1.Holling Type I Model
We consider model

$$\begin{aligned} u_t &= u_{xx} + u(a_1 - b_2 u - \omega_0 v) \\ v_t &= \mu_2 v_{xx} + v(-a_2 + \omega_1 u - \omega_2 \omega) \\ w_t &= \mu_3 v_{xx} + w(-a_3 + \omega_3 v) \end{aligned} \tag{1}$$

Now we consider uniform equilibrium state of (1). The zero-state equilibrium state is $E_0(u,v,w) = (0,0,0)$, the one-state equilibrium state is $E_1(u,v,w) = (1,0,0)$, and two-state equilibrium state is $E_2(\bar{u},\bar{v},0)$ exists where we can know

$$\bar{u} = \frac{a_2}{\omega_1}, \quad \bar{v} = \frac{a_1 - b_2\bar{u}}{\omega_0} \tag{2}$$

and it is only physical when $a_2 < \omega_1$ and $a_1 - b_2 < \omega_0$. Another three-state equilibrium state is $E_2(u_c, v_c, w_c)$ exists, where

$$u_c = \frac{a_1 - \omega_0 v_c}{b_2}, \quad v_c = \frac{\omega_3}{a_3}, \quad w_c = \frac{-a_2 + \omega_1 u_c}{\omega_2} \tag{3}$$

and it is only physical when $\omega_3 < a_3$, $a_1 - \omega_0 < b_2$ and $\omega_1 < \omega_2 + a_2$.

2.Nonlocality
We consider two specific manifestations of nonlocality appropriate for three-species model 1. We first use a stepfunction kernel

$$\varphi(x) = \begin{cases} \dfrac{1}{2\delta}, & |x| < \delta \\ 0, & |x| > \delta \end{cases} \tag{4}$$

and we can know the Fourier transform of this kernel

$$\hat{\varphi}(k) = \frac{\sin(\delta k)}{\delta k} \tag{5}$$

C-type Nonlocality. Species u competes with itself with nonlocality, and we can get the model

$$\begin{aligned} u_t &= u_{xx} + u(a_1 - b_2(\varphi * u) - \omega_0 v) \\ v_t &= \mu_2 v_{xx} + v(-a_2 + \omega_1 u - \omega_2 \omega) \\ w_t &= \mu_3 v_{xx} + w(-a_3 + \omega_3 v) \end{aligned} \tag{6}$$

where

$$\varphi * u = \int_{-\infty}^{\infty} \varphi(y - x) u(y) dy$$

P_ω-type Nonlocality. Species w preys on v with nonlocality, then we can get the model

$$\begin{aligned} u_t &= u_{xx} + u(a_1 - b_2 u - \omega_0 v) \\ v_t &= \mu_2 v_{xx} + v(-a_2 + \omega_1 u - \omega_2(\varphi * w)) \\ w_t &= \mu_3 w_{xx} - a_3 w + \omega_3 v(\varphi * w) \end{aligned} \tag{7}$$

3.Stability

3.1 Stability of E_1 for the ODE Model

We discuss the stability of the E_1 state for the ODE model with no spatial dependence. We consider the perturbation of the full (u, v, w) system, by considering u tends to 1 and v, w are small. We write this as

$$u \sim 1 + \epsilon\tilde{u} + \mathcal{O}(\epsilon^2), \quad v \sim \epsilon\tilde{v} + \mathcal{O}(\epsilon^2), \quad w \sim \epsilon\tilde{w} + \mathcal{O}(\epsilon^2)$$

where $\epsilon \ll 1$. We substitute these into original system and keep only therms that are linear in ϵ. This gives nonlinear equations

$$\begin{aligned} \tilde{u}_t &= (a_1 - 2b_2)\tilde{u} - \omega_0\tilde{v} \\ \tilde{v}_t &= -a_2\tilde{v} + \omega_1\tilde{v} \\ \tilde{\omega}_t &= -a_3\tilde{\omega} \end{aligned} \tag{8}$$

the last two equations decouple from the first one, thus can be considered first. And we can have long time behavior of v and w

$$v \sim e^{(\omega_1 - a_1)t}, \quad w \sim e^{-a_3 t} \tag{9}$$

So the E_1 state is stable if and only if

$$\omega_1 - a_1 < 0, \quad a_1 - 2b_2 < 0, \quad -a_3 < 0 \tag{10}$$

3.2 Stability of E_1 for the Nonlocal Problems

We consider perturbations in form of

$$u \sim 1 + \epsilon\tilde{u}(x,t), \quad v \sim \epsilon\tilde{v}(x,t), \quad \omega \sim \epsilon\tilde{\omega}(x,t)$$

where $\epsilon \ll 1$. For the local problem, this gives

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + (a_1 - 2b_2)\tilde{u} - \omega_0\tilde{v} \\ \tilde{v}_t &= \tilde{v}_{xx} - a_2\tilde{u} + \omega_1\tilde{v} \\ \tilde{w}_t &= \tilde{w}_{xx} - a_3\tilde{w} \end{aligned} \tag{11}$$

C-type Nonlocality. The C-type nonlocal problems yields equations of $\tilde{u}$

$$\tilde{u}_t = \tilde{u}_{xx} + (a_1 - 2b_2)\hat{\varphi}\tilde{u} - \omega_0\tilde{v} \tag{12}$$

The only instability of this system can only comes from the equation of $\tilde{u}$, we can assume that $\tilde{u} \sim e^{ikx}$, and consider in long time where $\tilde{v}, \tilde{\omega} = 0$. This can give us condition for stability

$$-k^2 + (a_1 - 2b_2)\hat{\varphi} < 0 \quad \Rightarrow \quad (a_1 - 2b_2)\frac{\sin(\delta k)}{\delta k} < k^2 \tag{13}$$

Introducing a new variable $\beta = \delta k$ so the equation of the critical condition for stability is

$$(a_1 - 2b_2)\frac{\sin(\beta)}{\beta} = -\frac{\beta^2}{\delta^2}$$

So the critical value of δ is

$$\delta_c = \sqrt{(a_1 - 2b_2)\frac{-\beta_c^3}{\sin(\beta_c)}} \tag{14}$$

We can find out the value of β to minimize that of δ_c, and we find that approximately $\beta = \pm 4.078$ will give the minimal value for δ, which gives $\delta_c \approx \sqrt{\pm 9.18(a_1 - 2b_2)}$.

3.3 Stability of E_3 for the Nonlocal Problems
We consider the stability of E_2 stat. We assume solutions of the form

$$u \sim \bar{u} + \epsilon\tilde{u}e^{\lambda t}e^{ikx}, \quad v \sim \bar{v} + \epsilon\tilde{v}e^{\lambda t}e^{ikx}, \quad w \sim \epsilon\tilde{w}e^{\lambda t}e^{ikx}$$

In the local case, we have the Jacobian

$$J_3 = \begin{bmatrix} j_1 & -\omega_0\bar{u} & 0 \\ \omega_1\bar{v} & j_2 & -\omega_2\bar{v} \\ 0 & 0 & -j_3 \end{bmatrix} \tag{15}$$

where

$$\begin{aligned} j_1 &= -k^2 + a_1 - 2b_2\bar{u} + \omega_0\bar{v} \\ &= -k^2 - b_2\bar{u} \\ j_2 &= -\mu_2 k^2 - a_1 + a_2 \\ j_3 &= -\mu_3 k^2 - a_3 + \omega_3\bar{v} \end{aligned} \tag{16}$$

with $\bar{v} = \frac{a_1 - b_2\bar{u}}{\omega_0}$. First two eigenvalues of J_3 are the same with the ω-free problem, and the third one is $\lambda_3 = -\mu_3 k^2 - a_3 + \omega_3\bar{v}$.

C-type Nonlocality. When we consider C-type nonlocality, only ω-free problem is affected by the nonlocality, and the third eigenvalue will no change. So we only need to consider two species u and v, and the Jacobian becomes

$$\tilde{J}_3 = \begin{bmatrix} -k^2 + a_1 - b_2\bar{u} - b_2\bar{u}\hat{\varphi} - \omega_0\bar{v} & -\omega_0\bar{u} \\ \omega_1\bar{v} & -\mu_2 k^2 - a_1 + a_2 \end{bmatrix} \tag{17}$$

Stability requires that the determinant of $\tilde{J}_3$ be positive. The first term of this matrix can be written as $-k^2 - b_2\bar{u}\hat{\varphi}$ with $\bar{v} = \frac{a_1 - b_2\bar{u}}{\omega_0}$. This gives

$$\frac{\sin(\delta k)}{\delta k} > \frac{a_2}{b_2\bar{u}}\left(\frac{-a_1 + b_0\bar{u}}{\mu_2 k^2 + a_1 - a_2}\right) - \frac{k^2}{b_2\bar{u}} \tag{18}$$

We rewrite (18) by using $\beta = \delta k$ and $D = 1/\delta^2$ to obtain critical condition

$$\begin{aligned} F(\beta, D) \equiv &\mu_2 \beta^4 D^2 + \left[b_2 \mu_2 \bar{u} \frac{\sin(\beta)}{\beta} - (a_2 - a_1) \right] \beta^2 D \\ &- b_2(a_1 - a_2)\bar{u} \frac{\sin(\beta)}{\beta} + a_2(a_1 - b_0 \bar{u}) = 0 \end{aligned} \tag{19}$$

3.3 Stability of E_2 for the Nonlocal Problems

We consider the stability of the E_3 state by seeking the solutions of the form

$$u \sim u_c + \epsilon \tilde{u} e^{\lambda t} e^{ikx}, \quad v \sim v_c + \epsilon \tilde{v} e^{\lambda t} e^{ikx}, \quad w \sim w_c + \epsilon \tilde{w} e^{\lambda t} e^{ikx}$$

and we can obtain the Jacobian

$$J_4 = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix} \tag{20}$$

where

$$\begin{aligned} &j_{11} = -k^2 + a_1 - 2b_2 u_c - \omega_0 v_c, \quad j_{12} = -\omega_0 u_c \\ &j_{21} = \omega_1 v_c, \quad j_{22} = -\mu_2 k^2 - a_2 + \omega_1 u_c - \omega_2 w_c, \quad j_{23} = -\omega_2 v_c \\ &j_{32} = \omega_3 w_c, \quad j_{33} = -\mu_3 k^2 - a_3 + \omega_3 v_c \end{aligned}$$

We know that the eigenvalues λ of J_4 satisfy following equations

$$\lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0 \tag{21}$$

where

$$\begin{aligned} B_1 &= -j_{11} - j_{22} - j_{33} \\ B_2 &= j_{22} j_{33} + j_{11} j_{22} + j_{11} j_{33} - j_{12} j_{21} - j_{23} j_{32} \\ B_3 &= j_{12} j_{21} j_{33} + j_{11} j_{23} j_{32} - j_{11} j_{22} j_{33} \end{aligned}$$

According to Routh-Huiwitz criterion for stability, we know that E_3 state is stable if and only if

$$B_1 > 0, \quad B_3 > 0, \quad B_1 B_2 - B_3 > 0 \tag{22}$$

C-type Nonlocality. The Jacobian of the system will be the same as (20), except the first one

$$j_{11} = -k^2 + a_1 - b_2(1 + \hat{\varphi}) u_c - \omega_0 v_c \tag{23}$$

We can use the same methods in 3.3 to find critical values of $\beta = \delta k$ and $D = 1/\delta^2$.

*P_ω-type Nonlocality.*Now for this problem, the Jacobian is the same as (20) except that

$$\begin{aligned} j_{22} &= -\mu_2 k^2 - a_2 + \omega_1 u_c - \omega_2 \hat{\varphi} w_c \\ j_{32} &= \omega_3 \hat{\varphi} w_c \end{aligned} \tag{24}$$