Homework 3 for Math 2370

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Problem 1

Proof. (1)Assume $l \in X'$ such that $l \in (Y+Z)^{\perp}$. Then we have $l(m) = 0, \forall m \in Y+Z$. Also, we know $Y \subset Y+Z$, so $l(y) = 0, \forall y \in Y$. Similarly, we have $l(z) = 0, \forall z \in Z$. Then we have $l \in Y^{\perp} \cap Z^{\perp}$, which implies $(Y+Z)^{\perp} \subset Y^{\perp} \cap Z^{\perp}$.

Now assume $l \in Y^{\perp} \cap Z^{\perp}$, then we have $l(y) = 0$ and $l(z) = 0$, for $\forall y \in Y, \forall z \in Z$. Then, for $\forall m \in Y+Z$ we have $l(m) = 0$, since $m = y+z$ for some $y \in Y$ and $z \in Z$. Thus, we have $l \in (Y+Z)^{\perp}$, which implies $Y^{\perp} \cap Z^{\perp} \subset (Y+Z)^{\perp}$. Now we proved that $(Y+Z)^{\perp} = Y^{\perp} \cap Z^{\perp}$.

(2)It is equivalent to show that $(Y \cap Z)^{\perp\perp} = Y \cap Z = (Y^{\perp} + Z^{\perp})^{\perp}$.

If $l \in (Y^{\perp} + Z^{\perp})^{\perp}$, we have $l(l_1 + l_2) = 0$, for $l_1 \in Y^{\perp}$ and $l_2 \in Z^{\perp}$. Also, we have $Y^{\perp} \subset Y^{\perp} + Z^{\perp}$, we have $l(l_1) = 0$ for $\forall l_1 \in Y^{\perp}$. Similarly, we have $l(l_2) = 0$ for $\forall l_2 \in Z^{\perp}$. Then $l \in Y^{\perp\perp} = Y$ and $l \in Z^{\perp\perp} = Z$. Thus, $l \in Y \cap Z$, which implies $(Y^{\perp} + Z^{\perp})^{\perp} \subset Y \cap Z$.

If $l \in Y \cap Z = (Y \cap Z)^{\perp\perp}$, we have $l \in Y \cap Z$, then $l \in Y = Y^{\perp\perp}$ and $l \in Z = Z^{\perp\perp}$. Then we have $l(l_1) = 0, l_1 \in Y^{\perp}$ and $l(l_2) = 0, l_2 \in Z^{\perp}$. Thus, we have $l(l_1 + l_2) = 0, l_1 + l_2 \in Y^{\perp} + Z^{\perp}$, which implies $l \in (Y^{\perp} + Z^{\perp})^{\perp}$. Then we have $Y \cap Z \subset (Y^{\perp} + Z^{\perp})^{\perp}$.

Finally, we have $(Y \cap Z)^{\perp} = Y^{\perp} + Z^{\perp}$. □

Problem 2

Proof. (1)For $\forall l \in Y'$ and $x_1, x_2 \in X$, we have

$$\begin{aligned} &(T'l)(x_1) = (T'l)(x_2) \\ \Rightarrow &(T'l)(x_1 - x_2) = 0 \\ &l(T(x_1 - x_2)) = 0 \end{aligned}$$

Since T is invertible, then it is one-to-one and onto, which means $x_1 = x_2$ if $T(x_1 - x_2) = 0$. Then we have T' is also one-to-one.

Also, if $(T'l)(x) = 0, x \in X$, then it implies $l(T(x)) = 0$. Since T is onto, then only x such that $l(T(x)) = 0$ is $x = 0$. Then T' is also onto. Thus, T' is invertible.

(2)Since T is invertible, then $T \circ T^{-1} = I$. For T', we denote

$$\begin{aligned} &(T \circ T^{-1})' = I' \\ \Rightarrow &(T^{-1})' \circ T' = I' \end{aligned}$$

We need to show $I = I'$. For $y \in Y$, we have

$$\begin{aligned} &(I'l, y) = (l, I(y)) = (l, y) \\ \Rightarrow &I' = I \end{aligned}$$

The we have $(T^{-1})' = (T')^{-1}$. □

Problem 3

Proof. Since X is n-dimensional linear space, and $T \in L(X, X)$, then T can be presented as an $n \times n$ matrix. Polynomials $p(T)$ can be viewed as an operator acting on the space of matrix T, which is n^2-dimensional, denoted by P. Since $\dim P = n^2$, then for any $T \in P$,

$1, T, T^2, \cdots, T^{n^2}$ must be linear dependent, since there are n^2+1 elements. So there exist $a_0, a_1, a_2, \cdots, a_{n^2}$ such that

$$p(T) = a_0 \cdot 1 + a_1 T + a_2 T^2 + \cdots + a_{n^2} T^{n^2} = 0$$

which is at most degree n^2. The proof is complete. $\square$

Problem 4

Proof. Since $ST \in L(U, W)$, we have

$$\begin{aligned} \dim N_{ST} &= \dim U - \dim R_{ST} \\ &= \dim N_T + \dim R_T - \dim R_{ST} \\ &\leq \dim N_T + \dim V - \dim R_{ST} \end{aligned}$$

since $L \in L(U, V)$ and $\dim R_T \leq \dim V$. Then we have

$$\begin{aligned} \dim N_{ST} &\leq \dim N_T + \dim V - \dim R_{ST} \\ &= \dim N_T + \dim V - \dim R_S + \dim R_S - \dim R_{ST} \\ &= \dim N_T + \dim N_S + \dim R_S - \dim R_{ST} \end{aligned}$$

Now we only need to prove $\dim R_S - \dim R_{ST} \geq 0$. For $\forall w \in R_{ST}$, we have $ST(u) = w$ for some $u \in U$. Then $S(Tu) = w$, where $Tu \in V$. Thus, for $\forall w \in R_{ST}$, there exists a $v = Tu \in V$ such that $S(v) = w$, then $w \in R_S$. Thus, we have $R_{ST} \subset R_S$, which means $\dim R_S - \dim R_{ST} \geq 0$. This implies $\dim N_{ST} \leq \dim N_T + \dim N_S$. $\square$

Problem 5

Proof. If the linear map P satisfies $Pu = u$ and $Pv = 0$, then we have $P^2(u) = P(Pu) = u$ and $P^2(v) = P(Pv) = P(0)$. Since it is in $\mathbb{R}^2$ space, linear map P can be presented as a matrix and $P(0) = 0$. Thus we have $P^2(v) = P(v) = 0$, then P is indeed a projection.

If there exists another projection P' such that $P'(u) = u, P'(v) = 0$. Then we have $(P - P')(u) = 0 = (P - P')(v)$. Since u, v are linearly independent, then $P - P' = 0$. So there exists a unique projection.

For $u = \left(\frac{3}{5}, \frac{4}{5}\right)$, $v = \left(\frac{4}{5}, \frac{4}{5}\right)$, we have

$$P = \begin{vmatrix} -\frac{9}{7} & \frac{12}{7} \\ -\frac{12}{7} & \frac{16}{7} \end{vmatrix}$$

$\square$