Homework 4 for Math 2370

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Problem 1

Proof. We can set

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} -2 & -8 \\ 1 & 4 \end{pmatrix}$$

Then we have

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, AB = \begin{pmatrix} -18 & -36 \\ 9 & 18 \end{pmatrix}$$

which satisfy the condition that $AB = 0, BA \neq 0$. □

Problem 2

Proof. The necessary and sufficient condition for A is that A is also a diagonal matrix. □

Problem 3

Proof. We have

$$\begin{aligned} &A(A^{-1} + B^{-1})B(A+B)^{-1} \\ =&(I + AB^{-1})B(A+B)^{-1} \\ =&(B+A)B(A+B)^{-1} \\ =&I \end{aligned}$$

Then we have

$$\begin{aligned} &A(A^{-1} + B^{-1})B(A+B)^{-1} = I \\ \Rightarrow&(A^{-1} + B^{-1})B(A+B)^{-1} = A^{-1} \\ \Rightarrow&(A^{-1} + B^{-1})B(A+B)^{-1}A = A^{-1}A \\ \Rightarrow&(A^{-1} + B^{-1})B(A+B)^{-1}A = I \end{aligned}$$

Then we showed $(A^{-1} + B^{-1})$ has inverse, which is $B(A+B)^{-1}A$. □

Problem 4

Proof. We have $\sigma(p) = -1^{(n-1)+(n-2)+\cdots+1} = -1^{n(n-1)/2}$. So $\sigma(p) = 1$ if $n \geq 4$, and $\sigma(p) = -1$ if $n \leq 4$. □

Problem 5

Proof. In general, it is not true. Pick

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Then we have

$$ABC = \begin{pmatrix} 5 & 4 \\ 13 & 12 \end{pmatrix}, ACB = \begin{pmatrix} -1 & 10 \\ -3 & 22 \end{pmatrix}$$

So we have $\mathrm{tr}ABC = 17 \neq \mathrm{tr}ACB = 21$. $\square$