Homework 1 for Math 2371

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Problem 1. Let $A > 0$. Show that

$$A + A^{-1} \geq 2I.$$

Proof. Since $A > 0$, then there exists a unique matrix U such that $A = U\Lambda U^*$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}.$$

and $\lambda_j > 0, 1 \leq j \leq n$. Then, we have

$$\begin{aligned} A + A^{-1} - 2I &= U\Lambda U^* + U\Lambda^{-1}U^* - 2UIU^* \\ &= U\left(\Lambda + \Lambda^{-1} - 2I\right)U^* \\ &= U \begin{pmatrix} \lambda_1 + \lambda_1^{-1} - 2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n + \lambda_n^{-1} - 2 \end{pmatrix} U^* \\ &= U\overline{\Lambda}U^*. \end{aligned}$$

and we can use the function $f(x) = x + \frac{1}{x} - 2$ to obtain that if $x > 0$, then $f(x) \geq 0$. Thus, we can know every diagonal entry of $\overline{\Lambda}$ is greater or equal to 0, which implies $A + A^{-1} - 2I = U\overline{\Lambda}U^* \geq 0$. □

Problem 2. Suppose $A, B > 0$. Show that

$$\det(A + B) \geq 2\sqrt{\det A \det B}.$$

Proof. With inequality $\det(tA + (1-t)B) \geq (\det A)^t(\det B)^{1-t}$, we can substitute A, B by $2A, 2B$, which also satisfy $2A, 2B > 0$ and choose $t = 1/2$, then we have

$$\begin{aligned} \det(A + B) &= \det\left(\frac{1}{2}(2A) + \frac{1}{2}(2B)\right) \\ &\geq (\det 2A)^{\frac{1}{2}}(\det 2B)^{\frac{1}{2}} \\ &= \sqrt{2^{n+1}\det A \det B} \\ &= 2^n\sqrt{\det A \det B} \\ &\geq 2\sqrt{\det A \det B}, \text{ for } n \geq 1. \end{aligned}$$

where in the third step we assume A, B are $n \times n$ matrices. Thus the proof is complelte. □

Problem 3. Let A, B be two real positive matrices. Suppose that $AB = BA$, show that $AB > 0$.

Proof. Since $Ab = BA$, we can know that A, B have the same eigensystem, i.e., if x_j is an eigenvector of A corresponding to eigenvalue λ_j, then Bx_j is also an eigenvector of A and x_j is also an eigenvector of B.

Let $x_1, \cdots, x_n$ be a basis consisting of eigenvectors of B corresponding to eigenvalues $\mu_1, \cdots, \mu_n$. Then we have

$$(x_j, ABx_j) = (x_j, A\mu_j x_j) = (x_j, \mu_j A x_j)$$

also, since $A, B > 0$, then all $\mu_j > 0$, which imlpies $(x_j, \mu_j A x_j) > 0$ for all x_j. Thus, $AB > 0$. □

Problem 4. Given m positive numbers $r_j, 1 \leq j \leq m$. Show that

$$G = \left(\frac{1}{r_i + r_j + 1}\right)_{m \times m}$$

is positive definite.

Proof. Let $f_j = x^{r_j}, 1 \leq j \leq m$, and we define $(f, g) = \int_0^1 f(x)g(x)dx$, then we have

$$(f_i, f_j) = \int_0^1 x^{r_i} x^{r_j} dx = \frac{1}{r_i + r_j + 1}.$$

Thus, G is a Gram matrix, then G is positive definite. □