*) so that

$$\begin{cases} F'(0-) = \epsilon^2, \\ F(-\infty) = 1, \\ \rho(t,x) := F(x - V_* t), & \forall (t,x), \\ \partial_t \rho = \frac{1}{2}\partial_x^2 \rho + f(\rho), & x < V_* t, \\ \rho = 0, & x \geq V_* t. \end{cases}$$

phase plane (PDE) argument

$$V_* \sim \epsilon^{-2\frac{1-p}{1+p}}.$$

Only a prediction.

Still need to analyze w when $V_* \sim \epsilon^{-2\frac{1-p}{1+p}}$.

on next page ...

Step 2: Analyze w.

$A_t = \int_0^t \dot{A}_s \, ds$ is the total mass of v killed at the boundary up to time t.

From $\partial_t u = \partial_x^2 u + f(u) + \epsilon \sqrt{u(1-u)}\, \dot{W}$

subtract $\partial_t v = \partial_x^2 v + f(v) + \epsilon \sqrt{v(1-v)}\, \dot{W}' - \dot{A}_t \delta_{V_t^*}(x)$,

We get $\partial_t w = \partial_x^2 w + f(u) - f(v) + \epsilon \sqrt{u(1-u)}\, \dot{W} - \epsilon \sqrt{v(1-v)}\, \dot{W}' + \dot{A}_t \delta_{V_t^*}(x)$.

Approximately $\partial_t w \approx \partial_x^2 w + c_f w^p + \epsilon \sqrt{w}\, \dot{W}'' + \dot{A}_t \delta_{V_t^*}(x)$

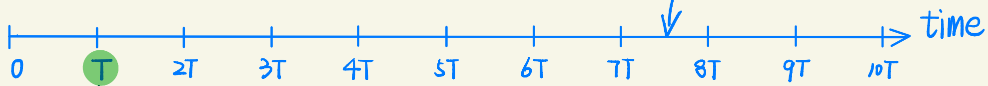
Girsanov transformation $\partial_t w \approx \partial_x^2 w + \epsilon \sqrt{w}\, \dot{W}'' + \dot{A}_t \delta_{V_t^*}(x)$

critical super-Brownian motion with immigration

☑ properties of support.

! Girsanov transformation doesn't preserve long time behavior.

So we perform the transformation on each of the small intervals $[nT, (n+1)T)$.



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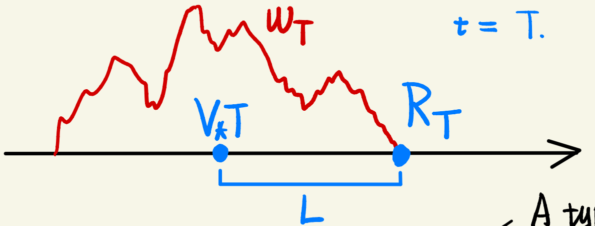
- can't be too large, otherwise: Transformed $w \not\approx$ Original w.
- can't be too small, otherwise: Small time random fluctuation $\xrightarrow{\text{Covers}}$ info. of propagation speed.

Step 3: Choose T.



A typical distance for the support of w to travel in a time interval with length T.

We want to choose T s.t.:

- L can be explained by the thermal diffusivity, i.e. $L \sim \sqrt{T}$.
- w does not give excess speed, i.e. $L/T \lesssim V_* \sim \epsilon^{-2\frac{1-p}{1+p}}$.
- T is as small as possible $\xleftarrow{\text{in order}}$ Transformed $w \approx$ Original w.

$\Rightarrow \quad T \sim \epsilon^{4\frac{1-p}{1+p}}.$

Further questions:

- $\lim_{u \downarrow 0} u^{-p} f(u)$ exists $\Rightarrow$ $\lim_{\epsilon \downarrow 0} \epsilon^{2\frac{1-p}{1+p}} V_f(\epsilon)$ exists.
- Duality for $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\,\dot{W}$ $(0<p<1)$.
- Weak uniqueness for $\partial_t u = \frac{1}{2}\partial_x^2 u + u^p(1-u) + \epsilon\sqrt{u(1-u)}\,\dot{W}$ $(0<p<\frac{1}{2})$.
- Speed of L_t = Speed of R_t $\left(\overset{\text{duality}}{=}\text{Speed of Branching-coalescing Brownian motion}\right)$.
- $(u_t(R_t+x) : x \in \mathbb{R}) \xrightarrow[t \to +\infty]{\text{weakly}}$ a stationary (random) wave profile.
- Speed of N-BBM & L-BBM with heavy-tailed offspring law (universality).

This talk is based on a joint work with

&

Leonid Mytnik

Clayton Barnes.

Manuscript is available on arXiv: 2107.09377

Thanks!!