Notes on Orthogonal and Symmetric Matrices
MENU, Winter 2013

These notes summarize the main properties and uses of orthogonal and symmetric matrices. We covered quite a bit of material regarding these topics, which at times may have seemed disjointed and unrelated to each other. However, the point is that there is much common ground here and having it all laid out in one place should be useful.

Most of this material should be good to look at when preparing for the midterm. Some of it takes a "higher" level view of some of these concepts which won't necessarily be tested on the midterm, but it doesn't hurt to understand this higher-level view as well. When in doubt, ask me if it's not clear whether something here is something you should know, but the answer will most likely be yes.

Orthogonal Projections

Let's start with the most important property of orthonormal bases:

Fact. *If $\vec{u}_1, \ldots, \vec{u}_k$ is an orthonormal basis of a subspace V of $\mathbb{R}^n$, then any $\vec{x}$ in V can be written as*

$$\vec{x} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_k)\vec{u}_k.$$

In other words, the coefficients needed to express $\vec{x}$ as a linear combination of $\vec{u}_1, \ldots, \vec{u}_k$ are obtained by taking dot products of $\vec{x}$ with the $\vec{u}$'s themselves.

This is a crucially important fact! Recall that usually to express a vector $\vec{x}$ as a linear combination of vectors $\vec{v}_1, \ldots, \vec{v}_k$ requires setting up the equation

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \quad \text{or} \quad \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \vec{x}$$

and using row operations to solve for $c_1, \ldots, c_k$. The point is that having an orthonormal set of vectors greatly simplifies this since no row operations are necessary, we only need to compute some dot products.

For this to be useful requires that we be able to find orthonormal bases of any subspace we want, but of course this is precisely what the Gram-Schmidt process allows us to do:

Fact. *The Gram-Schmidt process applied to a basis $\vec{v}_1, \ldots, \vec{v}_k$ of a subspace V of $\mathbb{R}^n$ produces an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_k$ of that same subspace.*

Now we're in business. For instance, we can now easily compute orthogonal projections:

Fact. *If $\vec{u}_1, \ldots, \vec{u}_k$ is an orthonormal basis of a subspace V of $\mathbb{R}^n$, the orthogonal projection of a vector $\vec{x}$ onto V is*

$$\operatorname{proj}_V \vec{x} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots (\vec{x} \cdot u_k)\vec{u}_k.$$

This can also be written as

$$\operatorname{proj}_V \vec{x} = QQ^T\vec{x} \quad \textit{where} \quad Q = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_k \\ | & & | \end{pmatrix}.$$

Thus the matrix of the orthogonal projection of $\mathbb{R}^n$ onto the subspace V is QQ^T where Q is the matrix having the given orthonormal basis vectors as its columns.

Note that this matrix QQ^T is always an $n \times n$ matrix and is symmetric since

$$(QQ^T)^T = (Q^T)^T Q^T = QQ^T$$

using the fact that the transpose of a product of matrices is the product of the transposes of each in the **reverse** order. This is our first contact between things related to orthogonality and things related to symmetry. We'll come back to this.

Note also that the right-hand side of the first expression in the Fact above is precisely what you get when you project $\vec{x}$ onto the line spanned by each basis vector and add the results together, so that the projection of $\vec{x}$ is obtained simply by adding together these individual projections. This **only** works when you have an *orthogonal* basis: for a basis which is orthogonal but not necessarily orthonormal all that changes is that the dot products on the right-hand side get replaced by fractions with denominators coming from the general formula for the projection of a vector onto a line.

So, to sum up, computing orthogonal projections involves the following steps:

Fact. *To compute orthogonal projections, you*

1. *Find a basis of the space you're projecting onto.*
2. *Apply the Gram-Schmidt process to that basis to get an orthonormal basis*
3. *Use that orthonormal basis to compute the projection as in the first part of the previous Fact, or use that orthonormal basis to compute the matrix of the projection as in the second part of the previous Fact.*

Least Squares

Now, why do we care about orthogonal projections? Here is a key point:

Fact. *Given a subspace V of $\mathbb{R}^n$ and a vector $\vec{x}$ in $\mathbb{R}^n$, the vector in V "closest" to $\vec{x}$ is the projection $\mathrm{proj}_V\,\vec{x}$. Here "closest" means that $\|\vec{x}-\vec{v}\|$ as v ranges through all things in V is minimized when $\vec{v} = \mathrm{proj}_V\,\vec{x}$.*

So, orthogonal projections come up in problems dealing with minimizing some quantity. Here is an example:

Example. We want to find the vector in the plane V spanned by

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$$

which is closest to $\left(\begin{smallmatrix}1\\-1\\-2\end{smallmatrix}\right)$. According to the above fact, this vector should be the orthogonal projection of $\left(\begin{smallmatrix}1\\-1\\-2\end{smallmatrix}\right)$ onto the given plane. To compute this projection, we first find an orthonormal basis of this plane. Applying the Gram-Schmidt process to the two given spanning vectors produces

$$\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$$

as an orthonormal basis. Hence the orthogonal projection of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ onto this plane (using a formula for orthogonal projections given in a previous Fact) is:

$$\begin{aligned}
\operatorname{proj}_V \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} &= \left[\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} + \left[\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \\
&= -\frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} + \frac{1}{\sqrt{6}} \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \\
&= \begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} 1/6 \\ -2/6 \\ 1/6 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix}.
\end{aligned}$$

If we want, the distance between this and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is

$$\left\| \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -3/2 \\ 0 \\ 3/2 \end{pmatrix} \right\| = \frac{\sqrt{18}}{2}.$$

Now, admittedly, this took quite a bit of work, but notice that there is another way to get this answer. Let A be the matrix with the spanning vectors of V as its columns:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then the plane V we're looking at is precisely the image of A, so the question is to find the vector in this image, i.e. of the form $A\vec{x}$, closest to $\vec{b} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. But this is precisely what the least-squares method gives us!

Fact. *The solution $\vec{x}$ of $A^T A\vec{x} = A^T\vec{b}$, i.e. the so-called "least-squares solution" of $A\vec{x} = \vec{b}$ **is** the vector $\vec{x}$ so that $A\vec{x}$ is as close as possible to $\vec{b}$ among all vectors in the image of A. In other words, the solution $\vec{x}$ is the one which makes $A\vec{x}$ the orthogonal projection of $\vec{b}$ onto the image of A.*

So, the vector $\vec{x}$ for which the vector $A\vec{x}$ in the plane $V = \operatorname{im} A$ is closest to $\vec{b} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is the one which solves $A^T A\vec{x} = A^T\vec{b}$, which in this case is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Solving this gives $\vec{x} = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$, so the orthogonal projection of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ onto the plane $V = \operatorname{im} A$ is

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix},$$

agreeing with what we found in the example doing it another way. The point is that while the least-squares method usually shows up in problems about best-fitting some given data points, it may also be useful in other types of length-minimizing problems; in particular ones dealing with subspaces which can be expressed as the image of some matrix.

Fact. *Many things dealing with minimizing some quantity or finding something which is closest to something else will involve either orthogonal projections or least squares.*

Orthogonal Matrices

Now we move on to consider matrices analogous to the Q showing up in the formula for the matrix of an orthogonal projection. The difference now is that while Q from before was not necessarily a square matrix, here we consider ones which are square.

Fact. *The following are equivalent characterizations of an orthogonal matrix Q:*

- *The columns of Q are orthonormal*
- $Q^T = Q^{-1}$*, which is the same as saying* $Q^T Q = I = QQ^T$
- *Q is length-preserving or dot product preserving in the sense that computing lengths or dot products after multiplying vectors by Q gives the same result as computing these before multiplying vectors by Q*
- *Q describes geometrically either a rotation or a reflection and which one you get is determined by the sign of* $\det Q$.

To elaborate on the last fact, note first that the second fact above implies that $\det Q = \pm 1$ for any orthogonal matrix Q; then the rotations are the ones for which $\det Q = 1$ and the reflections are the ones for which $\det Q = -1$. Note that we are not saying that *any* matrix such that $\det A = 1$ is a rotation or any one with $\det A = -1$ is a reflection: this only applies to matrices we already know are orthogonal.

Note also that the condition $QQ^T = I$ in the second statement above might seem strange since the matrices QQ^T describing orthogonal projections before did not necessarily equal the identity matrix. Again, the distinction is that here we are considering square matrices while before we weren't: for a matrix Q whose columns are orthonormal, QQ^T does not necessarily equal the identity matrix unless Q is square. However, $Q^T Q = I$ is true for **any** matrix whose columns are orthonormal, not just square ones. It's good practice to figure out why.

Symmetric Matrices

Now we come to properties of symmetric matrices, starting with a basic observation about transposes in general.

Fact. *For a square matrix A, the transpose A^T satisfies*

$$A\vec{u} \cdot \vec{v} = \vec{u} \cdot A^T \vec{v}$$

for any vectors $\vec{u}$ and $\vec{v}$. In other words, the transpose is what allows you to "move" the matrix from one factor in a dot product to the other factor. If A is symmetric, this gives

$$A\vec{u} \cdot \vec{v} = \vec{u} \cdot A\vec{v}$$

for any $\vec{u}$ and $\vec{v}$.

This might not seem all that important, but in actuality this is truly the key property of symmetric matrices which leads to their nice applications. In fact, in more advanced applications of linear algebra, it is generalizations of this property which defines a more general notion of "symmetric".

Here, then, are the crucial properties of symmetric matrices:

Fact. *For any symmetric matrix A:*

- *The eigenvalues of A all exist and are all real.*
- *Eigenvectors of A corresponding to different eigenvalues are automatically orthogonal.*
- *A is always diagonalizable, and in fact orthogonally diagonalizable.*

Recall that *orthogonally diagonalizable* means that we can diagonalize A as

$$A = QDQ^T$$

where D is diagonal and Q is orthogonal (so that $Q^{-1} = Q^T$). And of course, let us not forget the crucial observation that a matrix which is orthogonally diagonalizable must be symmetric since:

$$(QDQ^T)^T = (Q^T)^T D^T Q^T = QD^T Q^T = QDQ^T$$

where $D^T = D$ since diagonal matrices are symmetric. Thus we get the infamous spectral theorem:

Theorem (Spectral Theorem)**.** *A square matrix is orthogonally diagonalizable if and only if it is symmetric. In other words, "orthogonally diagaonlizable" and "symmetric" mean the same thing.*

Here are the steps needed to orthogonally diagonalize a symmetric matrix:

Fact. *To orthogonally diagonalize a symmetric matrix*

1. *Find its eigenvalues.*
2. *Find a basis for each eigenspace. Note that for an $n \times n$ symmetric matrix, you will **always** be able to find n linearly independent eigenvectors in total.*
3. *Apply the Gram-Schmidt process to each set of basis eigenvectors corresponding to the **same** eigenvalue. Don't worry about eigenvectors corresponding to different eigenvalues since they will automatically be orthogonal.*
4. *Use the resulting orthonormal eigenvectors as the columns of Q.*

Before giving some applications of this, let me say a bit about why this is so useful at a "higher" level. The point is that for a symmetric matrix A we can find an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A: say $\vec{u}_1, \ldots, \vec{u}_n$. The main reason why having such a basis is important is that it allows us to fully compute what A does to any vector even though we may *not* know what A explicitly is. The fact that the basis $\vec{u}_1, \ldots, \vec{u}_n$ is orthonormal makes it easy to write any vector in terms of this basis:

$$\vec{x} = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_n)\vec{u}_n.$$

Then the fact that the basis actually consists of eigenvectors of A makes it easy to compute what happens when we multiply by A:

$$\begin{aligned} A\vec{x} &= (\vec{x} \cdot \vec{u}_1)A\vec{u}_1 + \cdots + (\vec{x} \cdot \vec{u}_n)A\vec{u}_n \\ &= \lambda_1(\vec{x} \cdot \vec{u}_1)\vec{u}_1 + \cdots + \lambda_n(\vec{x} \cdot \vec{u}_n)\vec{u}_n \end{aligned}$$

where the λ's are the corresponding eigenvalues. So, we have an explicit formula for $A\vec{x}$ given only in terms of eigenvectors and eigenvalues. This has some amazing applications, but I digress.

Now, let's go back to the orthogonally diagonalized expression $A = QDQ^T$. Recalling properties of similar matrices, this means that A is the same transformation as the one represented by D only relative to a different basis, namely the one making up the columns of Q. Since Q is orthogonal, it is either a rotation or a reflection and thus so is Q^T. Hence, $A = QDQ^T$ means that the transformation A can be thought of as first performing the rotation/reflection given by Q^T, then performing the transformation given by D, and finally performing the rotation/reflection given by Q. Since D is diagonal, it describes scalings by different amounts in different directions (directions corresponding to the eigenvectors), so we get the following geometric description of symmetric matrices:

Fact. *Geometrically, a symmetric matrix is a scaling (by different amounts) in various directions, where the directions are obtained by rotating or reflecting the standard axes.*

In fact, we've already seen this at work when looking at quadratic forms: the ellipses/hyperbolas we get from the various quadratic forms we've seen are nothing but scalings and rotations/reflections of standard ellipses and hyperbolas. This is due to the above geometric characterization of symmetric matrices.

As an application, consider the following problem from the homework.

Example. 8.1.13. Consider a symmetric 3×3 matrix A with $A^2 = I$. Is the linear transformation $T(\vec{x}) = A\vec{x}$ necessarily the reflection about a subspace of $\mathbb{R}^n$? The answer is no, but almost, and it is instructive to see why.

Since A is symmetric, it is orthogonally diagonalizable so we can write it as

$$A = QDQ^T$$

for some orthogonal matrix Q and some diagonal matrix D. Then we can compute A^2 as:

$$A^2 = (QDQ^T)(QDQ^T) = QD(Q^TQ)DQ^T = QDDQ^T = QD^2Q^T.$$

If this is supposed to be I, we must have $QD^2Q^T = I$, which implies $D^2 = I$. Thus the diagonal entries of D satisfy $\lambda^2 = 1$ and so must all be either -1 or 1. Hence in

$$A = QDQ^T,$$

D has only 1's and -1's down the diagonal. Such a matrix describes either a reflection or a rotation by 180° around some axes (i.e. rotations such that doing the rotation twice has the effect of doing nothing overall), so A can be thought of as doing the rotation/reflection Q^T, followed by the reflection/special type of rotation D, followed by the rotation/reflection Q. Such a composition is itself always a reflection or the same special type of rotation, so A is one of these types of transformations.

This example illustrates something crucially important, which I will state as a fact:

Fact. *Whenever you see any question dealing with symmetric matrices and powers, it is always a good idea to write A in the form $A = QDQ^T$ and use this expression to say something about the powers.*

For instance, in addition to the above example, problems 8.1.36 and 8.1.42 in the book also use this idea.

Quadratic Forms

Finally, let's give a quick outline of the application of symmetric matrices we've seen to quadratic forms.

Fact. *Any quadratic form* $q(x_1, \ldots, x_n) = \vec{x} \cdot A\vec{x}$ *can be put into the form*

$$q(c_1, \ldots, c_n) = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$$

after a change of basis. Here, the λ*'s are the eigenvalues of* A *and the* c*'s are the coordinates relative to the corresponding orthonormal eigenvectors.*

Apart from being able to find the above eigenvalues and eigenvectors, here is the main important concept to know:

Fact. *The quadratic form* $q(x_1, \ldots, x_n)$ *is*

- positive definite *if all eigenvalues are positive*
- positive semi-definite *if all eigenvalues are nonnegative (zero is allowed)*
- negative definite *if all eigenvalues are negative*
- negative semi-definite *if all eigenvalues are nonpositive (zero is allowed)*
- indefinite *if it has eigenvalues of different signs*

An equation in 2-dimensions of the form $q(x_1, x_2) = 1$ *describes an ellipse when* q *is positive definite and a hyperbola when* q *is indefinite. The* principal axes *of these curves are determined by the eigenvectors corresponding to the different eigenvalues.*

And that's about it. Let me know if there's anything else you'd like to see summed up on here or anything you'd like to see expanded on.