NOTES ON DUAL SPACES

SANTIAGO CAÑEZ

In these notes we introduce the notion of a dual space. Dual spaces are useful in that they allow us to phrase many important concepts in linear algebra without the need to introduce additional structure. In particular, we will see that we can formulate many notions involving inner products in a way that does not require the use of an inner product.

Let V be a vector space over a field $\mathbb{F}$. Recall the following definition:

Definition 1. *A linear functional on V is a linear map $V \to \mathbb{F}$. In other words, a linear functional on V is an element of $\mathcal{L}(V, \mathbb{F})$.*

Being examples of linear maps, we can add linear functionals and multiply them by scalars. Also, there is a unique linear functional on V, called the zero functional, which sends everything in V to zero. All this gives the set of linear functionals the structure of a vector space.

Definition 2. *The dual space of V, denoted by V^*, is the space of all linear functionals on V; i.e. $V^* := \mathcal{L}(V, \mathbb{F})$.*

Proposition 1. *Suppose that V is finite-dimensional and let $(v_1, \ldots, v_n)$ be a basis of V. For each $i = 1, \ldots, n$, define a linear functional $f_i : V \to \mathbb{F}$ by setting*

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and then extending f_i linearly to all of V. Then $(f_1, \ldots, f_n)$ is a basis of V^*, called the dual basis of $(v_1, \ldots, v_n)$. Hence, V^* is finite-dimensional and $\dim V^* = \dim V$.

Proof. First we check that $(f_1, \ldots, f_n)$ is linearly independent. Suppose that $a_1, \ldots, a_n \in \mathbb{F}$ are scalars so that

$$a_1 f_1 + \cdots + a_n f_n = 0.$$

Note that the 0 on the right denotes the zero functional; i.e. the functional which sends everything in V to $0 \in \mathbb{F}$. The above equality above is an equality of maps, which should hold or any v we evaluate either side on. In particular, evaluating both sides on v_i, we have

$$(a_1 f_1 + \cdots + a_n f_n)(v_i) = a_1 f_1(v_i) + \cdots + a_n f_n(v_i) = a_i$$

on the left (by the definition of the f_i) and 0 on the right. Thus we see that $a_i = 0$ for each i, so $(f_1, \ldots, f_n)$ is linearly independent.

Now we show that $(f_1, \ldots, f_n)$ spans V^*. Let $f \in V^*$. For each i, let b_i denote the scalar $f(v_i)$. We claim that

$$f = b_1 f_1 + \cdots + b_n f_n.$$

Again, this means that both sides should give the same result when evaluating on any $v \in V$. By linearity, it suffices to check that this is true on the basis $(v_1, \ldots, v_n)$. Indeed, for each i we have

$$(b_1 f_1 + \cdots + b_n f_n)(v_i) = b_1 f_1(v_i) + \cdots + b_n f_n(v_i) = b_i = f(v_i),$$

again by the definition of the f_i and the b_i. Thus, f and $b_1 f_1 + \cdots + b_n f_n$ agree on the basis, so we conclude that they are equal as elements of V^*. Hence $(f_1, \ldots, f_n)$ spans V^* and therefore forms a basis of V^*. $\square$

Note in particular the following consequence of the above construction of the dual basis. If $v \in V$ is an element such that $f(v) = 0$ for all $f \in V^*$, then $v = 0$. To see this, let $(v_1, \ldots, v_n)$ be a basis of V and let $(f_1, \ldots, f_n)$ be the dual basis. Write v as

$$v = a_1 v_1 + \cdots + a_n v_n.$$

By assumption, we have that $f_i(v) = 0$ for all i. But by the definition of f_i, $f_i(v) = a_i$. Thus $a_i = 0$ for all i and so $v = 0$ as claimed.

Let V^{**} denote $(V^*)^*$ — i.e. the dual space of the dual space of V, often called the *double dual* of V. If V is finite-dimensional, then we know that V and V^* are isomorphic since they have the same dimension. However, in general writing down an actual isomorphism between V and V^* requires choosing a basis of V and constructing the dual basis of V^* — the required isomorphism the sends the ith basis vector of V to the corresponding dual basis vector of V^*. Similarly, since $\dim V^*$ also equals $\dim V^{**}$, we know that V and V^{**} are isomorphic. In this case however, there is an isomorphism between V and V^{**} which can be written down without the choice of a basis — such an isomorphism is said to be *natural*.

Proposition 2. *Suppose that V is finite-dimensional. The map $ev : V \to V^{**}$ defined by*

$$ev(v)(f) := f(v)$$

is an isomorphism.

Let us emphasize the definition of ev given above. For $v \in V$, $ev(v)$ should be an element of V^{**}, meaning that it should take as input an element of V^* and output a scalar. The definition says that $ev(v)$ takes an element $f \in V^*$ and spits out the scalar $f(v)$; the name of this map, ev, stands for "evaluation" since it sends $v \in V$ to the element of V^{**} which is "evaluation on v".

Proof of Proposition. We omit checking that ev is linear — this follows simply from the definition of addition and scalar multiplication in V^*. Since V and V^{**} have the same dimension, to show that ev is an isomorphism it is enough to show that ev is injective. Let $v \in \operatorname{null} ev$. Then $ev(v)$ is the zero element of V^{**}, meaning that

$$ev(v)(f) = 0 \text{ for all } f \in V^*.$$

Unwinding the definition of $ev(v)$, this means that

$$f(v) = 0 \text{ for all } f \in V^*.$$

Thus v has the property that evaluating any linear functional on it gives zero; the only element of V with this property is 0, so we conclude that $v = 0$. Hence $\operatorname{null} ev = \{0\}$, so ev is injective and is thus an isomorphism. □

If V is infinite-dimensional, then $ev : V \to V^{**}$ is still injective but is no longer surjective. Indeed, not every infinite-dimensional vector space is in fact isomorphic to its double dual.

Even though V and V^* are in general not *naturally isomorphic*, there is a nice situation in which they are; indeed, the following is exactly the statement of Theorem 6.45 in the book, now rephrased using the language of dual spaces:

Theorem 1. *Let V be an inner product space. Define a map $T : V \to V^*$ by*

$$Tv = \langle v, \cdot \rangle,$$

i.e. Tv is the linear functional on V whose value on $w \in V$ is $\langle v, w \rangle$. Then T is an isomorphism.

This is one of the main conceptual uses of inner products — they allow us to identity a vector space with its dual in a natural way, where again natural means "without the choice of a basis".

Now we look at maps between dual spaces.

Definition 3. *Let $T : V \to W$ be linear. The dual map (or transpose) of T is the map $T^* : W^* \to V^*$ defined by*

$$T^*g = gT \text{ for all } g \in W^*.$$

In other words, T^* sends a linear functional g on W to the composition gT, which is a linear functional on V.

Proposition 3. *Suppose that V is finite-dimensional and let $(v_1, \dots, v_n)$ be a basis of V. Let $(f_1, \dots, f_n)$ be the corresponding dual basis of V^*. For $T \in \mathcal{L}(V)$, the matrix of $T^* \in \mathcal{L}(V^*)$ with respect to $(f_1, \dots, f_n)$ is the transpose of the matrix of T with respect to $(v_1, \dots, v_n)$; i.e.*

$$\mathcal{M}(T^*) = \mathcal{M}(T)^t.$$

This gives the "true" meaning of taking the transpose of the matrix: if a matrix represents an operator on some finite-dimensional space, then its tranpose represents the dual operator. Note that under the identification of V with V^* given by an inner product, the dual map corresponds exactly to the adjoint of T with respect to that inner product; indeed, this is why the dual map and adjoint use the same notation.

Finally, we give one last useful construction. Note again that under the identification of a vector space with its dual given by an inner product, the following corresponds to taking the orthogonal complement of a subspace. As above, this is why the following construction uses the same notation as the orthogonal complement.

Definition 4. *Let U be a subspace of V. The annihilator of U in V^*, denoted by $U^\perp$, is the set of linear functionals on V which vanish on U; i.e.*

$$U^\perp = \{f \in V^* \mid f(u) = 0 \text{ for all } u \in U\}.$$

Note that $U^\perp$ is a subspace of V^*. Indeed, the zero functional sends everything in U to zero and so is in $U^\perp$, if $f(u) = 0$ and $g(u) = 0$, then $(f + g)(u) = f(u) + g(u) = 0 + 0 = 0$ so $U^\perp$ is closed under addition, and finally if also $a \in \mathbb{F}$, then $(af)(u) = af(u) = a0 = 0$ so $U^\perp$ is closed under scalar multiplication.

Using annihilators, we can give a nice description of the dual space to a quotient space.

Proposition 4. *There exists a natural isomorphism between $U^\perp$ and $(V/U)^*$. Hence we can identify linear functionals on V/U with elements of $U^\perp$.*

Proof. Let $f \in U^\perp$. Again, this means that f is a linear functional on V which vanishes on the subspace U. Define a linear functional Tf on V/U by

$$(Tf)(v + U) = f(v);$$

in other words, Tf sends the coset $v + U$ to the scalar $f(v)$. First we need to know that this definition of Tf is well-defined. Suppose that $v + U = v' + U$. We must check that evaluating Tf on either one gives the same result. Since $v+U = v'+U$, $v-v' \in U$. Thus since f vanishes on U, we have

$$0 = f(v - v') = f(v) - f(v'),$$

so $f(v) = f(v')$, showing that Tf is well-defined.

This then defines a map $T : U^\perp \to (V/U)^*$. From the definition of addition and scalar multiplication of linear functionals, it follows that T is linear. We claim that T is invertible.

To show that T is injective, suppose that $f \in \operatorname{null} T$. Then Tf is the zero functional on V/U so

$$0 = (Tf)(v + U) = f(v) \text{ for all } v \in V.$$

Thus f is the zero functional on V (i.e. the zero element of $U^{\perp}$) so $\operatorname{null} T = \{0\}$ and T is injective.

Finally, to show that T is surjective (note that we are not assuming V is finite-dimensional), let $g \in (V/U)^*$. Define an element $f \in V^*$ by

$$f(v) = g(v + U) \text{ for all } v \in V.$$

We claim that f is actually in $U^{\perp}$. Indeed, if $u \in U$, then $g(u + U) = g(U) = 0$ since U is the zero element of V/U and g is linear. Thus $f(v) = 0$, so $f \in U^{\perp}$. By definition of T, it follows that $Tf = g$ so that T is surjective and thus invertible. $\square$

These last few results in particular show the sense in which dual spaces can be used to rephrase many notions coming from inner products without actually using inner products. This can be quite useful for the following reason: choosing an inner product involves making a choice and putting yet more structure on a vector space, however, the dual space always exists without any additional choices so using it instead of an inner product can make certain things easier to keep track of.

We finish with one motivation for dual spaces coming from physics. A force in physics is usually represented by a vector F. The main thing we are interested in is what a force does when moving an object from point A to point B. If q is the position vector of the object, then F acts on it to produce a number called "work"; that is, F can be thought of us a linear functional

$$F : q \mapsto F(q).$$

Thus from this point of view, force should actually be thought of as an element of the dual space since it took a vector and produced a scalar. Thinking of forces in terms of dual spaces turns out to be incredibly useful and helps simplify many other concepts in physics. In particular, if you've taken a course in electromagnetism before, we finish by mentioning that Maxwell's equations are much simpler to express from this point of view.