Homework 8 for Math 2371
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Problem 1. Let K be the collection of all $n \times n$ stochastic matrices. Show that K is convex in the n^2 dimensional linear space of $n \times n$ real matrices. Find all extreme points of K.

Proof.

(1) Suppose $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are stochastic matrices. Then for any $t \in (0,1)$, we can have

$$\sum_{i=1}^{n} ta_{ij} + (1-t)b_{ij} = t\sum_{i=1}^{n} a_{ij} + (1-t)\sum_{i=1}^{n} b_{ij} = 1.$$

Then, $tA + (1-t)B$ is also stochastic matrix. Hence, K is a convex set.

(2) The permutation matrices are extreme points of K. Suppose permutation matrix $P = \frac{A+B}{2}$, where A and B are stochastic matrices. Since $a_{ij}, b_{ij} \in [0,1]$ and $P_{ij} = \frac{a_{ij}+b_{ij}}{2}$, then we have $A = B = P$. Thus, P is an extreme point.

Also, for any matrix $M = (m_{ij})_{n\times n}$ that is not a permutation matrix, it is not an extreme point. Indeed, there exist i, j such that $m_{ij} \in (0,1)$. Then there exist stochastic matrices A, B where $m_{ij} = \frac{a_{ij}+b_{ij}}{2}$. Thus, M is not an extreme point.

□

Problem 2. Let $P = (P_{ij})_{n\times n}$ be an entrywise positive matrix and λ be its dominant eigenvalue. Show that

$$\min_i \sum_{j=1}^{n} P_{ij} \le \lambda(P) \le \max_i \sum_{j=1}^{n} P_{ij}.$$

Proof. Suppose λ be its dominant eigenvalue, then $\lambda > 0$ and there exists eigenvector h with $h_i > 0$. Then, we have $Ph = \lambda h$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij}h_j = \sum_{i=1}^{n} \lambda h_j.$$

Then, with the change of order of the summation, we have

$$\left(\min_i \sum_{j=1}^{n} P_{ij}\right)\left(\sum_{j=1}^{n} h_j\right) \le \lambda \sum_{j=1}^{n} h_j \le \left(\max_i \sum_{j=1}^{n} P_{ij}\right)\left(\sum_{j=1}^{n} h_j\right),$$

and hence

$$\min_i \sum_{j=1}^{n} P_{ij} \le \lambda(P) \le \max_i \sum_{j=1}^{n} P_{ij}.$$

□

Problem 3. Let $P = (P_{ij})_{n\times n}$ be an entrywise positive matrix and λ be its dominant eigenvalue. Suppose $u, v \in \mathbb{R}^n$ are two positive vector such that

$$Pu = \lambda u, P^T v = \lambda v.$$

Show that

$$\lim_{k\to\infty} \frac{1}{\lambda^k} P^k = \frac{1}{(u,v)} uv^T.$$

Proof. For matrix P/λ, it has dominant eigenvalue 1. Now suppose w is a generalized eigenvector of P with eigenvalue β. With Perron theorem, we have $|\beta| < \lambda$, then

$$\lim_{k\to\infty} \left(\frac{P}{\lambda}\right)^k w = 0.$$

Then $(P/\lambda)^k$ converges to a matrix M which fixes u and v. We claim $M = \frac{1}{(u,v)} uv^T$.

First we note that $v^T M = \frac{1}{(u,v)} (v^T u) v^T = v^T$ and $Mu = u$. Then any other generalized eigenvector w for eigenvalue $\beta \neq \lambda$, we have $Mw = 0$. If not, $Mw = \frac{1}{(u,v)} uv^T w \neq 0$, which implies $v^T w \neq 0$, and then for all $k > 0$,

$$\lambda^k v^T w = v^T P^k w = v^T \left(P^k w\right),$$

and hence

$$v^T w = \frac{1}{\lambda^k} v^T \left(P^k w\right),$$

which is a contradiction, since

$$\lim_{k\to\infty} \frac{1}{\lambda^k} P^k w = 0.$$

Thus, we have

$$\lim_{k\to\infty} \frac{1}{\lambda^k} P^k = \frac{1}{(u,v)} uv^T.$$

□