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Analysis of Bifurcation of Spruce Budworm Model with Time Delay and Diffusion by Several Numerical Methods

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Abstract

The theory of differential equation holds great value in the physics, chemistry, population dynamics, space science and other fields, it obtained many scholars' attention and interest. Because of the delay, scholars introduced time delay and diffusion into the differential system to obtain better models to simulate real problems. But the number of equations could be solved is extremely limited. Thus scholars used various numerical methods, like Runge-Kutta method, in order to solve these equations approximately. Bifurcation is another important topic in the research of differential equations and dynamic systems, whose research focus is unstable systems with one or more variables. In this paper, the spruce budworm model and its condition of bifurcation are analyzed.

Spruce budworm has significant influence to the ecosystem, so since 70s, many schloars studied this specie and proposed many models. After decades, spruce budworm model has been greatly improved. Adding the terms of time delay and diffusion, the spruce budworm model can simulate its actual reproduction with good accuracy. Thus we have great interest in its bifurcation.

In this paper, forward Euler scheme is used to discretize the spruce budworm model with time delay and diffusion. Then, its condition of local stability of positive constant state and Neimark-Sacker bifurcation are proposed by analyzing the eigenvalues of Jacobi matrix of discrete system at internal positive constant state. And numerical simulation are constructed to verify the conditions we obtained.

Keywords— partial differential equations, forward Euler scheme, Neimark-Sacker bifurcation, spruce budworm model

Contents

Chapter 1

Introduction

1.1 Backgrounds and the Purpose of Study

The theory of differential equation is always significantly important in the physics, chemistry, population dynamics, space science and other industry fields. Around 1940s, scholars found out that not only the current situation would affect the moving of objects, but also the past one. Thus, more and more scholars pay attentions to differential equations with time delay. Differential equations with time delay are also called delay differential equations(DDEs). With the introduction of term of time delay, it becomes harder to solve it in both explicitly and numerical methods. However, the term of tie delay cannot be neglected, which makes the research of numerical solutions of delay differential equations more and more important.

The qualitative theory of differential equations is new breakthrough in this field, since scholars are more interested in the dynamical behavior of the system. And bifurcation is massively studied by scholars in the research of dynamical system, which indicates the change of topological structure brought by the changes of variables.

As for delay differential equations, scholars find that only a small part of it can be solved explicitly. Thus, besides qualitative theory, the theory of numerical methods has been developed. In 70s, scholars mainly focused on the construction of numerical solutions and stability and convergence of numerical methods. However, more important issue is that if the numerical methods preserve dynamical structure of the original systems when solve for numerical solutions of DDEs.

Besides, the research of dynamical behavior of the discrete systems enriched the study of DDE systems. When adding another term of diffusion, the depth and breadth of research of the bifurcation of DDEs cannot compete those of ODEs, thus in this paper DDEs with diffusion are analyzed. When using numerical methods to solve DDEs, one can obtain the discrete system of the original one, thereby can use the theory of finite dimensional dynamical systems to analyze dynamical behavior of discrete systems [13].

So summing up all we mentioned above, when analyzing discrete system of delay partial differential equation, whether or not the numerical methods can preserve the original structure is worth research. Thus, our purpose in this paper is using numerical methods to discretize a certain type of delay partial differential equation and then analyzing its bifurcation.

1.2 Recent Research

In this section, recent researches of numerical methods for delay differential equations, delay partial differential equations and bifurcation will be introduced.

1.2.1 Numerical Methods of Differential Equations with Time Delay

In ODEs, only a small amount can be solved explicitly, let alone delay differential equations. With the development of theory of DDEs, numerical methods for DDEs also have developed greatly. In 1984, Dekker[7] discussed the stability of Runge-Kutta methods for delay rigid nonlinear differential equations. In 1988, Bellen[1] discussed the stability of single step methods for DDEs of neutral type.

Numerical methods that can preserve some dynamical behavior of the original systems are massively studies in recent years. Hout[3] proved that, by proper numerical methods to discretize DDE, the discrete system has an invariant curve and the order of convergence to the periodic orbit is the same as the convergence order of the numerical methods.

1.2.2 Numerical Methods of Partial Differential Equations with Time Delay

Recent years, many scholars studied the numerical methods for delay partial differential equations. In 2001, Zubik-Kowal and Barbara[14] discussed the stability of multiple step method and Runge-Kutta method for linear parabolic equation with time delay

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t) + \mu u(x,t-\tau), t > 0, x \in \Omega \\ u(x,t) = u_0(x,t), t \in [-\tau, 0], x \in \Omega \\ u(x,t) = g(x,t), t > 0, x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a linear elliptic differential operator, $\tau > 0, \mu$ are given constants, Ω is a bounded domain in $\mathcal{R}^d$, u_0 and g_0 are initial condition and boundary condition respectively. By doing spatial semi-discrete for (1.1), he obtained

$$\begin{cases} U'(t) = AU(t) + \mu U(t-\tau) + r(t), t > 0 \\ U(t) = U_0(t), t \in [-\tau, 0] \end{cases} \tag{1.2}$$

where A is a $N \times N$ matrix, $U(t)$ is approximation to explicit solution $u(t)$, $r(t)$ is a functions depending on g and matrix A, which is also N dimensional. Using numerical methods solve (1.2) and we have

$$z_{n+1} = \varphi_0(hA)z_n + \cdots + \varphi_m(hA)z_{n-m} + hr_n, n \geq 0 \tag{1.3}$$

where $\varphi_0, \cdots, \varphi_m$ are constants, h is the step length, m is a positive integer such that $\tau \leq hm$, and z_n is approximation to $U(t)$ at grid points. Then equation (1.3) can be written as

$$Z_{n+1} = CZ_n + R_n, n \geq 0 \tag{1.4}$$

thus, the stability of system (1.2) can be obtained by analyzing (1.4).

In 1995, Higham[2] studied the numerical solutions of the equation, which involves diffusion and a nonlinear, delayed, reaction term. In this paper, the Laplace operator is discretized by central difference operator to obtain standard matrix A, and forward Euler scheme is used to semi-discretize the system in order to obtain the condition of stability. In 2008, Tian[9] studied following delay parabolic differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = k\dfrac{\partial^2 u(x,t)}{\partial x^2} + r\dfrac{\partial^2 u(x,t-\tau)}{\partial x^2}, t > 0, 0 \leq x \leq \pi \\ U(0,t) = U(\pi,t), t \geq -tau \\ u(x,t) = \Phi(x,t), -\tau \leq t \leq 0, 0 \leq x \leq \pi \end{cases} \tag{1.5}$$

and used forward and backward Euler scheme, and Crank-Nicolson difference to obtain the necessary and sufficient conditions for asymptotic stability of the system (1.5).

1.2.3 Theory of Bifurcation

The theory of bifurcation is an important topic in differential equations and dynamical system, which focuses on the research of unstable systems with one or more variables. The phenomenon of bifurcation is that the topological structure of the systems changes when variables occur small changes around certain constants. In bifurcation, Holp bifurcation is also important, where systems occur periodic solutions around stable points when the stability of equilibrium changes. Scholars verified the existence of local Holp bifurcation and the condition for stability of periodic solution of DDEs[6]. In conclusion, the term of delay will make positive solutions of the system unstable and the Holp bifurcation occurs in most cases.

Chapter 2

Discrete Bifurcation of the Spruce Budworm Model

2.1 Introduction of the Model

Spruce budworm is one kind of pest in the north American forest. Normally, this pest is always controlled by predators(normally birds). However, the rapid spread and reproduction of this budworm in recent decades aroused the attention of scientists. And since birds do not only feed on this budworm, the population of birds is independent to that of spruce budworm in some extent.

In 1978, Ludwig[5] proposed that the local density of budworm without spatial dependence should satisfy

$$u_t = ru\left(1 - \frac{u}{K}\right) - \frac{Bu^2}{A^2 + u^2} \tag{2.1}$$

where the first term in the right is regression term, proportional to the are of forest where the budworm is located, the second term is the number of the budworm as prey, and B is the saturated amount of prey, since the population of budworm and the hunting ability of predators are limited. In 1979, Ludwig[4] also proposed spruce budworm model with diffusion

$$u_t = d\Delta u + ru\left(1 - \frac{u}{K}\right) - \frac{Bu^2}{A^2 + u^2} \tag{2.2}$$

In 2003, Wang and Yeh[11] studied invariant point problem of this equation. Thus, this equation is widely accepted. However, the population of this budworm is not determined by the predators and the environmental carrying capacity, but also other factors. Thus, Vaidya and Wu[10] proposed following spruce budworm model with time delay

$$u_t = -Du(t) - \frac{\beta u^2(t)}{\gamma^2 + u^2(t)} + q_1 e^{-d\tau} u(t-\tau) e^{-\alpha_1 u(t-\tau)} \tag{2.3}$$

and they implemented numerical experiments and showed that this equation simulates the population of spruce budworm in the area of Green River, New Brunswick, Canada with great accuracy. In 2017, Wei and Xu[12] discussed the spruce budworm model with delay and diffusion under Dirichlet boundary condition

$$u_t = d_1 \Delta u(x,t) - Du(t) - \frac{\beta u^2(t)}{\gamma^2 + u^2(t)} + q_1 e^{-\tilde{d}\tau} u(t-\tau) e^{-\alpha_1 u(t-\tau)} \tag{2.4}$$

and analyzed the stability of its fixed points and the existence of the bifurcation around its positive fixed points. In this equation, d_1 is coefficient of diffusion, D is the average mortality of mature budworm, $\tilde{d}$ is the average mortality of immature budworm, β is the predation rate of predators, γ is population of budworm when the predation rate reaches half of its peak, and τ is the time delay of budworm's maturation. At last, $b(u) = q_1 u e^{-\alpha_1 u}, (q_1, \alpha_1 > 0)$ is the function of reproduction of spruce budworm.

2.2 Analysis of Bifurcation of Numerical Discrete System

As for (2.4), this chapter consider it under Dirichlet boundary condition. We can have nondimensional system of (2.4) with following transform

$$\hat{u}(x,\hat{t}) = \frac{1}{\gamma} u(x,t), \quad \hat{t} = \frac{\beta t}{\gamma}, \quad \hat{\tau} = \frac{\beta \tau}{\gamma} \tag{2.5}$$

and still write $\hat{u}$ as u

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - r_1 u - \dfrac{u^2}{1+u^2} + be^{-r_2\tau}u_\tau e^{-\alpha u_\tau}, t > 0, x \in \Omega, t > 0 \\ U(0,t) = U(\pi,t) = 0, t > 0 \\ u(x,t) = \eta(x,t) \geq 0, x \in \Omega, t \in [-\tau, 0] \end{cases} \tag{2.6}$$

where

$$d = \frac{\gamma d_1}{\beta}, \quad r_1 = \frac{\gamma D}{\beta}, \quad b = \frac{\gamma q_1}{\beta}, \quad r_2 = \frac{\gamma \tilde{d}}{\beta}, \quad \alpha = \gamma\alpha_1 \tag{2.7}$$

also, $\Omega = (0,\pi)$ and $\eta(x,t)$ is Hölder continuous with $\eta(x,0) \in C^1(\Omega)$.

There are two equilibriums in this system $u = 0$ and u_0, where u_0 satisfies

$$r_1 + \frac{u}{1+u^2} = be^{-r_2\tau}e^{-\alpha u} \tag{2.8}$$

when $be^{-r_2\tau} \leq r_1$, then $u = 0$ is the only of system(2.6), and when $be^{-r_2\tau} > r_1$ and $\alpha > 1$, the system has another positive equilibrium u_0. Now considering reaction diffusion equation

$$\frac{\partial u}{\partial t} = d\Delta u - r_1 u - \frac{u^2}{1+u^2} + be^{-r_2\tau}u_\tau e^{-\alpha u_\tau} \tag{2.9}$$

and substitute $u(x,t) = u(x,t\tau)$. Again writing $t\tau$ as t, we have

$$\frac{\partial u(x,t)}{\partial t} = d\tau\Delta u - r_1\tau u - \frac{\tau u^2}{1+u^2} + \tau be^{-r_2\tau}u(x,t-1)e^{-\alpha u(x,t-1)}$$

Now we transform u_0 to 0 with $\hat{u} = u - u_0$, and write $\hat{u}$ as u, we have

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} = d\tau\Delta u(x,t) - \tau \Bigg[& r_1(u+u_0) - \frac{(u+u_0)^2}{1+(u+u_0)^2} \\ & + be^{-r_2\tau}(u(x,t-1)+u_0)e^{-\alpha(u(x,t-1)+u_0)} \Bigg] \end{aligned} \tag{2.10}$$

In order to discretize the system, we set k as time step length and h as spatial step length, and $mk = 1$, where m is integer such that $m \geq 1$. Then we set grid points

$$\begin{aligned} t_n &= nk, n = -m, -m+1, \cdots \\ x_i &= ih, 1 = 0, 1, \cdots, N \end{aligned} \tag{2.11}$$

where $h = \pi/N$, N is a given integer. Using forward Euler scheme to discretize (2.10), we have

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{k} = d\tau \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \\ \qquad\qquad - \tau \left[r_1(u_i^n + u_0) - \dfrac{(u_i^n + u_0)^2}{1 + (u_i^n + u_0)^2} + be^{-r_2\tau}(u_i^{n-m} + u_0)e^{-\alpha(u_i^{n-m} + u_0)} \right] \\ u_0^n = u_N^n = 0, n = 1, 2, \cdots \\ u_i^n = \eta(x_i, t_n), i = 0, 1, 2, \cdots, N, n = -m, -m+1, \cdots, 0 \end{cases} \tag{2.12}$$

then we apply Taylor expansion and we can get linear terms of above system

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{k} = d\tau \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \\ \qquad\qquad - \tau \left[\left(r_1 + \dfrac{2u_0}{(1 + u_0^2)^2} \right) u_i^n - \left(r_1 + \dfrac{2u_0}{1 + u_0^2} \right) (1 - \alpha u_0) u_i^{n-m} \right] \\ u_0^n = u_N^n = 0, n = 1, 2, \cdots \\ u_i^n = \eta(x_i, t_n), i = 0, 1, 2, \cdots, N, n = -m, -m+1, \cdots, 0 \end{cases} \tag{2.13}$$

and system (2.12) can be written as

$$U^{n+1} = AU^n + CU^{n-m} \tag{2.14}$$

where $l = k/h^2$ and

$$\begin{cases} A = \left[1 - 2d\tau l - \tau k \left(r_1 + \dfrac{2u_0}{(1 + u_0^2)^2} \right) \right] I + d\tau l S \\ C = \tau k \left(r_1 + \dfrac{2u_0}{1 + u_0^2} \right) (1 - \alpha u_0) I \end{cases} \tag{2.15}$$

also, in this equation I is identity matrix and

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.16}$$

In order to study the trivial solutions of the system (2.13), we can write (2.14) as an augmented system of first order difference system

$$\Lambda^{n+1} = H\Lambda^{n} \tag{2.17}$$

where $\Lambda^n = (U^n, U^{n-1}, \cdots, U^{n-m})^T$, and

$$H = \begin{pmatrix} A & 0 & \cdots & 0 & -C \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \tag{2.18}$$

Now we can have have the characteristic polynomial

$$\begin{aligned} \det(\lambda I - H) = \Pi_{\lambda_j \in \delta(S)} \Bigg\{ & \lambda^{m+1} - \left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l \lambda_j\right]\lambda^m \\ & + \tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)\Bigg\} \end{aligned} \tag{2.19}$$

where λ_j is jth eigenvalues of matrix S, and we can have $\lambda_j = 2\cos(jh), j = 1, 2, \cdots, N-1$. So we need to consider equation

$$\lambda^{m+1} - \left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l \lambda_j\right]\lambda^m + \tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0) = 0 \tag{2.20}$$

Lemma 2.2.1 *For sufficiently small $\tau > 0$, when $u_0 \leq 1/\alpha$, all roots of equation (2.20) is smaller than 1.*

Proof As for $\tau = 0$, equation (2.20) can be written as

$$\lambda^{m+1} - \lambda^m = 0$$

and this equation has m roots $\lambda_1 = 0$ and one simple root $\lambda_2 = 1$.

Now consider $\lambda(\tau)$ such that $\lambda(0) = 1$, obviously,

$$\frac{d|\lambda|^2}{d\tau} = \lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}$$

we take derivative of both sides of equation (2.20) in term of τ, and we have

$$\frac{d\lambda}{d\tau} = \frac{\left[1 - 2dl - k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + dl\lambda_j\right]\lambda^m - k\left(r_1 + \frac{2u_0}{1+u_0^2}\right)(1-\alpha u_0)}{(M+1)\lambda^m - m\left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\lambda_j\right]\lambda^{m-1}}$$

and further more we can have

$$\left.\frac{d|\lambda|^2}{d\tau}\right|_{\lambda=1,\tau=0} = 2\left\{dl(\lambda_i - 2) - k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) - k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)\right\}$$

where $r_1 = \frac{\gamma D}{\beta} > 0(\gamma > 0, D > 0, \beta > 0)$ and $|\lambda_j| \leq 1$, thus we can have

$$dl(\lambda_j - 2) - k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) < 0$$

which left us $1 - \alpha u_0$ to discuss. When $1 - \alpha u_0 \geq 0$, then

$$\left.\frac{d|\lambda|^2}{d\tau}\right|_{\lambda=1,\tau=0} < 0$$

So as for sufficiently small $\tau > 0$, when $u_0 < 1/\alpha$, all roots of equation (2.20) are smaller than 1. And the proof is completed.

As for (2.8), it is obvious that that if $be^{-r_2\tau} \leq r_1$, then

$$r_1 + \frac{u}{1+u^2} \geq r_1 \geq be^{-r_2\tau} \geq be^{-r_2\tau}e^{-\alpha u}$$

for $u \in (0, \infty)$. Thus, when $be^{-r_2\tau} \leq r_1$, equation(2.6) has no positive equilibrium. Now we consider the case where $e^{-r_2\tau} > r_1$, which is $b > r_1$, then we have

$$0 \leq \tau < \tau_{max} := \frac{1}{r_2}\ln\frac{b}{r_1}$$

and assuming $\alpha > 1$, equation(2.6) only has one positive equilibrium. Now we multiply $1 + u^2$ to both sides of (2.8), and we denote

$$g_1(u) = r_1(1+u^2) + u, \quad g_2(u,\tau) = be^{-r_2\tau}(1+u^2)e^{-\alpha u} \tag{2.21}$$

where $g_1(0) = r_1, g_2(0,\tau) = e^{-r_2\tau}$. So u_0 is a positive root of the equation (2.8) if and only if u_0 is the positive root of equaiton $g_1(u) = g_2(u)$. From (2.21), we can have

$$g_1'(u) = 2r_1u + 1, \quad \frac{\partial g_2(u,\tau)}{\partial u} = e^{-r_2\tau}e^{-\alpha u}(2u - \alpha - \alpha u^2) \tag{2.22}$$

and when $u \in (0,\infty)$ and $\alpha > 1$, we know $(2u - \alpha - \alpha u^2)_{max} = \frac{1}{\alpha} - \alpha < 0$. Then we can know

$$g_1'(u) > 0, \quad \frac{\partial g_2(u,\tau)}{\partial u} < 0$$

and then, with $g_1(0) < g_2(0,\tau)$ and $u \to \infty$, we have $g_1 - g_2 \to \infty$ so that $g_1(u) = g_2(u,\tau)$ has only one positive root.

From Lemma 2.2.1, we cam know that u achieve its maximum $u_{max} = u_0(0)$ at $\tau = 0$ which satisfies

$$r_1 + \frac{u}{1+u^2} = be^{-\alpha u}$$

we denote $u_0 = u_0(\tau)$, based on the fact that u_0 satisfies equation(2.8) and depends on τ.

Lemma 2.2.2 *[12] If $b > r_1$ and $\alpha > 1$, then $u_0(\tau)$ is a strictly decreasing function on [a,b).*

Proof From above, we can know that $u_0(\tau)$ satisfies the equation $g_1(u) = g_2(u,\tau)$. We prove this by contradiction and assuming that for $\tau_i \in [0,\tau_{\max}), i = 1,2,$ u_i is the only solution of equation $g_1(u) = g_2(u,\tau_i), \tau_1 < \tau_2$. For this, we want to prove that $u_1 > u_2$.

Assuming $\tau_1 \leq \tau_2$, and based on definition of g_2 in (2.22), we have

$$g_2(u_1,\tau_2) < g_2(u_1,\tau_1) = g_1(u_1) \leq_1 (u_2) = g_2(u_2,\tau_2) \leq g_2(u_1,\tau_2)$$

which is contradicted with above assumption. And the proof is completed.

Theorem 2.1 *[12] Assuming $\alpha > 1$, and denoting $b_0 = er_1 + e\frac{\alpha}{1+\alpha^2}$*
(i) If $r_1 < b \leq b_0$, $u = u_0$ is locally asymptotically stable for any $\tau \in [0,\tau_{max})$ in the system (2.6).
(ii) If $b > b_0$, then there exists a $\hat{\tau} \in (0,\tau_{max})$ such that $g_1(1/\alpha) = g_2(1/\alpha,\hat{\tau})$. And for the system (2.6), $u = u_0$ is locally asymptotically stable for any $\tau \in (\hat{\tau},\tau_{max})$.

Proof (i) Assuming $\alpha > 0$ and $b > r_1$, then we can have

$$b \leq b_0 \Longleftrightarrow g_2\left(\frac{1}{\alpha}, 0\right) \leq g_1\left(\frac{1}{\alpha}\right) \Longleftrightarrow u_{\max} \leq \frac{1}{\alpha}$$

thus, we can get $u_0 \leq u_{\max} \leq 1/\alpha$. With lemma 2.2.2, we have $\dfrac{d|\lambda|^2}{d\tau} < 0$.

(ii) Assuming $b > b_0$, then we have $u_0 > 1/\alpha$. We can assume that there exist a $u_0(\hat{\tau})$ such that $u_0(\hat{\tau}) = 1/\alpha$, then

$$r_1 + \frac{\alpha}{1+\alpha^2} = be^{-r_2\hat{\tau}}e^{-1}$$

with $b_0 = er_1 + e\frac{\alpha}{1+\alpha^2}$, we can get

$$\hat{\tau} = \frac{1}{r_2}\ln\frac{b}{b_0}$$

Also, from lemma 2.2.1 that $u_0(\tau)$ is strictly decreasing on $[0,\tau_{\max})$, then we can obtain that for any $\tau \in [\hat{\tau},\tau_{\max})$, $u_0 \leq 1/\alpha$. From lemma 2.2.1, we immediately have $\dfrac{d|\lambda|^2}{d\tau} < 0$. And the proof is completed.

Now we define some conditions for convenience:

(I1) $\tau = \hat{\tau}$ and $u_0 = \frac{1}{\alpha}$

(I2) $\tau \in (\hat{\tau},\tau_{\max})$ which is $u_0 \leq \frac{1}{\alpha}$, and $r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) < 0$

(I3) $\tau \in (\hat{\tau},\tau_{\max})$ which is $u_0 \leq \frac{1}{\alpha}$, and $r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) > 0$

(I4) $\tau \in (0,\hat{\tau})$ which is $u_0 > \frac{1}{\alpha}$, and $r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) < 0$

(I5) $\tau \in (0,\hat{\tau})$ which is $u_0 > \frac{1}{\alpha}$, and $r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) > 0$

Lemma 2.2.3 *Assuming the spacial step length h is small enough and l is a constant, then eigenvalue equation* (2.20) *has no roots on the unit circle if one of conditions (I1),(I3),(I5) is satisfied.*

Proof Based on the theory of Neimark-Sacker bifurcation, there exists bifurcation when two conjugate complex roots of the eigenvalue equation (2.20) go through the unit circle.

We assume that the equation (2.20) has two conjugate roots and we denote them as $e^{iw}, w \in (-\pi,\pi]$. Since the equation (2.20) is real polynomial equation, w only need to satisfy the condition

$w \in (0,\pi]$. Dividing both sides by λ^m and plugging e^{iw} into equation, then we have

$$e^{iw} - \left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\lambda_j\right] + \tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)e^{-imw} = 0 \quad (2.23)$$

Separating the real and imaginary parts of above equation and we have

$$\begin{cases} \cos w - \left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\lambda_j\right] + \tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)\cos mw = 0 \\ \sin w + \tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)\sin mw = 0 \end{cases}$$

with relation $\sin^2 mw + \cos^2 mw = 1$, we have

$$\begin{aligned} \cos w &= 1 + \frac{\left[2d\tau l + \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) - d\tau l\lambda_j\right]^2 - \left[\tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)\right]^2}{2\left[1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\lambda_j\right]} \\ &= 2 + \frac{P}{2Q} \end{aligned} \quad (2.24)$$

where

$$\begin{aligned} Q &= 1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\cos(jh) \\ &= 1 - 2d\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + d\tau l\left(1 - \frac{(jh)^2}{2}\right) + \mathcal{O}(h^4) \\ &= 1 - \tau l h^2\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) - dl\tau(jh)^2 + \mathcal{O}(h^4) \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} P =& \left[k\tau r_1 + \frac{2k\tau u_0}{(1+u_0^2)^2} + dl\tau(jh)^2 + \tau kbe^{-r_2\tau-\alpha u_0}(\alpha u_0 - 1)\right]\cdot \\ & \left[k\tau r_1 + \frac{2k\tau u_0}{(1+u_0^2)^2} + dl\tau(jh)^2 - \tau kbe^{-r_2\tau-\alpha u_0}(\alpha u_0 - 1)\right] + \mathcal{O}(h^6) \\ =& (\tau lh)^2\left[r_1 + \frac{2u_0}{(1+u_0^2)^2} + dj^2 + be^{-r_2\tau-\alpha u_0}(\alpha u_0 - 1)\right]\cdot \\ & \left[r_1 + \frac{2u_0}{(1+u_0^2)^2} + dj^2 - be^{-r_2\tau-\alpha u_0}(\alpha u_0 - 1)\right] + \mathcal{O}(h^6) \end{aligned} \quad (2.26)$$

When spacial step length h is small enough and l is a constant, we can make sure that $Q > 0$. Now we need to discuss the conditions of $P > 0$, since when $P > 0$ and $Q > 0$, we have $\cos w > 1$, which is impossible. Thus, in order to make sure that $P > 0$, following inequality should be satisfied

$$r_1 + \frac{2u_0}{(1+u_0^2)^2} + dj^2 - be^{-r_2\tau - \alpha u_0}(\alpha u_0 - 1) > 0$$

So either one of conditions (I1),(I3),(I5) can make sure that eigenvalue equation has no root on unit circle. And the proof is completed.

If $Q > 0$ and either of (I2), (I4) is satisfied, then we have $|\cos w| < 1$ and

$$\tau = \frac{\sin w}{\sin mw} \cdot \frac{1}{k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)} \tag{2.27}$$

Lemma 2.2.4 *Assuming $\lambda_\xi(\tau) = \bar{r}_\xi e^{iw_\xi(\tau)}$ is a root of eigenvalue equation (2.20) near $\tau = \tau_\xi$, such that $\bar{r}_\xi(\tau_\xi) = 1$ and $w_\xi(\tau_\xi) = w_\xi$, then we have*

$$\left.\frac{d\bar{r}_\xi(\tau)}{d\tau}\right|_{\tau=\tau_\xi, w=w_\xi} > 0$$

Proof From equation (2.20), we can have

$$\lambda^m = \frac{-\tau k\left(r_1 + \frac{u_0}{1+u_0^2}\right)(1-\alpha u_0)}{\lambda - \left[1 - 2dl\tau l - \tau k\left(r_1 + \frac{2u_0}{(1+u_0^2)^2}\right) + dl\tau\lambda_j\right]}$$

Firstly, we solve for $\frac{d\lambda}{d\tau}$, and then

$$\left.\frac{d\bar{r}_\xi(\tau)}{d\tau}\right|_{\tau=\tau_\xi, w=w_\xi} = 2\Re\left.\left(\bar{\lambda}\frac{d\lambda}{d\tau}\right)\right|_{\tau=\tau_\xi, w=w_\xi}$$

Now we take derivative in term of τ of both sides of equation (2.20) and we have

$$\frac{d\lambda}{d\tau} = \frac{1}{\tau}\frac{\lambda(\lambda-1)}{\lambda(m+1) - mQ}$$

thus we can

$$
\begin{aligned}
\Re\left(\bar{\lambda}\frac{d\lambda}{d\tau}\right) &= \frac{1}{\tau}\frac{e^{iw}1}{e^{iw}(m+1)-mQ} \\
&= \frac{1}{\tau}\frac{\cos w + i\sin w - 1}{(\cos w + i\sin w)(m+1)-mQ} \\
\Rightarrow \left.\frac{d\bar{r}_\xi(\tau)}{d\tau}\right|_{\tau=\tau_\xi, w=w_\xi} &= \frac{2}{\tau}\frac{AC+BD}{C^2+D^2}
\end{aligned}
$$

where

$$
A = \cos w - 1, \quad B = \sin w, \quad C = (m+1)\cos w - mQ, \quad D = (m+1)\sin w
$$

At last, we can get

$$
\left.\frac{d\bar{r}_\xi(\tau)}{d\tau}\right|_{\tau=\tau_\xi, w=w_\xi} = \frac{2}{\tau}\frac{P[m(Q+1)+1]}{C^2+D^2} > 0 (P<0)
$$

and the proof is completed.

Theorem 2.2 *For the discrete system* (2.12)*, when h is small enough and l is a constant with $Q > 0$, we can have following conclusions*
(i) If either one of conditions (I1), (I3) and (I5) is satisfied, then the trivial solution of system (2.12) *is asymptotically stable.*
(ii) If either one of conditions (I2) and (I4) is satisfied, then the trivial solution is not asymptotically stable, and when $\tau = \tau_\xi, \xi = 1, 2, \cdots, \left[\frac{m-1}{2}\right]$, the system (2.12) *occurs Neimark-Sacker bifurcation at points $\tau = \tau_\xi$.*

Proof (i) Based on lemma 2.2.1 and 2.2.3, if either one of conditions (I1), (I3) and (I5) is satisfied, then eigenvalue equation (2.20) has no roots on unit circle. Using lemma 2.4 in [8], we can know that modulus of all roots is smaller than 1. So the proof is completed.

(ii) If either one of conditions (I2) and (I4) is satisfied, using lemma 2.2.2 and 2.2.4, we can know that at least the modulus of at least one pair of conjugate complex roots is larger than 1. And the proof is completed.

2.3 Conclusion

In this chapter, we used forward Euler scheme to analyze the discrete dynamical behavior of the spruce budworm model with time delay and diffusion terms under Dirichlet boundary condition., and we obtained the conditions of asymptotically stability of positive equilibrium and occurrence of Neimark-Sacker bifurcation.

Chapter 3

Numerical Experiments

3.1 Numerical Experiments

In this section, we implement numerical experiments for system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - r_1 u - \dfrac{u^2}{1+u^2} + be^{-r_2\tau}u_\tau e^{-\alpha u_\tau}, t > 0, x \in \Omega, t > 0 \\ U(0,t) = U(\pi,t) = 0, t > 0 \\ u(x,t) = \eta(x,t) \geq 0, x \in \Omega, t \in [-\tau, 0] \end{cases} \tag{3.1}$$

and verify if theorem 2.2 is correct.

Firstly, we use a set of coefficients as below

$$r_1 = 0.1980, \quad r_2 = 1.1, \quad b = 648649.8, \quad \alpha = 11.8572, \quad d = 0.03 \tag{3.2}$$

and under this condition, we can have $b > b_0 \approx 0.7658$ and $\alpha > 1$. And we can get

$$\hat{\tau} = 12.4089, \quad \tau_{\max} \approx 13.64$$

and the following plot

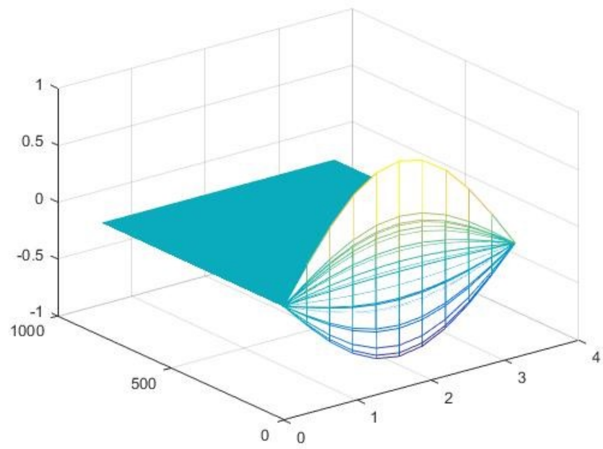

Figure 3.1: Numerical solution when $h = \pi/10$, $k = 1/4$, $\tau = 1$ and $u_0 = 0.3$

This plot shows the numerical solution of equations (3.1) by using forward Euler method when $\tau = 1$ and $u_0 = 0.3$, and we can see that the equilibrium is asymptotically stable. And under this condition, we can know that $u_0 > 1/\alpha$ and the coefficients satisfy condition (I5), which is

$$r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) > 0$$

and under condition (I5), trivial solution of this system is stable.

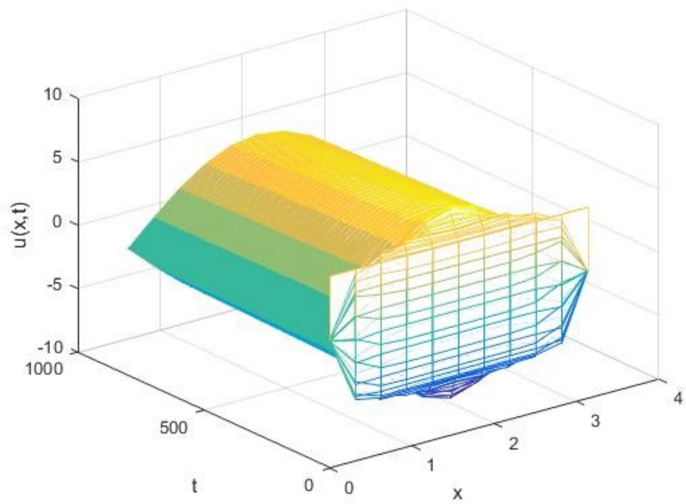

Figure 3.2: Numerical solution when $h = \pi/10$, $k = 1/4$, $\tau = 0.0617$ and $u_0 = 5$

Figure 3.2 is numerical solution when we change coefficients to $\tau = 0.0617$ and $u_0 = 5$, under which the equilibrium is unstable, since under this condition, $u_0 > 1/\alpha$, and coefficients satisfy condition (I4), which is

$$r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) < 0$$

and this assumption agree with the plot above.

Now we change the time step length to $k = 1/6$ with condition $\tau = 0.75, u_0 = 0.3$, under which the equilibrium is stable. And under this condition, we have that $u_0 > 1/\alpha$ and coefficients satisfy condition (I5), which is

$$r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) > 0$$

and this assumption agrees with the numerical solution as below

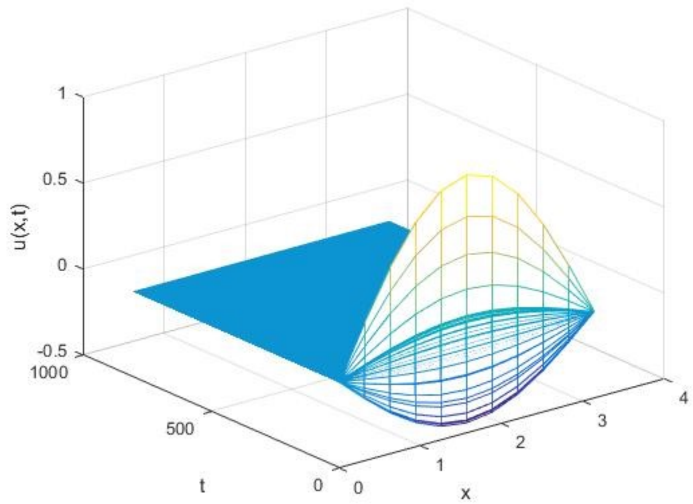

Figure 3.3: Numerical solution when $h = \pi/10$, $k = 1/6$, $\tau = 0.75$ and $u_0 = 0.3$

Finally, we change coefficients to $\tau = 0.0643, u_0 = 5$, under which the equilibrium is unstable. And we can have that $u_0 > 1/\alpha$ and coefficients satisfy condition (I4), which is

$$r_1 + \frac{2u_0}{1+u_0^2} + dlj^2 + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1) < 0$$

and also, this assumption agrees with the numerical solution as below

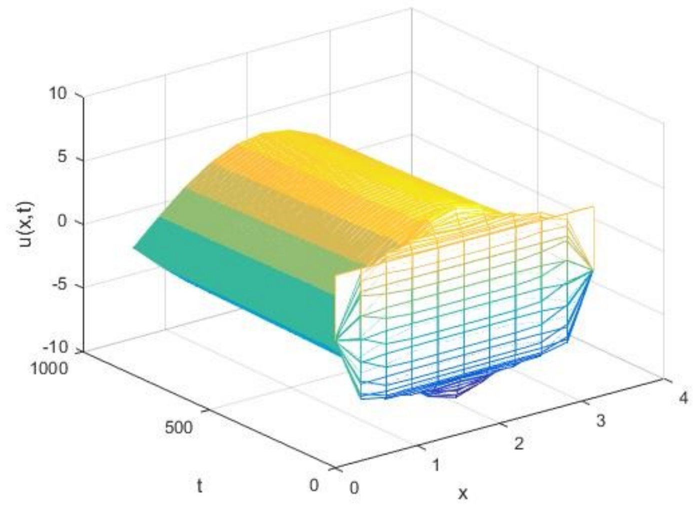

Figure 3.4: Numerical solution when $h = \pi/10$, $k = 1/6$, $\tau = 0.75$ and $u_0 = 0.3$

3.2 Conclusion

In this chapter, we use forward Euler scheme to simulate the system (3.1) and plot the numerical solutions, and we can tell that the theorem we obtain agrees with the numerical experiments.

Chapter 4

Conclusion

For the increasing need of numerical simulation and scientific computing, using numerical methods to study the dynamical behavior of delay partial differential equations is becoming more and more important. And bifurcation is a central topic in this. In this paper, we studied the discrete bifurcation of spruce budworm model with time Delay and diffusion.

In this paper, we studied the equation

$$\frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) - Du(x,t) - \frac{\beta u^2(x,t)}{\gamma^2 + u^2(x,t)} + q_1 e^{-\tilde{d}\tau} u(x,t-\tau) e^{-\alpha_1 u(x,t-\tau)}$$

and we used forward Euler scheme to discretize this equations in order to get the linear part of discrete formation. Then we used augmented matrix and eigenvalues to analyze the stability of its equilibrium and Neimark-Sacker bifurcation, and obtained the conditions of the stability of positive equilibrium and occurring of Neimark-Sacker bifurcation. At last, we chose different sets of step lengths and implemented the numerical experiments to verify that our theorem is correct.

In the future, we can further our study about spruce budworm model by using different numerical methods to discretize the system, such as backward Euler scheme and Crank-Nicolson method, and analyze the stability of the discrete system under these numerical methods.

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