Homework 7 for Math 2371

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Problem 1. Let

$$K = \Big\{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 2\Big\}.$$

Find its support function q_K. Here you can identify any $l \in (\mathbb{R}^2)'$ as $(l_1, l_2) \in \mathbb{R}^2$ such that

$$l(x + y) = l_1 x + l_2 y.$$

Proof. First we consider the region $x > 0, y > 0$ and split it into two regions $R_1 = \{x > 0, y < 2x\}$ and $R_2 = \{x > 0, y \geq 2x\}$. In R_1, for any point $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
(x, y) \cdot \left(1, \frac{y}{x}\right) &= x + \frac{y^2}{x} \\
&\leq x + \frac{2y^2}{y} \\
&= x + 2y = (x, y) \cdot (1, 2),
\end{aligned}$$

which implies $q_K(l) = x + 2y$, where $l_1 = 1$ and $l_2 = 2$. Similarly, we can have for region R_2, $q_K(l) = x + 2y$. With the same argument, we can have

$$q_K(l) = \begin{cases} x + 2y, & x > 0, y > 0 \\ -x - 2y, & x \leq 0, y < 0 \\ -a, & x < 0, b = 0 \\ a, & x > 0, b = 0 \\ x - 2y, & x > 0, b < 0 \\ -x + 2y, & x \leq 0, y > 0. \end{cases}$$

□

Problem 2. Let K be a bounded nonempty closed convex set of a finite dimensional linear space X. Show there exists at least one extreme point of K.

Proof. Since K is a bounded and closed convex set in finite dimensional space X, then we can find $x \in K$ such that $\|x\|$ is maximized. We prove by contradiction and suppose that there is a $y \neq 0$ such that $x + y \in K$, then

$$2\|x\|^2 \geq \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 > 2\|x\|^2,$$

which is a contradiction. Thus, K has at least one extreme point. □

Problem 3. Let K be a bounded nonempty closed convex set of a linear space X with $\dim X = n$. Suppose in addition that K contains at least one interior point. Show that K has at least $n + 1$ extreme points.

Proof. With Caratheodory theorem, we can know that every point in K can be represented by at most $n{+}1$ extreme points, then we can know that K at least $n{+}1$ extreme points. □