Homework 11 for Math 2371

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Problem 1. Suppose V is the space of complex polynomials in x and y of total degree at most 3, i.e.

$$V = \left\{ \sum_{i,j\geq 0,\, i+j\leq 3} a_{ij}x^i y^j \middle| a_{ij} \in \mathbb{C} \right\}.$$

Consider $T : V \to V$ defined by

$$T : p \mapsto y\frac{\partial p}{\partial y}.$$

Find the Jordan canonical form of T.

Proof. For any $p \in V$, we have

$$p = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{21}x^2y + a_{12}xy^2,$$

and

$$T(p) = a_{01}y + a_{11}xy + a_{21}x^2y + 2a_{12}xy^2. \tag{1}$$

Thus, T has the form

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

And the characteristic polynomial of T is

$$(\lambda - 1)^3(\lambda - 2) = 0,$$

then the eigenvalues are $\lambda = 0, 1, 2$. For $\lambda = 0$, we know that $\dim N_A = 2$, then there are two Jordan blocks corresponding to $\lambda = 0$. For $\lambda = 1$, $\dim N_{A-I} = 3$, then there are three Jordan blocks corresponding to $\lambda = 1$. Thus, the Jordan canonical form of T is

$$J_T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$\square$

Problem 2. Let A be an $n \times n$ complex matrix with $\det A \neq 0$. Suppose A^3 is diagonalizable. Prove that A^2 is diagonalizable. Provide a counter example if $\det A = 0$.

Proof.

(a) If $\det A \neq 0$, then $\det A^3 \neq 0$. Also, A^3 is diagonalizable, then A^3 has n nonzero distinct eigenvalues. Then the Jordan canonical form J_{A^3} of A^3 has n size one blocks, each corresponding to its eigenvalues $\lambda_j, 1 \leq j \leq n$.

Now consider the Jordan canonical form J_A of A, then there exists P such that $A = PJ_AP^{-1}$ and

$$PJ_A^3P^{-1} = PJ_{A^3}P^{-1} = A^3,$$

which implies each block in J_{A^3} is power to the block in J_A, making each block in J_A a size one block. Indeed, for each λ_j, if the multiplicity of λ_j is greater than one, then its corresponding Jordan block satisfies

$$\begin{pmatrix} \lambda_j & 1 & \cdots & \cdots & 0 \\ & \lambda_j & \cdots & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 1 \\ & & & & \lambda_j \end{pmatrix}^n = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}.$$

This shows that each Jordan block of A is size one. And so is A^2, then A^2 also has n nonzero eigenvalues. Hence A^2 is also diagonalizable.

(b) Take $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and then $\det A = 0$. Then, $A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and hence A^2 is not diagonalizable since it has repeated eigenvalues $\lambda = 2$. Also, $A^3 = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$, and its eigenvalues are $\lambda = 2\sqrt{2}, -2\sqrt{2}$, which implies A^3 is diagonalizable.

$\square$

Problem 3. Let A, B be $n \times n$ matrices. Suppose A is a Hermitian matrix.

(a) Show that $A^{2015}B = BA^{2015}$ implies that $A^{2014}B = BA^{2014}$.

(b) Is it true that $A^{2014}B = BA^{2014}$ implies that $A^{2015}B = BA^{2015}$?

Proof.

(a) Since A is a Hermitian matrix, then there exists invertible matrix P such that $A = P\Lambda P^{-1}$, where $\Lambda = [\lambda_1, \cdots, \lambda_n]$ is a diagonal matrix. Let $\widetilde{B} = P^{-1}\Lambda P$, then

$$\Lambda^{2015}\widetilde{B} = \widetilde{B}\Lambda^{2015}.$$

Then for ij-th entry, we have $\lambda_i^{2015}b_{ij} = b_{ij}\lambda_j^{2015}$. Then $\lambda_i = \lambda_j$, and hence $\lambda_i^{2014}b_{ij} = b_{ij}\lambda_j^{2014}$. Hence, $A^{2014}B = BA^{2014}$.

(b) It is not true. Since similarly to argument in (a), we have $\lambda_i^{2014} b_{ij} = b_{ij} \lambda_j^{2014}$, and then $\lambda_i = \lambda_j$ or $\lambda_i = -\lambda_j$. In the case $\lambda_i = -\lambda_j$, we have $\lambda_i^{2015} b_{ij} = -b_{ij} \lambda_j^{2015}$. Thus, $A^{2015} B \neq B A^{2015}$.

$\square$

Problem 4. Let A be a complex $n \times n$ matrix which commutes with all reflection matrices of the form

$$R = I - 2vv^T,$$

where $v \in \mathbb{R}^n$ is a unit vector. Is it true that $A = kI$ for some scalar k?

Proof. For any R, we have $A\left(I - 2vv^T\right) = \left(I - 2vv^T\right)A$, then we have $Avv^T = vv^T A$. Then, A maps $N_{vv^T - \lambda I} \to N_{vv^T - \lambda I}$, for some eigenvalue λ of vv^T. Indeed, for any $x \in N_{vv^T - \lambda I}$, we have $Avv^T x - \lambda A x = 0$, with $Avv^T = vv^T A$, we have $vv^T(Ax) = \lambda(Ax)$.

Then for all $x \in N_{vv^T - \lambda I}$, there exists $\lambda \in \mathbb{C}$, we have $Ax = \lambda x$. Then, $A = kI$, otherwise, it cannot have above property. $\square$