1.Holling Type II Model

We consider the nondimensionalized three-species predator-prey model

$$\begin{aligned} u_t &= u_{xx} + u\left(1 - u - \omega_0 \frac{v}{u + D_0}\right) \\ v_t &= \mu_2 v_{xx} + v\left(-a_2 + \omega_1 \frac{u}{u + D_1} - \omega_2 \frac{w}{v + D_2}\right) \\ w_t &= \mu_3 w_{xx} + w\left(-a_3 + \omega_3 \frac{v}{v + D_3}\right) \end{aligned} \tag{1}$$

where $\omega_0, a_2, \omega_1, \omega_2, a_3$ and ω_3 are all positive. Now we consider uniform equilibrium states of (11). The zero-state equilibrium state is $E_0(u,v,w) = (0,0,0)$, the one-state equilibrium state is $E_1(u,v,w) = (1,0,0)$, and two-state equilibrium state is $E_2(\bar{u},\bar{v},0)$ exists where we can know

$$\bar{u} = \frac{a_2 D_1}{\omega_1 - a_2}, \quad \bar{v} = \frac{\bar{u} + D_0 - \bar{u}^2 - D_0\bar{u}}{\omega_0} \tag{2}$$

and it is only physical when $a_2 < \omega_1$ and $a_2 D_1 < \omega_1 - a_2$. Another three-state equilibrium state is $E_3(u_c, v_c, w_c)$ exists, where

$$v_c = \frac{\omega_3 D_3}{\omega_3 - a_3}, \quad w_c = \frac{\omega_1 u_c (v_c + D_2)}{\omega_2 (u_c + D_1)} - \frac{a_2 (v_c + D_2)}{\omega_2} \tag{3}$$

and u_c is the solution of equation

$$F(u) = -u^2 + (1 - D_0)u + D_0 - \omega_0 v_c = 0$$

We want the solution u_c to be between $(0,1]$. When $D_0 - \omega_0 v_c \geq 0$, we can know $F(1) = -\omega_0 v_c$ which is always smaller than 0. So the solution will fall between $(0,1]$. When $D_0 - \omega_0 v_c < 0$, if $1 \leq D_0 \leq 3$, we can know that $F(1) = -\omega_0 v_c < D_0 - \omega_0 v_c = F(0)$ and the axis of symmetry is between $(0,1)$, u_c is physical only when $F\left(\frac{D_0 - 1}{2}\right) \geq 0$. If $D_0 < 1$, then there will no solution that falls between $(0,1)$. So we can know that u_c has physical meaning if $D_0 - \omega_0 v_c \geq 0$ or $F\left(\frac{D_0 - 1}{2}\right) \geq 0$ under the conditions $D_0 - \omega_0 v_c < 0$ and $1 \leq D_0 \leq 3$, v_c is physical meaningful if and only if $\omega_3 > a_3$, and w_c is meaningful if and only if $\omega_1 u_c < a_2(u_c + D_1)$.

2.Nonlocality

We consider two specific manifestations of nonlocality appropriate for three-species model 11. We first use a stepfunction kernel

$$\varphi(x) = \begin{cases} \dfrac{1}{2\delta}, & |x| < \delta \\ 0, & |x| > \delta \end{cases} \tag{4}$$

and we can know the Fourier transform of this kernel

$$\hat{\varphi}(k) = \frac{\sin(\delta k)}{\delta k} \tag{5}$$

C-type Nonlocality. Species u competes with itself with nonlocality, and we can get the model

$$
\begin{aligned}
u_t &= u_{xx} + u\left(1 - (\varphi * u) - \omega_0 \frac{v}{u + D_0}\right) \\
v_t &= \mu_2 v_{xx} + v\left(-a_2 + \omega_1 \frac{u}{u + D_1} - \omega_2 \frac{w}{v + D_2}\right) \\
w_t &= \mu_3 w_{xx} + w\left(-a_3 + \omega_3 \frac{v}{v + D_3}\right)
\end{aligned} \tag{6}
$$

where

$$
\varphi * u = \int_{-\infty}^{\infty} \varphi(y - x) u(y) dy
$$

P_ω-type Nonlocality. Species w preys on v with nonlocality, then we can get the model

$$
\begin{aligned}
u_t &= u_{xx} + u - u^2 - \omega_0 \frac{uv}{u + D_0} \\
v_t &= \mu_2 v_{xx} - a_2 v + \omega_1 \frac{uv}{u + D_1} - \omega_2 \frac{v(\varphi * w)}{v + D_2} \\
w_t &= \mu_3 w_{xx} - a_3 w + \omega_3 \frac{v(\varphi * w)}{v + D_3}
\end{aligned} \tag{7}
$$

3.Stability

3.1 Stability of E_1 for the ODE Model

We discuss the stability of the E_1 state for the ODE model with no spatial dependence (local problem). We linearized the system and get the system

$$
\begin{aligned}
u_t &= -u + 1 - \frac{\omega_0 v}{1 + D_0} \\
v_t &= \left(-a_2 + \frac{\omega_1}{1 + D_1}\right) v \\
w_t &= -a_3 w
\end{aligned} \tag{8}
$$

Then we can get the Jacobian

$$
J_1 = \begin{bmatrix} -1 & -\frac{\omega_0}{1+D_0} & 0 \\ 0 & -a_2 + \frac{\omega_1}{1+D_1} & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \tag{9}
$$

So the E_1 state is stable if and only if

$$
-a_2 + \frac{\omega_1}{1 + D_1} < 0 \tag{10}
$$

3.2 Stability of E_1 for the Nonlocal Problems
We consider perturbations in form of

$$u \sim 1 + \epsilon\tilde{u}(x,t), \quad v \sim \epsilon\tilde{v}(x,t), \quad \omega \sim \epsilon\tilde{\omega}(x,t)$$

where $\epsilon \ll 1$. For the local problem, this gives

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} - \tilde{u} - \frac{\omega_0 \tilde{v}}{1+D_0} \\ \tilde{v}_t &= \mu_2 \tilde{v}_{xx} + \left(-a_2 + \frac{\omega_1}{1+D_1}\right)\tilde{v} \\ \tilde{w}_t &= \mu_3 \tilde{w}_{xx} - a_3 \tilde{w} \end{aligned} \tag{11}$$

then the state E_1 is stable if and only if $-a_2 + \frac{\omega_1}{1+D_1} < 0$.

C-type Nonlocality. The C-type nonlocal problems yields equations of $\tilde{u}$

$$\tilde{u}_t = \tilde{u}_{xx} - \hat{\varphi}\tilde{u} - \frac{\omega_0}{1+D_0}\tilde{v} \tag{12}$$

The only instability of this system can only comes from the equation of $\tilde{u}$, we can assume that $\tilde{u} \sim e^{ikx}$, and consider in long time where $\tilde{v}, \tilde{\omega} = 0$. This can give us condition for stability

$$-k^2 - \hat{\varphi} < 0 \quad \Rightarrow \quad \frac{\sin(\delta k)}{\delta k} > -k^2 \tag{13}$$

Introducing a new variable $\beta = \delta k$ so the equation of the critical condition for stability is

$$\frac{\sin(\beta)}{\beta} = -\frac{\beta^2}{\delta^2}$$

So the critical value of δ is

$$\delta_c = \sqrt{\frac{-\beta_c^3}{\sin(\beta_c)}} \tag{14}$$

We can find out the value of β to minimize that of δ_c, and we find that approximately $\beta = 4.078$ will give the minimal value for δ, which gives $\delta_c \approx 9.18$.

3.3 Stability of E_2 for the Nonlocal Problems
We consider the stability of E_2 state firstly in local case. We assume solutions of the form

$$u \sim \bar{u} + \epsilon\tilde{u}e^{\lambda t}e^{ikx}, \quad v \sim \bar{v} + \epsilon\tilde{v}e^{\lambda t}e^{ikx}, \quad w \sim \epsilon\tilde{w}e^{\lambda t}e^{ikx}$$

and we can have the system

$$\begin{aligned} \lambda\tilde{u} &= -k^2\tilde{u} + \tilde{u} - 2\bar{u}\tilde{u} - \omega_0 \frac{\bar{u}\tilde{v} + \bar{v}\tilde{u}}{\bar{u} + D_0} \\ \lambda\tilde{v} &= -\mu_2 k^2 \tilde{v} - a_2\tilde{v} - \omega_1 \frac{\bar{u}\tilde{v} + \bar{v}\tilde{u}}{\bar{u} + D_1} - \omega_2 \frac{\bar{v}\tilde{w}}{\bar{v} + D_2} \\ \lambda\tilde{w} &= -\mu_3 k^2 \tilde{w} - a_3\tilde{w} + \omega_3 \frac{\bar{v}\tilde{w}}{\bar{v} + D_3} \end{aligned} \tag{15}$$

In the local case, we have the Jacobian

$$J_2 = \begin{bmatrix} j_1 & -\omega_0 \frac{\bar{u}}{\bar{u}+D_0} & 0 \\ \omega_1 \frac{\bar{v}}{\bar{u}+D_1} & j_2 & -\omega_2 \frac{\bar{v}}{\bar{v}+D_0} \\ 0 & 0 & j_3 \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} j_1 &= -k^2 + 1 - 2\bar{u} + \omega_0 \frac{\bar{v}}{\bar{u}+D_0} \\ j_2 &= -\mu_2 k^2 - a_2 - \omega_1 \frac{\bar{u}}{\bar{u}+D_1} \\ j_3 &= -\mu_3 k^2 - a_3 + \omega_3 \frac{\bar{v}}{\bar{v}+D_3} \end{aligned} \tag{17}$$

First two eigenvalues of J_2 are the same with the ω-free problem, and the third one is $\lambda_3 = -\mu_3 k^2 - a_3 + \omega_3 \frac{\bar{v}}{\bar{v}+D_3}$. This system is stable if and only if the real part of all eigenvalues of J_2 is negative, which gives us

$$\begin{aligned} &[(j_1 - \lambda)(j_2 - \lambda) + \omega_0 \omega_1 \bar{u}\bar{v}/(\bar{u}+D_0)(\bar{u}+D_1)] = 0 \\ &\Rightarrow \lambda^2 - (j_1 + j_2)\lambda + j_1 j_2 + \omega_0 \omega_1 \frac{\bar{u}\bar{v}}{(\bar{u}+D_0)(\bar{u}+D_0)} = 0 \end{aligned} \tag{18}$$

So, if we want to make sure that the real part of all eigenvalues is negative, we have three conditions

$$\begin{aligned} \lambda_1 + \lambda_2 &= -(j_1 + j_2) < 0 \\ \lambda_1 \lambda_2 &= j_1 j_2 + \omega_0 \omega_1 \frac{\bar{u}\bar{v}}{(\bar{u}+D_0)(\bar{u}+D_0)} > 0 \\ \lambda_3 &= -\mu_3 k^2 - a_3 + \omega_3 \frac{\bar{v}}{\bar{v}+D_3} < 0 \end{aligned} \tag{19}$$

The state E_2 in local case is stable if and only if all three conditions of (19) are satisfied.

C-type Nonlocality. When we consider C-type nonlocality, only ω-free problem is affected by the nonlocality, and the third eigenvalue will no change. So we only need to consider two species u and v, and the Jacobian becomes

$$\tilde{J}_2 = \begin{bmatrix} -k^2 + (1-\bar{u}) - \bar{u}\hat{\varphi} - \omega_0 \frac{\bar{v}}{\bar{u}+D_0} & -\omega_0 \frac{\bar{u}}{\bar{u}+D_0} \\ \omega_1 \frac{\bar{v}}{\bar{u}+D_1} & j_2 \end{bmatrix} \tag{20}$$

Stability requires that the real part of all eigenvalues of $\tilde{J}_2$ are negative, so the determinant of $\tilde{J}_2$ should be positive

$$\begin{aligned} &\left(k^2 - (1-\bar{u}) + \bar{u}\hat{\varphi} + \omega_0 \frac{\bar{v}}{\bar{u}+D_0}\right) j_2 + \frac{\omega_0 \omega_1 \bar{u}\bar{v}}{(\bar{u}+D_0)(\bar{u}+D_0)} > 0 \\ \Rightarrow &\frac{\sin(\delta k)}{\delta k} < \frac{1}{\bar{u}} \left[(1-\bar{u}) - k^2 - \omega_0 \frac{\bar{v}}{\bar{u}+D_0} - \frac{\omega_0 \omega_1 \bar{u}\bar{v}}{(\bar{u}+D_0)(\bar{u}+D_0)} \right] \end{aligned} \tag{21}$$

this is the condition for stability of E_2 in nonlocal case.

3.3 Stability of E_3 for the Nonlocal Problems
We consider the stability of the E_3 state by seeking the solutions of the form

$$u \sim u_c + \epsilon \tilde{u} e^{\lambda t} e^{ikx}, \quad v \sim v_c + \epsilon \tilde{v} e^{\lambda t} e^{ikx}, \quad w \sim w_c + \epsilon \tilde{w} e^{\lambda t} e^{ikx}$$

and we can obtain the Jacobian

$$J_3 = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{22}$$

where

$$\begin{aligned}
&j_{11} = -k^2 + 1 - 2u_c - \omega_0 \frac{v_c}{u_c + D_0}, \quad j_{12} = -\omega_0 \frac{u_c}{u_c + D_0} \\
&j_{21} = \omega_1 \frac{v_c}{u_c + D_1}, \quad j_{22} = -\mu_2 k^2 - a_2 + \omega_1 \frac{u_c}{u_c + D_1} - \omega_2 \frac{w_c}{v_c + D_2}, \quad j_{23} = -\omega_2 \frac{v_c}{v_c + D_2} \\
&j_{32} = \omega_3 \frac{w_c}{v_c + D_3}, \quad j_{33} = -\mu_3 k^2 - a_3 + \omega_3 \frac{v_c}{v_c + D_3}
\end{aligned}$$

We know that the eigenvalues λ of J_4 satisfy following equations

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \tag{23}$$

where

$$\begin{aligned}
B_1 &= -j_{11} - j_{22} - j_{33} \\
B_2 &= j_{22} j_{33} + j_{11} j_{22} + j_{11} j_{33} - j_{12} j_{21} - j_{23} j_{32} \\
B_3 &= j_{12} j_{21} j_{33} + j_{11} j_{23} j_{32} - j_{11} j_{22} j_{33}
\end{aligned}$$

According to Routh-Huiwitz criterion for stability, we know that E_3 state is stable if and only if

$$B_1 > 0, \quad B_3 > 0, \quad B_1 B_2 - B_3 > 0 \tag{24}$$

C-type Nonlocality. The Jacobian of the system will be the same as (22), except the first one

$$j_{11} = -k^2 + 1 - (1 + \hat{\varphi}) u_c - \omega_0 \frac{v_c}{u_c + D_0} \tag{25}$$