Geometry of differential **geometry**: connections, flows, Lie brackets

Evgenii Samutichev

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Outline

Objects of study and how they look

Warning: a little bit different notation

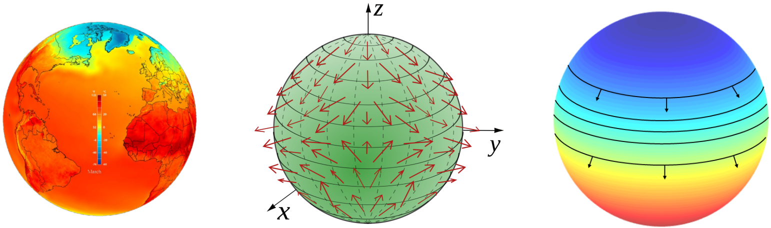
- **Warning** Throughout this presentation I'll be using a different notation for the differential, namely

$$\mathrm{D}_x f[v] := \mathrm{D}f(x)[v] \tag{1}$$

- I believe this makes equations more readable.

Tensor fields

- Scalar fields $f : \mathcal{M} \to \mathbb{R}$, living in $\mathfrak{F}(\mathcal{M})$, (0, 0)-tensor fields
- Vector fields $V : \mathcal{M} \to \mathrm{T}\mathcal{M}$, living in $\mathfrak{X}(\mathcal{M})$, (1, 0)-tensor fields
- Covector fields $\alpha : \mathcal{M} \to \mathrm{T}\mathcal{M}^*$, where $\alpha(x) \in (\mathrm{T}_x\mathcal{M})^*$, (0, 1)-tensor fields. Example $x \to \mathrm{D}_x f[\cdot]$ for $f \in \mathfrak{F}(\mathcal{M})$.



(a) A scalar field (b) A vector field (c) A covector field

Riemannian metric

- **Riemannian metric** is a $(0, 2)$-tensor field, that is

$$\langle \cdot, \cdot \rangle : \mathcal{M} \to T\mathcal{M}^* \times T\mathcal{M}^*, \text{ as } x \to \langle \cdot, \cdot \rangle_x \tag{2}$$

- Judging by the signature, it "consumes" two vectors that are (1, 0)-tensors and returns a scalar. So $\langle V, W \rangle$ we defined on the last lecture is clearly a scalar field.
- Why should we care? The chosen metric affects lengths of curves on the manifold as for $c : I \to \mathcal{M}$ its length is

$$L(c) = \int_I \sqrt{\langle c'(t), c'(t) \rangle_{c(t)}} dt \tag{3}$$

- Thus, choosing the metric makes our rubber-sheet geometric structure of the manifold more rigid by fixing distances which are defined through the lengths of geodesic curves.

Previously defined operations

- Scalar field $f(x)$ to covector field $\mathrm{D}_x f$
- Covector field $\mathrm{D}_x f$ to vector field $\mathrm{grad} f(x)$ as

$$\mathrm{D}_x f[u] = \langle \mathrm{grad} f(x), u \rangle_x \tag{4}$$

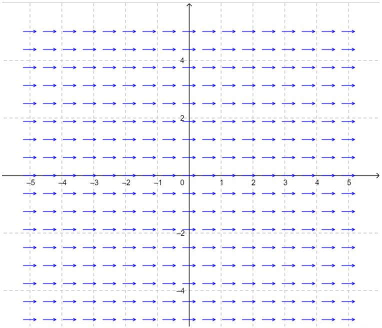
- **Definition 5.5 (3)** Two vector fields V, W to scalar field $\langle V, W \rangle$ as

$$\langle V, W \rangle(x) = \langle V(x), W(x) \rangle_x \tag{5}$$

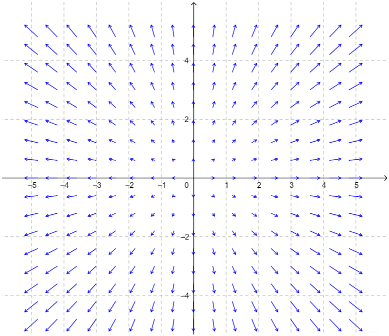
- Today we will discuss more of them.

Parallel transport problem

Change in vector fields on a plane



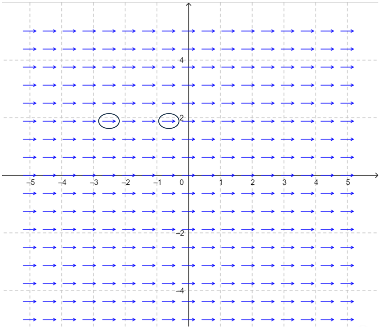
(a) A constant vector field



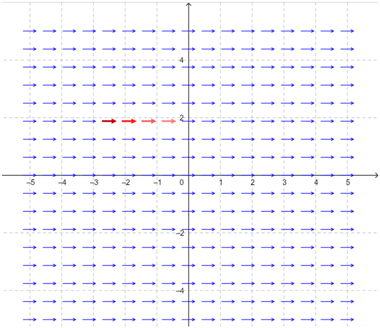
(b) A non-constant vector field

- But what does it mean to be constant?

Parallel transport on the plane



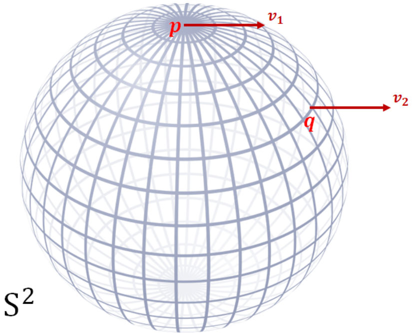
(a) Need to compare these two vectors



(b) Parallel transport of a vector

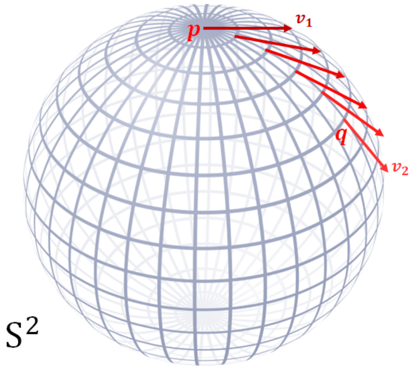
- To compare two vectors on the plane we need **parallel transport**. How to generalize this on manifolds?

Parallel transport on a sphere (the wrong)



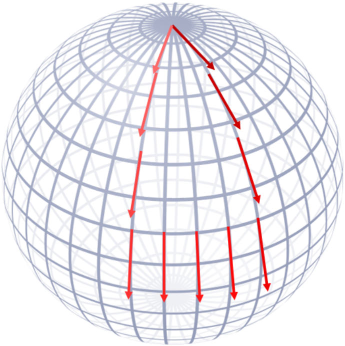
- While $v_1 \in \mathrm{T}_p S^2$, clearly $v_2 \notin \mathrm{T}_q S^2$, so we can't identify them.

Parallel transport on a sphere (the correct)



- Looks better, right? But there is still something to learn.

Parallel transport on a sphere (the weird)



- We thought that we kept the vector constant, but along the loop it doesn't make sense. Then what does it mean?

Formalization

- Parallel transport keeps vectors as constant as possible.

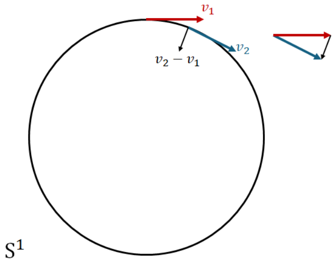


Figure: The change is only along the normal to the manifold

- Given a path $c(t)$ on a manifold, a parallel transported vector field $\vec{V}(x)$ should satisfy

$$\frac{d\vec{V}}{dt}(c(t)) = \vec{n}, \text{ or } \frac{d\vec{V}}{dt}(c(t)) - \vec{n} = 0 \quad (6)$$

Connections ignore parallel transport

- Thus, when we measure change, we should substract the normal component.
- Equation (5.4) in the book describing a possible connection is nothing more than a very natural implementation of this idea

$$\nabla_u V(x) = \mathrm{Proj}_x(\mathrm{D}_x \bar{V}[u]) \tag{7}$$

- **Definition** If $c : I \to \mathcal{M}$ is a smooth curve and $v \in \mathrm{T}_x\mathcal{M}$ where $x = c(0)$, then we say that a vector field V along c satisfying

$$\nabla_{c'(t)} V(c(t)) = 0, \forall t \in I \text{ and } V(c(0)) = v \tag{8}$$

is a **parallel transport of** v along c.

The formal definition of connection

- We will mostly use the equation (5.4) as a connection, but it's possible to take something else. Well, in this case it should at least enjoy similar properties.
- **Definition 5.1** A **connection** on a manifold $\mathcal{M}$ is an operator

$$\nabla : \mathrm{T}\mathcal{M} \times \mathfrak{X}(\mathcal{M}) \to \mathrm{T}\mathcal{M}, \quad (u, V) \mapsto \nabla_u V \tag{9}$$

such that $\nabla_u V \in \mathrm{T}_x\mathcal{M}$ for $u \in \mathrm{T}_x\mathcal{M}$ and which satisfies

1. smoothness: $(\nabla_U V)(x) : \nabla_{U(x)} V$ is a smooth vector field
2. linearity in u: $\nabla_{au+bw} V = a\nabla_u V + b\nabla_w V$
3. linearity in V: $\nabla_u(aV + bW) = a\nabla_u V + b\nabla_u W$
4. Leibniz' rule: $\nabla_u(fV) = \underbrace{\mathrm{D}_x f[u]}_{\text{scalar}} V(x) + f(x)\nabla_u V$

- **Theorem 5.2** Shows that $\nabla_u V = \mathrm{Proj}_x(\mathrm{D}_x \bar{V}[u])$ satisfies these properties, serving as a prototype.

Geometric motivation to algebraic operations

Vector fields as derivations (1 / 2)

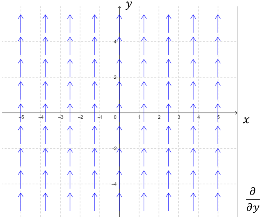
- **Definition** A **derivation** on $\mathcal{M}$ is a linear map $\mathcal{D} : \mathfrak{F}(\mathcal{M}) \to \mathfrak{F}(\mathcal{M})$ that also satisfies the Leibniz' rule

$$\mathcal{D}(fg) = g\mathcal{D}(f) + f\mathcal{D}(g) \tag{10}$$

- Typical examples on $\mathbb{R}^n$ are

$$\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n \tag{11}$$

- There is a vector field $V_i(x) = e_{x_i}$ associated with each of these examples as $V_i f(x) = D_x f[e_{x_i}]$



$\frac{\partial}{\partial y}$ vector field

Vector fields as derivations (2 / 2)

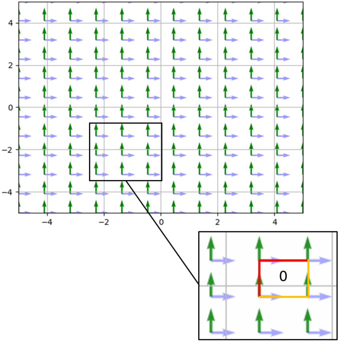
- Let's generalize this approach.
- **Definition 5.5 (1)** Let $V \in \mathfrak{X}(\mathcal{M})$ be a vector field on a manifold $\mathcal{M}$, then for a scalar field $f \in \mathfrak{F}(\mathcal{M})$ we can get another scalar field $Vf \in \mathfrak{F}(\mathcal{M})$ as

$$Vf(x) = \mathrm{D}_x f[V(x)] \tag{12}$$

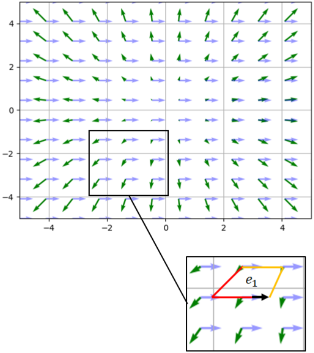
- This is a derivation which represents the rate of change of f along the direction of V at every point x.
- This is a special case of the **Lie derivative** which represents the rate of change of a tensor field along the flow of a vector field. We will soon see another example.

Lie bracket motivation

- One vector field V produces the flow, the other W changes along it. We want to quantify this as $\mathcal{L}_V W$.



(a) $V = e_2, W = e_1$



(b) $V = xe_1 + ye_2, W = e_1$

Flows (1 / 2)

- **Definition** Let $c : I \to \mathcal{M}$ be a smooth curve, we say that c is a **integral curve** of some vector field V if and only if

$$c'(t) = V(c(t)), \forall t \in I \tag{13}$$

- Then **Definition 5.5 (1)** can alternatively be written as

$$Vf(x) = \frac{d}{dt}(f \circ c)\Big|_{t=0} = \lim_{t \to 0} \frac{f(c(t)) - f(x)}{t} = \mathrm{D}_x f[V(x)] \tag{14}$$

where c is a flow s.t. $c(0) = x$. The last equality follows from **Definition 3.34** as $c'(0) = V(x)$ and $c(0) = x$.

Flows (2 / 3)

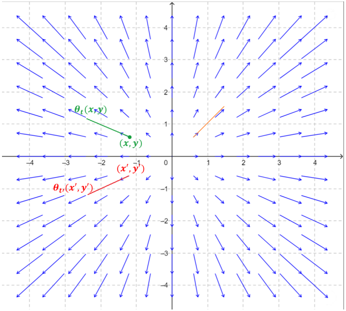
- Suppose that for each point $p \in M$ there exists a unique integral curve of V starting at p and defined on $I = \mathbb{R}$. We denote it as $\theta^{(p)} : \mathbb{R} \to \mathcal{M}$.
- We define a map $\theta_t : \mathcal{M} \to \mathcal{M}$ sending each $p \in M$ to the point obtained by following for time t the integral curve startint at p

$$\theta_t(p) = \theta^{(p)}(t) \tag{15}$$

- Notice that $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ and $\theta_0(p) = \theta^{(0)}(p) = p$.
- Thus, we can create a map $\theta : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ which satisfies nice algebraic properties, while also having a geometric interpretation. We call it a **global flow**.

Flows (3 / 3)

- Let's compute the flow on a plane for $V(x, y) = xe_1 + ye_1$. Start with an integral curve
$$\frac{dc_x}{dt} = c_x(t), \quad \frac{dc_y}{dt} = c_y(t) \tag{16}$$
then $c(t) = (ae^t, be^t)$.
- Then $\theta^{(x,y)}(t) = (xe^t, ye^t)$.



Lie bracket (1 / 4)

- Having the global flow θ associated with V, we can now redefine Vf again as

$$Vf(x) = \underbrace{(\mathcal{L}_V f)}_{\text{Lie derivative}}(x) := \frac{d}{dt}(f \circ \theta_t)(x)\Big|_{t=0} \tag{17}$$

- What about the Lie derivative $\mathcal{L}_V W$ of some other vector field W? For a fixed t, $\theta_t : \mathcal{M} \to \mathcal{M}$, and we can differentiate it getting $D_x\theta_t : \mathrm{T}_x M \to \mathrm{T}_{\theta_t(x)}\mathcal{M}$. Then

$$(\mathcal{L}_V W)(x) = \frac{d}{dt}(D_x\theta_t)^{-1}[W \circ \theta_t(x)]\Big|_{t=0} \tag{18}$$

- Notice how we are computing this in $\mathrm{T}_x\mathcal{M}$.

Lie bracket (2 / 4)

- Consider two vector fields on a plane $V(x, y) = xe_1 + ye_2$ and $W(x, y) = e_1$. We already know what $\theta_t(x, y)$ is in this case, $\theta_t(x, y) = (xe^t, ye^t)$, and

$$\mathrm{D}_{(x,y)}\theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}, \quad \mathrm{D}_{(x,y)}\theta_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \tag{19}$$

- Notice how $(\mathrm{D}_{(x,y)}\theta_t)^{-1} = \mathrm{D}_{(x,y)}\theta_{-t}$, this is not a coincidence. Finally

$$(\mathcal{L}_V W)(x) = \frac{d}{dt}\begin{pmatrix} e^t \\ 0 \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \tag{20}$$

Lie bracket (3 / 4)

- **Definition 5.5 (2)** Let $V, W \in \mathfrak{X}(\mathcal{M})$. A **Lie bracket** is a vector field $[V, W] = \mathcal{L}_V W$ that represents the derivative of Y along the flow generated by X.
- It's possible to show that for $f \in \mathfrak{F}(\mathcal{M})$

$$[V, W]f = V(Wf) - W(Vf) \tag{21}$$

For example, in $\mathbb{R}^n$

$$[V_i, V_j]f = V_i(V_j f) - V_j(V_i f) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = 0 \tag{22}$$

where $V_i = e_i$. This is a result of **Clairaut's theorem**.

Lie bracket (4 / 4)

- Returning back to our example with $V(x,y) = xe_1 + ye_2$ and $W(x,y) = e_1$ we have

$$Wf(x,y) = \frac{\partial f}{\partial x}, \quad Vf(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \tag{23}$$

and

$$V(Wf)(x,y) = x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x} \tag{24}$$

$$W(Vf)(x,y) = \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} \tag{25}$$

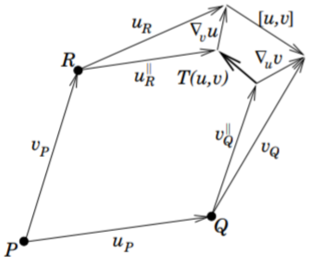
so

$$[V,W]f = \frac{\partial f}{\partial x} \tag{26}$$

- This is precisely the action of $\mathcal{L}_V W = e_1$ we computed previously on f.

Torsion

- Two vector fields u and v.
- Compute $\nabla_v u$ and $\nabla_u v$ using parallel transport
- Compute the Lie bracket $[u, v]$
- Closure failure $T(u, v) = \nabla_u v - \nabla_v u - [u, v]$ is known as **torsion**



The canonical Euclidean connection

- **Definition** Let $U, V, W \in \mathfrak{X}(\mathcal{M})$. A connection ∇ is called
  1. **Torsion-free** or **symmetric** if and only if
  $$T(V, W) = \nabla_V W - \nabla_W V - [V, W] = 0 \tag{27}$$
  2. **Compatible with the metric**
  $$U\langle V, W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle \tag{28}$$
  this one implies that the metric is constant w.r.t. ∇, i.e. $\nabla_U \langle \cdot, \cdot \rangle = 0$.
- **Theorem 5.7** The Riemannian connection on a Euclidean space $\mathcal{E}$ with any Euclidean metric $\langle \cdot, \cdot \rangle$ is $\nabla_u V = \mathrm{D}_x V[u]$: the canonical Euclidean connection.

Proof of Theorem 5.7

- Take $U, V, W \in \mathfrak{X}(\mathcal{M})$. We start by showing compatability with the metric, that is

$$U\langle V, W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle \qquad (29)$$

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- Taylor:

$$\begin{aligned} V(x + tU(x)) &= V(x) + t\mathrm{D}V(x)[U(x)] + O(t^2) = \\ &= V(x) + t(\nabla_U V)(x) + O(t^2) \end{aligned}$$

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- We can do the same thing for W.

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- Then

$$(Uf)(x) := D_x f[U(x)] =$$
$$= \lim_{t \to 0} \frac{\langle V(x + tU(x)), W(t + tU(x)) \rangle - \langle V(x), W(x) \rangle}{t}$$

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$$
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&= \lim_{t \to 0} \frac{\langle V(x + tU(x)), W(t + tU(x)) \rangle - \langle V(x), W(x) \rangle}{t} = \\
&= \lim_{t \to 0} \frac{\langle V(x) + t(\nabla_U V)(x), W(x) + t(\nabla_U W)(x) \rangle - \langle V(x), W(x) \rangle}{t} = \\
&= \left(\langle \nabla_U V, W \rangle + \langle V, \nabla_U W \rangle \right)(x)
\end{aligned}
$$

Proof of Theorem 5.7

- Now, let's prove that it's symmetric. First, notice that

$$(Vf)(x) = \mathrm{D}f(x)[V(x)] = \langle \mathrm{grad} f(x), V(x) \rangle_x \tag{30}$$

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- $\mathrm{grad} f(x)$ is a vector field, so

$$U(Vf) = U\langle \mathrm{grad} f, V \rangle = \langle \nabla_U(\mathrm{grad} f), V \rangle + \langle \mathrm{grad} f, \nabla_U V \rangle \tag{31}$$

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- But $\nabla_U (\mathrm{grad} f) = \mathrm{Hess} f[U]$ is a vector field s.t.

$$(\mathrm{Hess} f[U])(x) = \mathrm{Hess} f(x)[U(x)] = \nabla_{U(x)} (\mathrm{grad} f) \tag{32}$$

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- We have for $f \in \mathfrak{F}(\mathcal{M})$

$$U(Vf) = \langle \mathrm{Hess} f[U], V\rangle + \langle \mathrm{grad} f, \nabla_U V\rangle$$
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- But Hess$f$ is self-adjoint (Clairaut's theorem), so

$$\langle \mathrm{Hess} f[U], V\rangle = \langle \mathrm{Hess} f[V], U\rangle \tag{34}$$

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- But Hess$f$ is self-adjoint (Clairaut's theorem), so

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- Thus

$$[U, V]f = \langle \mathrm{grad} f, \nabla_U V - \nabla_V U \rangle = (\nabla_U V - \nabla_V U)f \tag{35}$$

and so

$$[U, V] = \nabla_U V - \nabla_V U \tag{36}$$

Symmetricity of connection (5.4)

- **Theorem 5.8** Let $\mathcal{M}$ be an embedded submanifold of a Euclidean space $\mathcal{E}$. The connection ∇ defined by

$$\nabla_u V = \mathrm{Proj}_x(\mathrm{D}_x \bar{V}[u]) \tag{37}$$

is symmetric.

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- Let O be a neighborhood of $\mathcal{M}$ in $\mathcal{E}$.
- Consider $U, V \in \mathfrak{X}(\mathcal{M})$ and $f \in \mathfrak{F}(\mathcal{M})$ with smooth extensions $\bar{U}, \overline{V} \in \bar{X}(O)$ and $\bar{f} \in \mathfrak{F}(O)$.

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- Then

$$
\begin{aligned}
[U, V]f = U(Vf) - V(Uf) = U(\bar{V}\bar{f}|_f) - V((\bar{U}\bar{f}|_{\mathcal{M}})) = \\
= (\bar{U}(\bar{V}\bar{f}))|_{\mathcal{M}} - (\bar{V}(\bar{U}\bar{f}))|_{\mathcal{M}} = ([\bar{U}, \bar{V}]\bar{f})|_{\mathcal{M}} = \\
= (\underbrace{(\bar{\nabla}_{\bar{U}}\bar{V} - \bar{\nabla}_{\bar{V}}\bar{U})}_{\bar{W}}\bar{f})_{\mathcal{M}} = (\bar{W}\bar{f})|_{\mathcal{M}}
\end{aligned}
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- Thus $\bar{W}(x) \in \ker \mathrm{D}\bar{h}(x)$ which is equivalent to $\bar{W}(x) \in \mathrm{T}_x\mathcal{M}$

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- Finally, we can write

$$W = \bar{W}|_{\mathcal{M}} = \mathrm{Proj}(\bar{W}) = \mathrm{Proj}(\bar{\nabla}_{\bar{U}}\bar{V} - \bar{\nabla}_{\bar{V}}\bar{U}) \triangleq \nabla_U V - \nabla_V U$$

- But then

$$[U, V]f = (\bar{W}\bar{f})|_{\mathcal{M}} = Wf = (\nabla_U V - \nabla_V U)f \tag{40}$$