Homework 5 for Math 2370
Zhen Yao

Problem 1. Let A, B, C be $n \times n$ matrices satisfying $AB = BA$. Show that

$$\det(A + BC) = \det(A + CB).$$

Proof. (1)Since $AB = BA$, if B is invertible, then we have $A = B^{-1}AB$. Then we have

$$\begin{aligned}\det(A + BC) &= \det(B^{-1}(A + BC)B) \\ &= \det(B^{-1}AB + CB) \\ &= \det(A + CB)\end{aligned}$$

(2)If B is not invertible. We can set a new matrix $M = \begin{pmatrix} C & -I \\ A & B \end{pmatrix}$, and we can solve for the determinant of this matrix. Since $AB = BA$, then $\det(M) = \det(CB - (-I)A) = \det(CB + A)$. Also, we have $-IB = B(-I)$, then the determinant can be presented as $\det(M) = \det(BC - (-I)A) = \det(BC + A)$. Then we have $\det(A + BC) = \det(A + CB)$. The proof is complete. □

Remark: In (2), we used if $AB = BC$, then $\det(M) = \det(CB - (-I)A)$. We should give proper proof to this. Suppose matrix $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ and we have $RS = SR$. Then, if S is invertible, we have

$$\begin{aligned}\det\begin{pmatrix} P & Q \\ R & S \end{pmatrix} &= \det\begin{pmatrix} P - RS^{-1}Q & 0 \\ R & S \end{pmatrix} \\ &= \det(PS - SRS^{-1}Q) \\ &= \det(PS - RSS^{-1}Q) \\ &= \det(PS - RQ)\end{aligned}$$

If S is not invertibe, then there exists $\varepsilon_k \to 0$ such that $\det S_k = \det(B + \varepsilon_k I) \neq 0$ and $S_k R = RS_k$. Then $\det\begin{pmatrix} P & Q \\ R & S_k \end{pmatrix} = \det(PS_k - QR)$. Taking $k \to \infty$ will prove this case. The proof is complete. Similarly, we can prove that if $QS = SQ$, then $\det M = \det(SP - QR)$.

Problem 2. Let A, B, C be $n \times n$ matrices. Is it always true that

$$\det(A + BC) = \det(A + CB)?$$

Prove or find a counter example.

Proof. In general, it is not true. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix}$. Then we have $\det(A + BC) = 76$ and $\det(A + CB) = 85$. □

Problem 3. Let $n \geq 2$. Given $(2n-1)$ scalars $x_1, \cdots, x_{n-1}$ and $y_1, \cdots, y_n$, we can define an $n \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} = x_j \text{ if } i > j,$$
$$a_{ij} = y_j \text{ if } i \leq j.$$

Show that

$$\det A = y_n \prod_{k=1}^{n-1} (y_k - x_k) .$$

Proof. We can know that A has the form

$$A = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ x_1 & y_2 & y_3 & \cdots & y_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & y_n \end{pmatrix}$$

We can do elementary row operations that starting from the first row, and then apply $\text{row}_i = \text{row}_i + (-1)\text{row}_{i+1}$. Then we get new matrix

$$A = \begin{pmatrix} y_1 - x_1 & 0 & 0 & \cdots & 0 \\ 0 & y_2 - x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & y_n \end{pmatrix}$$

Then it is obvious that $\det(A) = y_n \prod_{k=1}^{n-1}(y_k - x_k)$. $\square$