Homework 12 for Math 2371
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Problem 1. Let Π be the plane in $\mathbb{R}^3$ defined by

$$x + 2y + 3z = 0.$$

Let T be the reflection in $\mathbb{R}^3$ about the plane Π. Find the matrix representation of T.

Proof. The normal vector, which perpendicular to the plane Π, is $n = (1, 2, 3)$. For any vector $r = (x, y, z)$ and its reflection $r' = (x', y', z')$, then the normal component of r'_n with respect to the plane is

$$r'_n = \frac{r \cdot n}{n \cdot n} n,$$

and then the reflected vector is

$$\begin{aligned} r' &= r - 2\frac{r \cdot n}{n \cdot n} n \\ &= \left(x - \frac{x + 2y + 3z}{7}, y - 2\frac{x + 2y + 3z}{7}, z - 3\frac{x + 2y + 3z}{7} \right). \end{aligned}$$

Thus the matrix reperesentation is

$$T = \begin{pmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{pmatrix}.$$

$\square$

Problem 2. Let V be the complex linear space consists of all 2×2 complex matrices, hence $\dim V = 4$. Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We define a linear map $T : V \to V$ such that

$$T(V) = VE - EV.$$

(a) Find all eigenvalues of the linear map T.

(b) Find the Jordan canonical form the linear map T.

Proof.

(a) For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$, we have $T(A) = \begin{pmatrix} -c & a - d \\ 0 & c \end{pmatrix}$. We can write A as $A = \begin{pmatrix} a & b & c & d \end{pmatrix}^T$, and then T can be expressed as

$$T = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalue of T is 0 with multiplicity 4.

(b) Also, $\dim N_{T-0I} = 3$, then there are three Jordan blocks in its Jordan canonical form, which is

$$J_T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

Problem 3. Let

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}.$$

Calculate

$$\sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1}.$$

Proof. The characteristic polynomial of A is $\lambda^2 - 3\lambda + 3 = 0$. Then the eigenvalues are $\lambda = 1$, and $\lambda = 2$. Also, the eigenvectors are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively. Denote by

$$S = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

then $A = S\Lambda S^{-1}$. Then,

$$\begin{aligned} \sin A &= S \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \Lambda^{2k+1} \right) S^{-1} \\ &= S \begin{pmatrix} \sin 1 & 0 \\ 0 & \sin 2 \end{pmatrix} S^{-1} \\ &= \begin{pmatrix} -\sin 1 - 2\sin 2 & -\sin 1 - \sin 2 \\ -2\sin 1 - 2\sin 2 & -2\sin 1 - \sin 2 \end{pmatrix}. \end{aligned}$$

□