Homework 10 for Math 2371
Zhen Yao

Problem 1. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be linear. Show that for any subspace W of V,

$$\dim T^{-1}(W) \leq \dim N_T + \dim W.$$

Proof. Suppose $U \subset V$ such that $T(U) = W$, then for $T|_U \colon U \to W$, we have

$$\dim T(U) + \dim N_{T|_U} = \dim U = \dim T^{-1}(W).$$

Also, with $\dim N_{T|_U} \leq \dim N_T$, we have

$$\dim T^{-1}(W) \leq \dim N_T + \dim T(U) = \dim N_T + \dim W.$$

$\square$

Problem 2. Suppose A and B are $n \times n$ matrices, and $A + B$ is invertible. Prove that

$$\operatorname{rank} A + \operatorname{rank} B \geq n.$$

Also, show that

$$\operatorname{rank} A + \operatorname{rank} B = n$$

if and only if

$$R_A \cap R_B = \{0\}.$$

Proof.

(a) Since $A + B$ is invertible, then $A + B$ is full rank, which implies $\operatorname{rank}(A + B) = n$ and $N_{A+B} = \{0\}$. Then, $\dim N_{A+B} = 0$, and we have

$$\dim(N_A + N_B) = \dim N_A + \dim N_B - \dim(N_A \cap N_B).$$

Also, for $x \in N_A \cap N_B$, then $(A + B)x = 0$, hence $N_A \cap N_B \subset N_{A+B}$. Then we have $\dim(N_A \cap N_B) = 0$, which yields

$$\dim N_A + \dim N_B = \dim(N_A + N_B) \leq n.$$

With rank-nullity theorem, we have

$$\operatorname{rank} A + \operatorname{rank} B = n - \dim N_A + n - \dim N_B \geq n.$$

(b) 1) If $\operatorname{rank} A + \operatorname{rank} B = n$, with the fact that $R_{A+B} \subset R_A + R_B$, then, $\dim(R_A + R_B) = n$,

$$n = \dim(R_A + R_B) = \operatorname{rank} A + \operatorname{rank} B - \dim(R_A \cap R_B),$$

which implies $\dim(R_A \cap R_B)$. Hence, $R_A \cap R_B = \{0\}$.

2) If $R_A \cap R_B = \{0\}$, then $\dim(R_A \cap R_B)$. Thus,

$$\operatorname{rank} A + \operatorname{rank} B = \dim(R_A + R_B) - \dim(R_A \cap R_B) = n - 0 = n.$$

$\square$

Problem 3. Suppose A, B, C, D are $n \times n$ matrices satisfying

$$AB = DB, AC = 2DC.$$

Show that

$$\operatorname{rank} A + \operatorname{rank} B + \operatorname{rank} C \leq 2n.$$

Proof. Since $AB = DB$, then we have $(A - D)B = 0$ and thus $R_B \subset N_{A-D}$. Similarly, we have $R_C \subset N_{A-2D}$. Then,

$$\begin{aligned} \operatorname{rank} B &\leq \dim N_{A-D} = n - \operatorname{rank}(A - D), \\ \operatorname{rank} C &\leq \dim N_{A-2D} = n - \operatorname{rank}(A - 2D). \end{aligned}$$

Then, we want to prove that $\operatorname{rank} A \leq \operatorname{rank}(A - D) + \operatorname{rank}(A - 2D)$. $\square$

Problem 4. Suppose that $A_{n\times n}, B_{n\times m}, C_{m\times n}$ and $D_{m\times m}$ are matrices such that $\det A \neq 0$. Show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det \left(D - CA^{-1}B\right).$$

Proof. With elementary row operation, multiplying $-CA^{-1}$ with the first row and adding it to the second row yields

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B, \end{pmatrix}$$

since $\det A \neq 0$, and hence A^{-1} exists. And it is obviously that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det \left(D - CA^{-1}B\right).$$

$\square$

Problem 5. Let A, B, C, D be $n \times n$ matrices and

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

(a) Prove that

$$\det E = \det(AD - BC)$$

when all matrices A, B, C, D are diagonal.

(b) Prove that

$$\det E = \det(AD - BC)$$

when all matrices A, B, C, D are upper triangular.

(c) Prove that

$$\det E = \det(AD - BC)$$

when all matrices A, B, C, D commute.

Proof.

(a) 1) If A is invertible, then, with Problem 4, we have

$$\begin{aligned}\det E &= det A \det\left(D - CA^{-1}B\right)\\ &= \det\left(AD - ACA^{-1}B\right)\\ &= \det\left(AD - CAA^{-1}B\right)\\ &= \det\left(AD - CB\right)\\ &= \det\left(AD - BC\right),\end{aligned}$$

where in the last two step we used the fact that $AC = CA$ and $BC = CB$ since A, C, B are diagonal.

2) If A is not invertible, then there exist $\varepsilon_k \to 0$ such that

$$\det A_k = \det(A + \varepsilon_k I) \neq 0.$$

Then, we have $A_k C = CA_k$. Thus, with similar argument in 1),

$$\det E = \lim_{k\to\infty} \begin{pmatrix} A_k & B \\ C & D \end{pmatrix} = \lim_{k\to\infty} \det\left(A_k D - BC\right) = \det\left(AD - BC\right).$$

(b) If D is invertible, then similar to Problem 4, we have $\det E = \det(A - BD^{-1}C)\det(D)$. Since D is upper triangular, then so is D^{-1}. Then,

$$\begin{aligned}\det\left(A - BD^{-1}C\right) &= \prod\left(A_{ii} - B_{ii}D_{ii}^{-1}C_{ii}\right)\\ &= \prod\left(A_{ii} - B_{ii}C_{ii}D_{ii}^{-1}\right)\\ &= \det\left(A - BCD^{-1}\right).\end{aligned}$$

It follows that

$$\det E = \det\left(A - BCD^{-1}\right)\det(D) = \det(AD - BC).$$

If D is not invertible, with the similar argument in (a) 2), the result follows easily.

(c) If D is invertible, and A, B, C, D commute, then

$$\begin{aligned}\det E &= \det\Big(A - BD^{-1}C\Big)\det(D) \\ &= \det\Big(A - BD^{-1}CD\Big) \\ &= \det\Big(A - BD^{-1}DC\Big) \\ &= \det(A - BC).\end{aligned}$$

If D is not invertible, with the similar argument in (a) 2), the result follows easily.

□