Homework 4 for Math 2371

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Problem 1. Suppose A and B are 2×2 nilpotent matrices. Prove that AB is diagonalizable.

Proof.

(1) If A or B is zero, then it is trivial.

(2) If $A, B \neq 0$, since A, B are nilpotent, then $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$ and the characteristic polynomials of A, B are λ^2. Then let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$, where $-a^2 = bc, -d^2 = ef$ and $a, b, c, d, e, f \neq 0$. Then we have $AB = \begin{pmatrix} ad + bf & ae - bd \\ cd - afc & ce + ad \end{pmatrix}$, and we can have its characteristic polynomial

$$P_{AB}(\lambda) = \lambda^2 - \left(\sqrt{bf} + \frac{ad}{\sqrt{bf}}\right)\lambda.$$

Hence, AB has two different eigenvalues. Thus, AB is diagonalizable.

□

Problem 2. Let Q be a reflection clockwise rotation with angel θ about the origin in $\mathbb{R}^2$. Find the matrix representation of Q in the standard basis and express Q as a composition of two reflections.

Proof. For any $u \in \mathbb{R}^2$, it can be represented as $u = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$, and we can have $Qr = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix}$, then we have

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now consider the reflection about the line $y = \tan\frac{\theta}{2}x$ and it can be represented as

$$R_{\frac{\theta}{2}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

then we have

$$R_{\frac{\alpha}{2}}R_{\frac{\beta}{2}} = \begin{pmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & -\cos(\alpha-\beta) \end{pmatrix}.$$

Let $\alpha = \theta - \beta$, then we can have $Q = R_{\frac{\alpha}{2}}R_{\frac{\beta}{2}}$. □

Problem 3. Let R be a reflection and Q be an orthogonal matrix in $\mathbb{R}^n$, show that QRQ^T is also a reflection. Explain the connection of two reflections geometrically.

Proof. Since Q is orthogonal matrix, then Q can be written as a composition of at most n reflections. Then the composition QRQ^T is also reflection. $\square$

Problem 4. Show that every orthogonal map Q is the composition of at most k hyperplane reflections, where

$$k = n - \dim(\ker(Q - I)).$$