Homework 6 for Math 2371
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Problem 1. Let A be a 2×2 complex matrix. Show that $\|A\|= r(A)$ if and only if A is normal. Show that the statement is not true if A is an $n \times n$ matrix with $n \geq 3$.

Proof.

(1) Suppose $A_{2\times 2}$ is complex matrix and $AA^* = A^*A$.

a) $r(A) \leq \|A\|$. Indeed, for eigenvalue λ_j and the corresponding eigenvector x_j, we have $Ax_j = \lambda_j x_j$ and

$$\|Ax_j\|= |\lambda_j| \, \|x_j\|\leq \|A\|\|x_j\|.$$

Hence, $\max|\lambda_j|= r(A) \leq \|A\|$.

b) A is normal, then there exists orthogonal basis of X consisting of eigenvectors of A. For any $x \in X$, $x = \sum_{i=1}^n c_j x_j$, applying A to x gives

$$Ax = \sum_{i=1}^n c_j \lambda_j x_j.$$

Then we have

$$\frac{\|Ax\|}{\|x\|} = \left(\frac{\sum_{i=1}^n c_j^2 \lambda_j^2}{\sum_{i=1}^n c_j^2} \right)^{\frac{1}{2}} \leq \max \lambda_j = r(A).$$

Hence, $\|A\|= \frac{\|Ax\|}{\|x\|} \leq r(A)$.

Thus, $\|A\|= r(A)$.

(2) If $\|A\|= r(A)$, then we set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Also, we have $\|A\|= \sqrt{A^*A}$. Then after computing, we have $b = c$, which implies A is normal.

(3) Now take $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, then $\|A\|= 2 = \sqrt{r(A^*A)}$. However, $AA^* - A^*A \neq 0$, which implies A is not normal.

□

Problem 2. Construct a continuous matrix function $A(t)$ where $A(t)$ is an anti-symmetric real 3×3 matrix for each t such that the unique solution to

$$\begin{cases} M_t = A(t)M, \\ M(0) = I, \end{cases}$$

is not give by $\exp\left(\int_0^t A(s)\, ds\right)$.

Proof. We can set A as

$$A = \begin{pmatrix} 0 & t & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we can solve for M, which is

$$M = \begin{pmatrix} \sin\frac{t^2}{2} + \cos\frac{t^2}{2} & \sin\frac{t^2}{2} & 0 \\ \cos\frac{t^2}{2} - \sin\frac{t^2}{2} & \cos\frac{t^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also, we have

$$e^{\int_0^t A(s)\,ds} = \begin{pmatrix} 1 & \frac{1}{2}t^2 & 0 \\ -\frac{1}{2}t^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then when A depends on t and $A(t)$ does not commute with $\int_0^t A(s)\,ds$, the solution cannot be represented by $e^{\int_0^t A(s)\,ds}$. □

Problem 3. Let

$$K = \left\{(x, y) \in \mathbb{R}^2 : |x|< 1, |y|< 1\right\}.$$

Show that K is convex and find its gauge function p_K.

Proof.

(1) For any $x = (x_1, x_2), y = (y_1, y_2) \in K$ and $t \in (0, 1)$, we define $z = (tx_1 + (1 - t)y_1, tx_2 + (1 - t)y_2)$. And we have

$$|tx_1 + (1 - t)y_1|\leq t|x_1|+(1 - t)|x_2|\leq \max\{x_1, x_2\} < 1,$$

similarly, $|tx_2 + (1 - t)y_2|< 1$. Thus we have $z \in K$, and hence K is convex.

(2) $p_K = \max\{x, y\}$.

□