MATH2370 HOMEWORK 1

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Problem 1

Proof: (1) We set $u_1 \in U \cap (V + W)$, then there exist some $v_1 \in V$ and $w_1 \in W$ such that $u_1 = v_1 + w_1$ since $u_1 \in U$ and also $u_1 \in (V + W)$. Also, $u_1 \in U$ and $W \subset U$ which means $w_1 \in U$, then we have $v_1 \in U$ since U is a subspace which is closed under addition. Then from $v_1 \in U$, we have $v_1 \in (U \cap V)$. Based on the fact that $u_1 = v_1 + w_1$ and $w_1 \in W$, we have $u_1 \in (U \cap V + W)$. This implies that $U \cap (V + W) \subset (U \cap V + W)$.

(2)Now we set $u_2 \in (U \cap V + W)$, then there exist some $\lambda \in U \cap V$ and $w_2 \in W$ such that $u_2 = \lambda + w_2$. And we have $\lambda + w_2 \in V + W$, since $\lambda \in U \cap V$ and $w_2 \in W$. Also, we know that $W \subset U$, then we have $\lambda + w_2 \in U$. Thus we can have $\lambda + w_2 \in U \cap (V + W)$. Hence, $u_2 \in U \cap (V + W)$, which implies $(U \cap V + W) \subset U \cap (V + W)$.

Then we showed that $U \cap (V + W) = U \cap V + W$.

Problem 2

(i)Set $P_1, P_2 \in Y$ and they have form $P_i = (t - t_1)(t - t_2) \cdots (t - t_m)q_i(t)$ that are zero at distinct $t_1, t_2, \cdots, t_m \in K$. Then we have

$$P_1 + P_2 = \sum_{i=1}^{2} (t - t_1)(t - t_2) \cdots (t - t_m)q_i(t)$$
$$aP_1 = a(t - t_1)(t - t_2) \cdots (t - t_m)q_1(t)$$

where $a \in K$. And we can know that both $P_1 + P_2$ and aP_1 are zero at points $t_1, t_2, \cdots, t_m \in K$. So Y is closed under addition and multiplication. Hence, Y is a subspace of X.

(ii)To satisfy being zero at distinct $t_1, t_2, \cdots, t_m \in K$ where $m < n$, the polynomial $P_Y(t) \in Y$ has the form

$$P_Y(t) = (t - t_1)(t - t_2) \cdots (t - t_m)q(t)$$

where $q(t)$ is not determined. Also, we know that the space of all polynomials is degree less than n, which means that $q(t)$ is degree less than $n - m$.

Since the polynomials $P(t)$ in the space X are degree less that n, it can be presented by the form

$$P(t) = \sum_{k=0}^{n-1} c_k t^k$$

So the basis of X can be written as $1, t, t^2, \cdots, t^{n-1}$, and we have $\dim X = n$. Now we can present $q(t)$ by utilizing this basis as

$$q(t) = \sum_{k=0}^{n-m-1} c_k t^k$$

then the basis for subspace Y can be presented as

$$\left\{\prod_{i=1}^{m}(t-t_i), t\prod_{i=1}^{m}(t-t_i), \cdots, t^{n-m-1}\prod_{i=1}^{m}(t-t_i)\right\}$$

and we can check that the linear combination of this basis is equal to zero if and only if all coefficients are all zero. So we have dim $Y = n - m$.

(iii) From theorem, we can know that dim $X/Y =$ dim $X-$ dim $Y = m$. Now we set a basis that spans the subspace X/Y.

We firstly set $P_1(t) = (t-t_2)\cdots(t-t_m)q(t)$ and the class $\{P_1\}$ of P_1 is the space $\{P(t) \in X : P(t) - P_1(t) \in Y\}$, and then set $P_2(t) = (t-t_1)(t-t_3)\cdots(t-t_m)q(t)$ and the class $\{P_2\}$ in the same way. And we continue this process where we get rid of $(t-t_i)$ in class P_i until we finally have $P_m(t) = (t-t_1)\cdots(t-t_{m-1})q(t)$ and the class $\{P_m\}$. Then we can check that $(\{P_1\}, \{P_2\}, \cdots, \{P_m\})$ is the span of X/Y.

Problem 3

The statement is not true.

Here is a counterexample. Let's consider three lines U, V, W in $\mathbb{R}^2$ such that they intersect in one point. So we have

$$\dim(U+V+W) = 2$$

and

$$\begin{aligned}
&\dim(U) + \dim(V) + \dim(W) - \dim(U\cap V) - \dim(U\cap W) - \\
&\dim(V\cap W) + \dim(U\cap V\cap W) \\
&= 3 - 0 - 0 - 0 + 0 \\
&= 3
\end{aligned}$$

the left and right sides are not the same.

Problem 4

If $\dim U_1 = \cdots = \dim U_k = n$, then we can simply take $V = \emptyset$.

If $\dim U_1 = \cdots = \dim U_k = n-1$, based on the fact that space X cannot be presented as a finite union of its proper subspace, there exists a $v \notin U_k$ for every k, and we can define the complement of $U_1, U_2, \cdots, U_k$ as $U^c = \operatorname{span}(v)$.

Then we consider the case when $\dim U_1 = \cdots = \dim U_k = n-2$, also there exists a $v \notin U_k$ for every k. Then we can define a new subspace $\tilde{U}_i = \operatorname{span}(u_i, v)$ for $1 \le i \le k$, and we can immediately know that $\dim\tilde{U}_i = n-1$. Then we can get a complement of $\tilde{U}_i$, denoted by U_i^c and we have $\dim U_i^c = 1$. In particular, $v \notin U_i^c$, so we can take $\operatorname{span}(U_i^c, v)$, which is dimension 2 and a complement of $U_1, U_2, \cdots, U_k$.

Now we can continue this induction and consider $\dim U_1 = \cdots = \dim U_k = m, m < n$, and we can find $v \notin U_k$ for every k. Then we can define $\tilde{U}_i = \operatorname{span}(u_i, v)$ for $1 \le i \le k$, and we can immediately know that $\dim\tilde{U}_i = m+1$. Then we can get a complement of $\tilde{U}_i$, denoted by U_i^c and we have $\dim U_i^c = n-m-1$. In particular, $v \notin U_i^c$, so we can take $\operatorname{span}(U_i^c, v)$, which is dimension $n-m$ and a complement of $U_1, U_2, \cdots, U_k$.