Homework 6 for Math 2370

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Problem 1. Let A be an invertible $n \times n$ matrix, show that there exists a polynomial g such that

$$A^{-1} = g\left(A\right).$$

Proof. Since A is invertible, then A has no engenvalues. Thus, the characteristic polynomial $P(x)$ for A has constant terms, which can be written as $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Also, we know that $P(A) = 0$, thus we have

$$\begin{aligned} &A^n + a_{n-1}A^{n-1} + \cdots + a_0 = 0 \\ \Rightarrow &A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1) = g(A) \end{aligned}$$

Then $A^{-1} = g(A)$, the proof is complete. $\square$

Problem 2. Let

$$A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}.$$

Show that the minimal polynomial m_A is the least common multiple of m_{A_1} and m_{A_2}.

Proof. From the form of A, we can know that $\det(\lambda - IA) = \det(\lambda - IA_1)\det(\lambda - IA_1)$. Then, for any polynomial $T(x)$ such that $T(A) = 0$, then we have $T(A_1) = 0$ and $T(A_2) = 0$. And since m_A, m_{A_1} and m_{A_2} are minimal polynomials corresponding to A, A_1 and A_2, then we have $T(x) = m_1 m_{A_1}(x)$ and $T(x) = m_2 m_{A_1}(x)$ for some m_1, m_2. Also, we have $m_A(x)|T(x)$, then we have $m_{A_1}(x)|m_A(x)$ and $m_{A_1}(x)|m_A(x)$, then m_A is the least common multiple of m_{A_1} and m_{A_2}. $\square$

Problem 3. Find the minimal polynomial m_A for

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Proof. The characteristic polynomial for A is that $P(\lambda) = (\lambda - 1)(\lambda - 2)^2$. Then the minimal polynomial is $m_A = (\lambda - 1)(\lambda - 2)$. $\square$

Problem 4. Let A be an $n \times n$ matrix where $n \geq 2$ satisfying rank $A = 1$.
(i) Show that there exists two column vectors a, b such that $A = ab^T$.
(ii) Show that the minimal polynomial

$$m_A = \lambda^2 - \left(a^T b\right)\lambda.$$

Proof. (i)Since rank$A = 1$, then the image of A is one-dimensional. Thus, there exist $u, v \in \mathbb{R}^n$ such that $Au = kv$ for a fixed v. It also holds for a basis for $\mathbb{R}^n$, then every column of A is a multiple of v. Then there exists $(w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$, such that

$$A = v(w_1, w_2, \cdots, w_n)$$

then we denote $v = a$, and $(w_1, w_2, \cdots, w_n) = b^T$, where $a, b \in \mathbb{R}^n$.Then $A = ab^T$.

(ii)We have $A^2 = ab^Tab^T = a(b^Ta)b^T = (b^Ta)ab^T = (b^Ta)A$, which implies $q(A) =$

$A^2 - (b^T a)A = 0$. This polynomial satisfies that $q(A) = 0$, then $m_a | q(\lambda) = \lambda^2 - (b^T a)\lambda$. Also, m_A cannot be λ or $\lambda - (b^T a)$, since this means A is a scalar. Thus, $m_A = \lambda^2 - (b^T a)\lambda$. The proof is complete. □