**Homework 9 for Math 2371**

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**Problem 1.** Let $S_n$ be the linear space consisting of $n \times n$ real symmetric matrices. For any $A \in M_n$, denoted by $r(A)$ the spectral radius of $A$. Is $r$ a norm for $S_n$?

*Proof.* $r$ is a norm for $S_n$. Suppose $A \in S_n$, then we have $A^T = A$. Also, we already knew that $\|A\|_2 = \sqrt{r(A^T A)}$. Symmetric matrix has real eigenvalues, then $r(A^T A) = r(A^2) = r(A)^2$, since

$$A^2 v = A(Av) = A(\lambda v) = \lambda^2 v$$

for some eigenvalue $\lambda$ of $A$ and its corresponding eigenvector $v$. Thus, we have $\|A\|_2 = r(A)$. □

**Problem 2.** Let $X = \mathbb{R}^n$ be the normed linear space with $l^p$ norm for some $1 \leq p \leq n$. Let $T \in X'$ be such that

$$Tx = \sum_{k=1}^{n} k x_k.$$

Find the operator norm $\|T\|$.

*Proof.*

(1) First, if $\|x\|_p = 1$, with Hölder's inequality, we have

$$\begin{aligned} |Tx| &\leq \sum_{k=1}^{n} |k x_k| \\ &\leq \left( \sum_{k=1}^{n} k^q \right)^{\frac{1}{q}} \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Then we have

$$\|T\|_{op} = \sup_{\|x\|_p = 1} |Tx| \leq \left( \sum_{k=1}^{n} k^q \right)^{\frac{1}{q}}.$$

(2) Also, we can choose $x$ such that $\|x\|_p = 1$, then we have

$$\|T\|_{op} \geq |Tx| = \left( \sum_{k=1}^{n} k^q \right)^{\frac{1}{q}}.$$

Thus, we have

$$\|T\|_{op} = \left( \sum_{k=1}^{n} k^q \right)^{\frac{1}{q}}.$$

where $\frac{1}{q} + \frac{1}{p} = 1$. □

**Problem 3.** Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ be normed linear spaces with $l^\infty$ norm. Let $T \in \mathcal{L}(X, Y)$ represented by the $m \times n$ matrix $(t_{ij})_{m\times n}$, i.e.,

$$y_i = \sum_{j=1}^{n} t_{ij} x_j.$$

Find the operator norm $\|T\|$.

*Proof.*

(1) First, we have

$$\begin{aligned}|y_i| &\leq \sum_{j=1}^{n} |t_{ij}| \, |tx_j| \\ &\leq \left( \sum_{j=1}^{n} |t_{ij}| \right) \|x\|_\infty.\end{aligned}$$

Then we have

$$\|T\|_{op} \leq \sup_{\|x\|_\infty = 1} \|Tx\|_\infty \leq \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |t_{ij}| \right\}.$$

(2) Second, suppose the maximum of the right hand side of the above equation is attained at $i = i_0$. Let $x$ be the vector such that $x_j = \operatorname{sgn} t_{i_0 j}$, where

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

then we have $\|x\|_\infty = 1$ and

$$\|Tx\|_\infty = \sum_{j=1}^{n} |t_{i_0 j}|.$$

Since $\|T\|_{op} \geq \|Tx\|_\infty$, we have

$$\|T\|_{op} \geq \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |t_{ij}| \right\}.$$

Thus, we have

$$\|T\|_{op} = \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |t_{ij}| \right\}.$$

□