Homework 10 for Math 2370

Zhen Yao

Problem 1. Suppose A and B are normal complex $n \times n$ matrices. Prove that

$$r(AB) \leq r(A)r(B).$$

Here $r(\cdot)$ is the spectral radius of a matrix. Find a counter example if A or B is not normal.

Proof. We have $r(AB) \leq \|AB\|$, since if λ be an eigenvalue of AB, then for $x \in \mathbb{C}^n, x \neq 0$ being corresponding eigenvector, we have

$$\begin{aligned} ABx &= \lambda x \\ \Rightarrow \|AB\|\|x\| &\geq \|ABx\| = |\lambda|\|x\| \\ \Rightarrow \|AB\| &\geq |\lambda| \end{aligned}$$

Also, we have $\|AB\| \leq \|A\|\|B\|$. And with A, B being normal matrices, we know $\|A\| = r(A)$ and $\|B\| = r(B)$. Thus, with all the results above, we have

$$r(AB) \leq \|AB\| \leq \|A\|\|B\| = r(A)r(B)$$

The proof is complete.

Take $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and A, B are not normal. We can compute that $r(AB) = \sqrt{3}$ and $r(A)r(B) = 1 \cdot 1 = 1 < r(AB)$. This is a counter example if A and B are not normal. □

Problem 2. What is the operator norm of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix}$$

in the standard Euclidean structures of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Proof. Denote the matrix above by A, then the operator norm of A is $\sqrt{r(A^*A)} = \sqrt{\frac{15+\sqrt{137}}{2}}$. □

Problem 3. Let $\{\lambda_i\}_{i=1}^n$ be eigenvalues of matrix $A = (a_{ij})_{n\times n}$. Show that

$$\sum_{j=1}^n |\lambda_j|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2.$$

Proof. With Schur decomposition, we cnould know that $A = QUQ^*$, where Q is unitary and U is upper triangular and its diagonal entries are engenvalues of A, since A and U are similar. And we can show that Hilbert-Schwarz norm norm $\|A\|_{HS} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$ is invariant under unitary matrix multiplication:

$$\|QA\|_{HS}^2 = \operatorname{tr}\left((QA)^*(QA)\right) = \operatorname{tr}\left(A^*Q^*QA\right) = \operatorname{tr}\left(A^*A\right) = \|A\|_{HS}^2$$

then we can have

$$\|A\|_{HS}^2 = \|QAQ^*\|_{HS}^2 = \|U\|_{HS}^2$$

Also we can know that

$$\sum_{j=1}^{n} |\lambda_j|^2 \leq \sum_{i,j=1}^{n} |u_{ij}|^2 = \|A\|_{HS}^2$$

since the square sum of all diagonal entries of U is smaller than that of all entries of U. $\square$

Problem 4. Let $A = (a_{ij})_{n\times n}$ be normal. Show that

$$r(A) \geq \max_{1\leq i\leq n} |a_{ii}| .$$

Proof. Since A is normal matrix, then we have $\|A\| = r(A)$. Also, we have known that for all a_{ij}, $|a_{ij}| \leq \|A\|$. Thus, we have $r(A) \geq \max_{1\leq i\leq n} |a_{ii}|$. $\square$