Homework 7 for Math 2370

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Problem 1. Suppose $1 \leq k \leq n$ and $x_1, x_2, \cdots x_k$ are k vectors in $\mathbb{R}^n$ satisfying for any $1 \leq i, j \leq k$,

$$(x_i, x_j) = \delta_{ij}.$$

For each $1 \leq j \leq k$, let a_j be the first component of x_j. Show that

$$\sum_{j=1}^{k} a_j^2 \leq 1.$$

Proof. Since $(x_i, x_j) = \delta_{ij}, 1 \leq i, j \leq n$, then we can arrange $x_1, x_2, \cdots, x_n$ into a matrix and denote it by $A = (x_1, x_2, \cdots, x_n)$, then we have A is an orthogonal matrix with determinant 1. Then $\det A^* = 1$.

Now we pick a vector $z = (1, 0, \cdots, 0)^T \in \mathbb{R}^n$. Then we have $A^* z = (a_1, a_2, \cdots, a_n)^T$, and therefore the first component of the vector AA^*z is $\sum_{j=1}^k a_j^2$, which means $AA^*z = \left(\sum_{j=1}^k a_j^2, \cdots\right)^T$. Also, we have $\|AA^*z\| \leq \|Iz\| = 1$. We denote other components of AA^*z as $w_2, w_3, \cdots, w_n$, then we have

$$\sum_{j=1}^{k} a_j^2 \leq \|AA^*z\|^{1/2} = \sqrt{\sum_{j=1}^{k} a_j^2 + w_2^2 + \cdots + w_n^2} = 1$$

$$\Rightarrow \sum_{j=1}^{k} a_j^2 \leq 1$$

The proof is complete. □

Problem 2. Let A be an $m \times n$ matrix, c_j $1 \leq j \leq n$ be column vectors of A and r_i, $1 \leq i \leq m$ be row vectors of A, show that

$$\|A\| \geq \max_{1 \leq j \leq n} \|c_j\| \text{ and } \|A\| \geq \max_{1 \leq i \leq m} \|r_i\|.$$

Here we view A as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof. For jth column c_j of A, we pick a unit vector $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^T \in \mathbb{R}^n$, where jth entry is 1, others are all zero. Then we can have $Ae_j = c_j$. Thus, we have

$$\|c_j\| \leq \|A\| \|e_j\| = \|A\|$$

since this is true for all $1 \leq j \leq n$, then we have $\max_{1 \leq j \leq n} \|c_j\| \leq \|A\|$.

For ith row r_i of A, we can consider $A^* = (r_1, r_2, \cdots, r_m)$. And still, we pick a vector $e_i' = (0, \cdots, 0, 1, 0, \cdots, 0)^T \in \mathbb{R}^m$, where ith entry is 1, others are all zero. And then we take $A^* e_i' = r_i$, which gives us

$$\|r_i\| \leq \|A^*\| \|e_i'\| = \|A^*\| = \|A\|$$

in the last step we used the fact that $\|A^*\| = \|A\|$. This is true for all $1 \leq i \leq m$, then we have $\max_{1 \leq i \leq m} \|r_i\| \leq \|A\|$. The proof is complete. □

Problem 3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Find the spectral radius, operator norm and Hilbert-Schmidt norm of $A : \mathbb{R}^n \to \mathbb{R}^n$.

Proof. The eigenvalues of A are 1 and 3, and then we can know that the spectral radius is $r(a) = \max|\lambda| = 3$. The operator norm of A is the largest eigenvalues of AA^T, which is

$$AA^T = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

And the charasteristic polynomial is $\lambda^2 - 14\lambda + 9 = 0$, which gives us norm of A is $\max_{j=1,2} \lambda_j = 7 + 2\sqrt{10}$. The Hilbert-Schmidt norm of A is $\|A\| = \left(\sum_{i,j} |a_{ij}^2|\right)^{1/2} = \sqrt{14}$. $\square$