Homework 7 for Math 2370
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Problem 1. Let A_k, $1 \leq k \leq K$ be $n \times n$ matrices satisfying

$$A_iA_j = A_jA_i \text{ for any } 1 \leq i, j \leq k.$$

Show the existence of a basis of $\mathbb{C}^n$ which consists of eigenvector and generalized eigenvectors of A_k for each $1 \leq k \leq K$.

Proof. Let $\{\lambda_j\}_j^J = 1$ be J distinct eigenvalues of A_1, and then we have

$$\mathbb{C}^n = \bigoplus_{j=1}^{J} N_j$$

where $N_j = N_{(A_1 - \lambda_j I)^{d_j}}$, d_j is index of jth eigenvalue λ_j. For $\forall x \in \mathbb{C}^n$, since $A_1A_i = A_iA_1, 2 \leq i \leq K$, then we have $(A_1 - \lambda_j I)^{d_j} A_i = A_i(A_1 - \lambda_j I)^{d_j}$. Thus, if $x \in N_k$

$$(A_1 - \lambda_j I)^{d_j} A_i x = A_i(A_1 - \lambda_j I)^{d_j} x = 0$$

which means $A_i x \in N_j$. Thus, A_i is a mapping from N_j to N_j. Now we apply Spectral Theorem to the linear mapping A_i and we know that N_j has a basis consisting of eigenvevtors and generalized eigenvevtors of A_i. And it is true for all $A_i, 2 \leq i \leq K$. Thus, a basis of $\mathbb{C}^n$ consists of eigenvectors and generalized eigenvectors of A_j for each $1 \leq j \leq K$. The proof is complete. $\square$

Method 2 of proof

Proof. We will prove it by induction on $\dim V$ and $1 \leq k \leq K$. And assume we have pairwise commuting operators $A_1, A_2, \cdots, A_K$ on V.

When $\dim V = 1$, and in this case, all A_i are scalars. Take $B = \{1\}$, and k be arbitrary. Assume the result is true whenever $\dim V < l$, and $A_1, A_2, \cdots, A_k$ are pairwise commuting operators on V if $\dim V < l$. And we want to show that if $\dim V = l$, and $A_1, \cdots, A_{k+1}$ are pairwise commuting operators on V, then there exists a basis of generalized eigenvectors.

For A_1, we have

$$\mathbb{C}^l = \bigoplus_{j=1}^{m} N_{\lambda_j}(A_1)$$

since $A_1A_i = A_iA_1, 2 \leq i \leq k+1$, we have $A_i : N_{\lambda_j}(A_1) \to N_{\lambda_j}(A_1), \forall j = 1, \cdots, m$ and $\forall i = 1, 2, \cdots, k+1$.

Case I. If $m = 1$, then there exists a basis B of $\mathbb{C}^1$ consisting of generalized eigenvectors for $A_2, \cdots, A_k, A_{k+1}$. Any vectors in $\mathbb{C}^l$ is a generalized eigenvectors for A_1 because $\mathbb{C}^l = N_{\lambda_1}(A_1)$, then any vectors in B is a a generalized eigenvectors for $A_1, A_2, \cdots, A_{k+1}$.

Case II. If $m > 1$, then $N_{\lambda_j}(A_1) \neq \mathbb{C}^l, \forall j = 1, \cdots, k+1$. On $N_{\lambda_j}(A_1)$, we have $A_1|_{N_{\lambda_j}(A_1)}, A_2|_N \lambda_j(A_1), \cdots, A_{k+1}|_N \lambda_j(A_1)$. By induction, there exists a basis $\beta_j, j =$

$1, 2, \cdots, m$ of $N_{\lambda_j}(A_1)$, which are generalized engenectors for $A_1|_{N_{\lambda_j}(A_1)}, \cdots, A_{k+1}|_{N\lambda_j}(A_1)$. Take

$$\beta = \bigcup_{j=1}^{m} \beta_j$$

Then β is a basis for $\mathbb{C}^l$ consisting of generalized eigenvectors for $A_1, A_2, \cdots, A_{k+1}$ $\square$

Problem 2. Let λ be an eigenvalue of an $n \times n$ matrix A. Suppose that

$$\begin{aligned} \dim N_1(\lambda) = 2, \dim N_2(\lambda) = 4 \\ \text{and } \dim N_3(\lambda) = \dim N_4(\lambda) = 5, \end{aligned}$$

Find the Jordan blocks of A corresponding to λ.

Proof. Since $\dim N_3(\lambda) = \dim N_4(\lambda) = 5$, then we can know that the index $d(\lambda) = 3$, then we can know the Jordan blocks of A corresponding to λ is

$$J = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

We can verify that this is the Jordan blocks we want. We can compute $N_{(J-\lambda I)}$, $N_{(J-\lambda I)^2}$, $N_{(J-\lambda I)^3}$ and $N_{(J-\lambda I)^4}$. We have

$$J - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and it is obvious that $\dim N_{(J-\lambda I)} = 2$, since there are two 0 column vectors. Similarly, we have

$$(J - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (J - \lambda I)^3 = (J - \lambda I)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we can know that $\dim N_{(J-\lambda I)^2} = 4$ and $\dim N_{(J-\lambda I)^3} = \dim N_{(J-\lambda I)^3} = 5$. The proof is complete. $\square$

Problem 3. Let A be a 5×5 rank one matrix, find all possible Jordan canonical forms of A. The order of Jordan blocks should be ignored.

Proof. Since A is rank one matrix, then there exists two column vectors a, b such that $A = ab^T$, also we know that the minimal polynomial for A is $m_A(\lambda) = \lambda^2 - \alpha\lambda$. So A

has eigenvalue 0 with multiplicity 4 and α with multiplicity 1. There are several possible Jordan forms for eigenvalue 0, which are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

Since the null space of $A - 0I$ has dimension 4 and one of them is generated by eigenvalue α. Thus, $\dim N_{A-0I} = 3$, which means that there are 3 blocks corresponding to eigenvalue 0. Thus, we can know that all possible Jordan canonical forms of A are

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{and} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

□

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find its eigenvectors and generalized eigenvectors. Find its Jordan canonical form J and the corresponding matrix S so that

$$A = SJS^{-1}.$$

Proof. Taking $A - \lambda I = 0$, we can have characteristic polynomial $p_A(\lambda) = (1-\lambda)^3$, which gives us eigenvalues 1. Now we determine the null space of $A - 1 \ldots I$

$$A - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So this eigenspace is dimensional-2. Hence there are two Jordan blocks corresponding to the eigenvalue 1 in the Jordan form. So we have its Jordan canonical form

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can know the eigenvectors corresponding to 1 are

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Each of these will give Jordan chain and we compute $(A - I)w_1 = v_1$ and $(A - I)w_2 = v_2$. The second equation does not have solution, so we can know that

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Then we have the engenvectors and generalized engenvectors, which are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Thus, we can find S, such that $AS = JS$, and we have

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\square$

Problem 5. Let P be the linear space of polynomials with real coefficients equipped with the scalar product

$$(f, g) = \int_0^1 f(x)\, g(x)\, dx.$$

(a) Using Gram-Schmidt process to generate an orthonormal basis of the span of vectors $\{1, x^2\}$.
(b) Find the projection of polynomial x on the span of vectors $\{1, x^2\}$.

Proof. (a)Set $y_1 = 1$ and $y_2 = x^2$, using Gram-Schmidt process, we can have

$$x_1 = \frac{y_1}{\|y_1\|} = \frac{1}{\sqrt{\int_0^1 1dx}} = 1$$

$$x_2 = \frac{y_2 - (y_2, x_1)x_1}{\|y_2 - (y_2, x_1)x_1\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_0^1 (x^2 - 1/3)^2 dx}} = \frac{3\sqrt{5}x^2 - \sqrt{5}}{2}$$

(b)Finding the projection of polynomial x on the span of vectors $\{1, x^2\}$ is equivalent to findind the solution for a, b in the equations

$$(1, x - (a + bx^2)) = 0$$
$$(x^2, x - (a + bx^2)) = 0$$

which gives us $b = \frac{15}{16}, a = \frac{3}{16}$. Thus, the projection is $\left(\frac{3}{16}, \frac{15}{16}\right)$. $\square$

Problem 6. Find the least squares solution to the over-determined system

$$\begin{aligned} 3x - y &= 1, \\ x + y &= 1, \\ 2x + 3y &= 2. \end{aligned}$$

Proof. Writting these equations into $AX = b$, where $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, then the least square least solution can be determined by $z = (A^T A)^{-1} A^T b = \begin{pmatrix} 0.4638 \\ 0.3768 \end{pmatrix}$. $\square$