Homework 2 for Math 2370

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Problem 1

Proof. (1)First, we check that $\{e_1, e_2, \cdots, e_n\}$ is linear independent. Suppose that there exist $a_1, a_2, \cdots, a_n \in K$ such that

$$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0$$

then for $\forall x_i \in X$, we have

$$(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)(x_i) = a_1 e_1(x_i) + \cdots + a_n e_n(x_i) = a_i = 0$$

So $a_i = 0$, for $\forall a_i$, which means $\{e_1, e_2, \cdots, e_n\}$ are linear independent.

(2)Then, we need to show that $\text{span}\{e_1, e_2, \cdots, e_n\} = X'$. For any $f \in X'$, let $b_i = f(x_i)$, and $f = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n$. Then, for $\forall x_i$, we have

$$f(x_i) = (b_1 e_1 + b_2 e_2 + \cdots + b_n e_n)(x_i) = b_i$$

Thus, f can be presented by $\{e_1, e_2, \cdots, e_n\}$. The proof is complete. □

Problem 2

Proof. (1)Since $T : X \to R$, then there $x_1 \in X$ such that $Tx \neq 0$. And we let $x_2 = \frac{x_1}{f(x_1)}$, then we have $f(x_2) = 1$. And for $\forall x \in X$we can know

$$T(x - T(x) \cdot x_2) = T(x) - T(x) = 0$$

Since T and S have the same null space, then we have $S(x - T(x) \cdot x_2) = 0$, then

$$\begin{aligned} S(x) &= S(x - T(x) \cdot x_2 + T(x) \cdot x_2) \\ &= S(x - T(x) \cdot x_2) + S(T(x) \cdot x_2) \\ &= 0 + S(x_2)T(x) \end{aligned}$$

Let $\lambda = S(x_2)$, then we proved that $S = \lambda T$.

(2)If $S = \lambda T$, then for $\forall x \in N_P$, we have $T(x) = \frac{1}{\lambda} S(x) = 0$, which means $N_P \subset N_T$. And for $\forall x \in N_T$, $S(x) = 0$, which means $N_T \subset N_P$. So $N_T = N_P$. The proof is complete. □

Problem 3

Proof. (1)Assume $(x_1, x_2, \cdots, x_n)$ is a basis of X, then we have

$$\text{dim} R_T = \text{dim} X - \text{dim} N_T$$

If $T \in L(X, U)$ is one-to-one, then $\text{dim} N_T = 0$. We can have $\text{dim} R_T = \text{dim} X = \text{dim} U$. Then $R_T = U$, which implies that T is an isomorphism. Then T is onto

(2)If T is onto, and $\text{dim} X = \text{dim} N$, then the only element $x \in X$ satisfying $Tx = 0$ is $x = 0$. So $\text{dim} N_T = 0$. Then $\text{dim} R_T = \text{dim} U$, which means $R_T = U$. Then T is an isomorphism and T is of course one-to-one. The proof is complete. □

Problem 4

Proof. Since $T \in L(X,X)$, which means $R_T = X$ and $\dim R_T = \dim X$. Also, with $\dim R_{T^2} = \dim R_T$, we have

$$\begin{aligned}
&\dim R_{T^2} + \dim N_{T^2} = \dim X \\
&\dim R_T + \dim N_{T^2} = \dim X \\
&\Rightarrow \dim N_{T^2} = 0
\end{aligned}$$

and with T being isomorphism, we have $N_{T^2} = \{0\}$.

Also, $N_{T^2} = \{Tx | T^2(x) = T(Tx) = 0\}$, $R_T = \{y | Tx = y, x \in X\}$ and $N_T = \{x | Tx = 0, x \in X\ \}$. Then we can immediately know that $N_{T^2} = R_T \cap N_T$, since for $\forall Tx \in X$, it is in R_T satisfying $y = Tx$, and it is also in N_T satisfying $T(Tx) = 0$. And since $N_{T^2} = \{0\}$, we can know $R_T \cap N_T = \{0\}$. $\square$