Homework 2 for Math 2371

Zhen Yao

Problem 1. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find all of its positive singular values.

Proof. Singular values of A are eigenvalues of $AA^* = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$, which are $\lambda_1 = 7$ and $\lambda_2 = 3$. $\square$

Problem 2. Let $m \geq n \geq 2$ and A be an $m \times n$ matrix. Suppose $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are singular values of A. Suppose $u_1 \in \mathbf{R}^m, v_1 \in \mathbf{R}^n$ are unit vectors such that

$$(u_1, Av_1) = \sigma_1.$$

Show that

$$\sigma_2 = \max_{\substack{\|u\|=\|v\|=1 \\ (u,u_1)=(v,v_1)=0}} (u, Av).$$

Proof. Suppose $\sigma_2 = (u_2, Av_2)$ and $\|u_2\| = \|v_2\| = 1$. Define for any vector g, h:

$$f(s,t) = \frac{(u_2 + sg, A(v_2 + th))}{\|u_2 + sg\| \cdot \|v_2 + th\|},$$

and we have

$$\begin{aligned} \left.\frac{\partial f}{\partial s}\right|_{(0,0)} &= \frac{(g, Av_2) - (u_2, Av_2)(u_2, g)}{\|u_2 + sg\|^2 \cdot \|v_2 + th\|^2} \\ &= \frac{(g, Av_2) - \sigma_2(u_2, g)}{\|u_2 + sg\|^2 \cdot \|v_2 + th\|^2}. \end{aligned}$$

Since $\sigma_2 = (u_2, Av_2)$, then we have $(u_2, \sigma_2 u_2) = (u_2, Av_2)$, which implies $\sigma_2 u_2 = Av_2$. Then, $\left.\frac{\partial f}{\partial s}\right|_{(0,0)} = 0$. Similarly, we have $\left.\frac{\partial f}{\partial t}\right|_{(0,0)} = 0$. Thus, $f(x,t)$ obtain its maximum at $(0,0)$ if $\sigma_2 = (u_2, Av_2)$. And with $(u_1, Av_1) = \sigma_1$, we have $Av_1 = \sigma_1 u_1$ and if $(u_2, u_1) = (v_2, v_1) = 0$, we can have

$$(Av_1, v_2) = (\sigma_1 u_1, v_2) = 0,$$

which implies $(u_1, v_2) = 0$. Thus, we have $\sigma_2 = \max_{\substack{\|u\|=\|v\|=1 \\ (u,u_1)=(v,v_1)=0}} (u, Av)$. $\square$