Homework 2 for Math 2371
Zhen Yao

Problem 1. Let Y be a subspace of a finite dimensional linear space X such that

$$\dim Y = \dim X - k.$$

Let Z be a subspace of X. Show that

$$\dim(Z \cap Y) \geq \dim Z - k.$$

Proof. With $\dim(Y+Z) = \dim Y + \dim Z - \dim(Y \cap Z)$ and $\dim(Y+Z) \leq \dim X$, then we have

$$\begin{aligned} \dim(Y \cap Z) &= \dim Y + \dim Z - \dim(Y+Z) \\ &\geq \dim Z + \dim X - k - \dim X \\ &\geq \dim Z - k. \end{aligned}$$

□

Problem 2. Let $v_1, v_2, \cdots, v_k, k \geq 2$, be vectors in $\mathbf{R}^n$ and $1 \leq s < k$. Show that

$$\det G(v_1, \cdots, v_k) \leq \det G(v_1, \cdots, v_s) \det G(v_{s+1}, \cdots, v_k).$$

where $G(v_1, \cdots, v_k)$ is the Gram matrix of vectors $v_1, \cdots, v_k$ with the standard inner product.

Proof. We denote $G(v_1, \cdots, v_k) = \begin{pmatrix} G(v_1, \cdots, v_s) & B \\ B^* & G(v_{s+1}, \cdots, v_k) \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$. We assume $D > 0$, and with Schur complement, we have

$$\frac{G(v_1, \cdots, v_k)}{D} = A - BD^{-1}B^*.$$

Then we have $\det G(v_1, \cdots, v_k) = \det D \det(A - BD^{-1}B^*)$. □

Problem 3. Find the polar decomposition of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Proof. We have $A^T A = \begin{pmatrix} 10 & 8 \\ 8 & 8 \end{pmatrix}$, and the eigenvalues are $9 + \sqrt{65}, 9 - \sqrt{65}$. In polar decomposition, $A = QS$, and we have $S = \begin{pmatrix} \sqrt{9+\sqrt{65}} & 0 \\ 0 & \sqrt{9+\sqrt{65}} \end{pmatrix}$. Thus, we have $Q = AS^{-1} = \begin{pmatrix} \frac{1}{\sqrt{9+\sqrt{65}}} & \frac{2}{\sqrt{9-\sqrt{65}}} \\ \frac{2}{\sqrt{9+\sqrt{65}}} & \frac{3}{\sqrt{9-\sqrt{65}}} \end{pmatrix}$. □

Problem 4. Let A be self-adjoint. Show that the singular values of A are absolute values of eigenvalues of A.

Proof. In singular value decomposition, we have $A = WDV$, where W, V are unitary, and $D \geq 0$ is diagonal. Then we have

$$AA^* = WDVV^*DW^* = WD^2W^*,$$

which implies the singular values are eigenvalues of AA^*, i.e., $\sigma\left(D^2\right) = \sigma(AA^*)$.

Also, A is self-adjoint, and suppose λ is an eigenvalue of A with corresponding eigenvector v. Then $\overline{\lambda}$ is an eigenvalue of A^* with the same eigenvector v. Suppose $\lambda_1, \cdots, \lambda_n$ are eigenvalues of A with eigenvectors $v_1, \cdots, v_n$, then we have

$$AA^*v_j = A\overline{\lambda_j}v_j = \lambda_j\overline{\lambda_j}v_j = |\lambda_j|^2 v_j,$$

which implies that the eigenvalues of AA^* are $|\lambda_1|^2, \cdots, |\lambda_n|^2$, then we have $\sigma(D) = |\lambda_j|, 1 \leq j \leq n$. $\square$