Homework 9 for Math 2370
Zhen Yao

Problem 1. Let

$$q\left(x\right) = 2x_1x_2 - 6x_2x_3 + 2x_1x_3.$$

Find an invertible matrix L, such that

$$q\left(L^{-1}y\right) = d_1y_1^2 + d_2y_2^2 + d_3y_3^2$$

where $d_i = 0$ or ± 1.

Proof. We have $q(x) = (x, Hx)$, where

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix}$$

Now we need to normalize the matrix H, and we can compute for its eigenvalues, which are $\lambda = 3, \frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}$, with eigenvectors

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3-\sqrt{17}}{2} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3+\sqrt{17}}{2} \\ 1 \\ 1 \end{pmatrix},$$

Now we can normalize these vectors and we get

$$\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{17-3\sqrt{17}}{34}} \\ \frac{2}{\sqrt{17-3\sqrt{17}}} \\ \frac{2}{\sqrt{17-3\sqrt{17}}} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{17+3\sqrt{17}}{34}} \\ \frac{2}{\sqrt{17+3\sqrt{17}}} \\ \frac{2}{\sqrt{17+3\sqrt{17}}} \end{pmatrix},$$

And we arrange eigenvectors into a matrix, denoting it by

$$C = \begin{pmatrix} 0 & -\sqrt{\frac{17-3\sqrt{17}}{34}} & \sqrt{\frac{17+3\sqrt{17}}{34}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17-3\sqrt{17}}} & \frac{2}{\sqrt{17+3\sqrt{17}}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17-3\sqrt{17}}} & \frac{2}{\sqrt{17+3\sqrt{17}}} \end{pmatrix}$$

We can verify that $C^*HC = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3-\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{3+\sqrt{17}}{2} \end{pmatrix}$. Now we denote $z = Cx = (z_1, z_2, z_3)$, where

$$\begin{aligned} z_1 &= -\sqrt{\frac{17-3\sqrt{17}}{34}}x_2 + \sqrt{\frac{17+3\sqrt{17}}{34}}x_3 \\ z_2 &= -\frac{1}{\sqrt{2}}x_1 + \frac{2}{\sqrt{17-3\sqrt{17}}}x_2 + \frac{2}{\sqrt{17+3\sqrt{17}}}x_3 \\ z_3 &= \frac{1}{\sqrt{2}}x_1 + \frac{2}{\sqrt{17-3\sqrt{17}}}x_2 + \frac{2}{\sqrt{17+3\sqrt{17}}}x_3 \end{aligned}$$

and we need to change variable to get the quadratic form $q\left(L^{-1}y\right) = d_1y_1^2 + d_2y_2^2 + d_3y_3^2$. We make the change of variable

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{3}}z_1 \\ y_2 &= \sqrt{\frac{2}{3-\sqrt{17}}}z_2 \\ y_3 &= \sqrt{\frac{2}{3+\sqrt{17}}}z_3 \end{aligned}$$

and we can denote this transform by matrix E, where

$$E = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3-\sqrt{17}}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3+\sqrt{17}}} \end{pmatrix}$$

then we can know that $L^{-1} = CE$, which are defined above. And finally, $L = (CE)^{-1}$. $\square$

Problem 2. Show that the congruence is an equivalence relation for symmetric matrices. Find the total number of equivalence classes for $n \times n$ symmetric matrices.

Proof. We denote the relation of congruence by $\sim$.

(1) For A is a symmetric matrix, then we have $A \sim A$, since $A = I^TAI$, where I is identity matrix.

For A, B are symmetric matrices, we have if $A \sim B$, then $B \sim A$. Since if $A = S^TBS$, where S is invertible, then we have $B = (S^T)^{-1}AS^{-1}$, which means $B \sim A$.

For A, B and C are symmetric matrices, we have if $A \sim B, B \sim C$, then $A \sim C$. Since if we have $A = S^TBS$ and $B = P^TCP$, then we have $A = S^TP^TCPS = (PS)^TCPS$, which implies $A \sim C$. Then we proved the congruence is an equivalence relation.

(2) Suppose $A = S^TBS$, and S is invetible. Also, we have $R_{BS} \subseteq R_B$ with equality when S is invertible, since S is full rank. Then we have, in this case, $\dim B = \dim BS$. Then we have S^T is also full rank and $\dim A = \dim S^TBS = \dim B$. So we can know that for symmetric matrices A and B, if they are congruent then they have the same rank, which means there are $n+1$ equivalebce classes, since there are matrix with rank $0, 1, 2, \cdots, n$, which is $n+1$ possibilities. $\square$

Problem 3. Let A, B be two $n \times n$ real orthogonal matrices satisfying

$$\det A + \det B = 0.$$

Show there exists a unit vector x such that

$$Ax = -Bx.$$

Proof. Since A and B are orthogonal matrices, then we have $\det A = \det B = \pm 1$ and $A^TA = B^TB = I$. We prove by contradiction, and suppose that there does not exist $x \in \mathbb{R}^n$ and $\|x\| = 1$ such that $Ax = -Bx$.

Then for $\forall x$, $(A+B)x \neq 0$, then $(A^T+B^T)(A+B)x \neq 0$. Then we have, for $\forall x \neq 0$

$$\begin{aligned}
&(A^TA + A^TB - B^TA - B^TB)x \neq 0 \\
\Rightarrow &(A^TB - B^TA)x \neq 0 \\
\Rightarrow &A^TB - B^TA \neq 0 \\
\Rightarrow &\det A^TB \neq \det B^TA \\
\Rightarrow &\frac{1}{\det A}\det B \neq \frac{1}{\det B}\det A \\
\Rightarrow &(\det A)^2 \neq (\det B)^2
\end{aligned}$$

Since $\det A + \det B = 0$, and $\det A = \det B = \pm 1$, without losing generality, we can assume $\det A = 1$ and $\det B = -1$. Then this contradicts $(\det A)^2 \neq (\det B)^2$. The proof is complete. $\square$