Homework 9 for Math 2370
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Problem 1. Let
$$q(x) = 2x_1x_2 - 6x_2x_3 + 2x_1x_3.$$
Find an invertible matrix L, such that
$$q\left(L^{-1}y\right) = d_1y_1^2 + d_2y_2^2 + d_3y_3^2$$
where $d_i = 0$ or ± 1.

Proof. We have $q(x) = (x, Hx)$, where
$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{pmatrix}$$
Now we need to normalize the matrix H, and we can compute for its eigenvalues, which are $\lambda = 3, \frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}$, with eigenvectors
$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3-\sqrt{17}}{2} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3+\sqrt{17}}{2} \\ 1 \\ 1 \end{pmatrix},$$
Now we can normalize these vectors and we get
$$\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{17-3\sqrt{17}}{34}} \\ \frac{2}{\sqrt{17-3\sqrt{17}}} \\ \frac{2}{\sqrt{17-3\sqrt{17}}} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{17+3\sqrt{17}}{34}} \\ \frac{2}{\sqrt{17+3\sqrt{17}}} \\ \frac{2}{\sqrt{17+3\sqrt{17}}} \end{pmatrix},$$
And we arrange eigenvectors into a matrix, denoting it by
$$C = \begin{pmatrix} 0 & -\sqrt{\frac{17-3\sqrt{17}}{34}} & \sqrt{\frac{17+3\sqrt{17}}{34}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17-3\sqrt{17}}} & \frac{2}{\sqrt{17+3\sqrt{17}}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17-3\sqrt{17}}} & \frac{2}{\sqrt{17+3\sqrt{17}}} \end{pmatrix}$$
We can verify that $C^*HC = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3-\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{3+\sqrt{17}}{2} \end{pmatrix}$. Now we denote $z = Cx = (z_1, z_2, z_3)$, where
$$\begin{aligned} z_1 &= -\sqrt{\frac{17-3\sqrt{17}}{34}}x_2 + \sqrt{\frac{17+3\sqrt{17}}{34}}x_3 \\ z_2 &= -\frac{1}{\sqrt{2}}x_1 + \frac{2}{\sqrt{17-3\sqrt{17}}}x_2 + \frac{2}{\sqrt{17+3\sqrt{17}}}x_3 \\ z_3 &= \frac{1}{\sqrt{2}}x_1 + \frac{2}{\sqrt{17-3\sqrt{17}}}x_2 + \frac{2}{\sqrt{17+3\sqrt{17}}}x_3 \end{aligned}$$

and we need to change variable to get the quadratic form $q\left(L^{-1}y\right) = d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2$. We make the change of variable

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{3}} z_1 \\ y_2 &= \sqrt{\frac{2}{3-\sqrt{17}}} z_2 \\ y_3 &= \sqrt{\frac{2}{3+\sqrt{17}}} z_3 \end{aligned}$$

and we can denote this transform by matrix E, where

$$E = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3-\sqrt{17}}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3+\sqrt{17}}} \end{pmatrix}$$

then we can know that $L^{-1} = CE$, which are defined above. And finally, $L = (CE)^{-1}$. $\square$

Problem 2. Show that the congruence is an equivalence relation for symmetric matrices. Find the total number of equivalence classes for $n \times n$ symmetric matrices.

Proof. We denote the relation of congruence by $\sim$.

(1) For A is a symmetric matrix, then we have $A \sim A$, since $A = I^T A I$, where I is identity matrix.

For A, B are symmetric matrices, we have if $A \sim B$, then $B \sim A$. Since if $A = S^T B S$, where S is invertible, then we have $B = (S^T)^{-1} A S^{-1}$, which means $B \sim A$.

For A, B and C are symmetric matrices, we have if $A \sim B, B \sim C$, then $A \sim C$. Since if we have $A = S^T B S$ and $B = P^T C P$, then we have $A = S^T P^T C P S = (PS)^T C P S$, which implies $A \sim C$. Then we proved the congruence is an equivalence relation.

(2) Suppose $A = S^T B S$, and S is invetible. Also, we have $R_{BS} \subseteq R_B$ with equality when S is invertible, since S is full rank. Then we have, in this case, $\dim B = \dim BS$. Then we have S^T is also full rank and $\dim A = \dim S^T B S = \dim B$. So we can know that for symmetric matrices A and B, if they are congruent then they have the same rank, which means there are $n+1$ equivalebce classes, since there are matrix with rank $0, 1, 2, \cdots, n$, which is $n+1$ possibilities. $\square$

Problem 3. Let A, B be two $n \times n$ real orthogonal matrices satisfying

$$\det A + \det B = 0.$$

Show there exists a unit vector x such that

$$Ax = -Bx.$$

Proof. Since A and B are orthogonal matrices, then we have $\det A = \det B = \pm 1$ and $A^T A = B^T B = I$. Also, with $\det A + \det B = 0$, we have $\det A \det B = -1$. Now consider

$$\begin{aligned} \det(A+B) &= \det(A(A^T + B^T)B) = \det A \det(A^T + B^T) \det B \\ &= -\det(A^T + B^T) = -\det(A+B)^T = -\det(A+B) \end{aligned}$$

Then we have $\det(A+B)=0$, which means $A+B$ is not full rank. Then we can find a vector $y \in N_{A+B}$ such that $(A+B)y=0$. Now we pick $x=\frac{y}{\|y\|}$, this is the unit vector we need. $\square$