Homework 5 for Math 2371

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Problem 1. Let U and V be two real linear spaces with $\dim U, \dim V \geq 2$. Show that there exists $w \in U \otimes V$ such that for any $u \in U$ and $v \in V$, $w \neq u \otimes v$.

Proof. Consider $U = V = \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$. Then the tensor product any two elements $u = (a_1, a_2), v = (b_1, b_2)$ of $\mathbb{R}^2$ is

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) = \sum_{i,j} a_i b_j (e_i \otimes e_j).$$

Then we consider $w = e_1 \otimes e_1 + e_2 \otimes e_2$, and we can have $a_1 b_1 = a_2 b_2 = 1$, and $a_1 b_2 = 0$. Then there is a contradiction, since we cannot find u, v such that $a_1 b_1 = a_2 b_2 = 1, a_1 b_2 = a_2 b_1 = 0$. Then there exists $w \in U \otimes V$ such that for any $u \in U$ and $v \in V$, $w \neq u \otimes v$. $\square$

Problem 2. Let U and V be two real linear spaces with $\dim U, \dim V \geq 2$. Suppose u_1, u_2 are linearly independent in U and u_1, u_2 are linearly independent in V. Show that the four vectors $u_1 \otimes v_1, u_1 \otimes v_2, u_2 \otimes v_1$ and $u_2 \otimes v_2$ are linearly independent.

Proof.

(1) If $\dim U = \dim V = 2$, then we are done, since $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are basis for U and V respectively.

(2) Otherwise, we assume $\dim U = m$ and $\dim V = n$. Then we can extend $\{u_1, u_2\}$ and $\{v_1, v_2\}$ into $\{u_1, \cdots, u_m\}$ and $\{v_1, \cdots, v_n\}$, which are basis for U and V respectively. Then it is obvious that $\{u_j \otimes v_j\}$ are linearly independent.

$\square$

Problem 3. Let A be an $n \times n$ matrix. Suppose that for any $1 \leq i \leq n$,

$$|a_{ii}| > \sum_{1 \leq j \leq n, j \neq i} |a_{ij}|.$$

Show that A is invertible.

Proof. Suppose there exists $x = (x_1, \cdots, x_n) \neq 0$ such that $Ax = 0$, then we have $\sum_{j=1}^n a_{ij} x_j = 0, i = 1, 2, \cdots, n$. Let $x_k = \max_j |x_j|$, and we have

$$\begin{aligned} a_{kk} x_k &= -\sum_{j \neq k} a_{kj} x_j \\ \Rightarrow a_{kk} &= -\sum_{j \neq k} a_{kj} \frac{x_k}{x_j} \end{aligned}$$

which implies

$$|a_{kk}| = \left| \sum_{j \neq k} a_{kj} \frac{x_k}{x_j} \right| \leq \sum_{j \neq k} |a_{kj}| \left| \frac{x_k}{x_j} \right| < \sum_{j \neq k} |a_{kj}|,$$

where we used the fact that $\left| \frac{x_k}{x_j} \right| \leq 1$. Then this is a contradiction to the assumption. Then 0 is not an eigenvalue of A, hence A is invertible. $\square$

Problem 4. Let $A = (a_{ij})_{n\times n}$ be a matrix such that

$$|a_{ij}| \leq 1, \quad \text{for all } 1 \leq i, j \leq n.$$

Show that there exists a constant $\Lambda > 0$ such that

$$|\lambda| \leq \Lambda$$

holds for any eigenvalue λ of A. Find the smallest such Λ and justify your answer.

Proof. With the norm of A and the spectral radius $r(A)$, we have

$$|\lambda| \leq |r(A)| \leq \|A\| \leq \|A\|_{HS} = \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}} \leq n,$$

which implies that there exists a constant $\Lambda > 0$ such that $|\lambda| \leq \Lambda$. $\square$