Solving the Wide-band Inverse Scattering Problem via Equivariant Neural Networks

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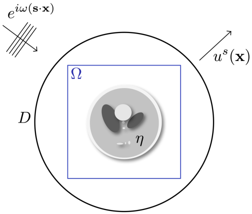
Outline

Forward problem

- Helmholtz equation setting, domain of interest Ω, Sommerfeld radiation condition
- The media is described by $\eta(\mathbf{x}) = n(\mathbf{x}) - 1$ where n is the refractive index
- Probing wave $u^{\text{in}} = e^{i\omega \mathbf{s}\cdot\mathbf{x}}$ of frequency ω and direction $\mathbf{s} \in \mathbb{S}^1$ triggers the scattered wave field $u^{\text{sc}}(\mathbf{x};\mathbf{s})$

$$\Delta u^{\text{sc}}(\mathbf{x}) + \omega^2(1+\eta(\mathbf{x}))u^{\text{sc}}(\mathbf{x}) = -\omega^2\eta(\mathbf{x})u^{\text{in}}(\mathbf{x}) \qquad (1)$$



The measurements are taken on the circle D of radius R:

$$\Lambda^\omega(\mathbf{s},\mathbf{r}) = u^{\text{sc}}(R\mathbf{r};\mathbf{s})$$

$$\textbf{FP: } \Lambda^\omega = \mathcal{F}^\omega[\eta]$$

Inverse problem

- The **inverse problem** is to infer the media η from the measured data $\{\Lambda^\omega\}_{\omega\in\bar{\Omega}}$ where $\bar{\Omega}$ is a set of frequencies

$$\eta^* = \mathcal{F}^{-1}(\{\Lambda^\omega\}_{\omega\in\bar{\Omega}}) \tag{2}$$

- A classical way is to recast this problem to PDE-constrained optimization

$$\eta^* = \arg\min_{\nu} \sum_{\omega\in\bar{\Omega}} \|\mathcal{F}^\omega[\nu] - \Lambda^\omega\|^2 \tag{3}$$

which can be solved by highly tailored gradient-descent optimization techniques.

Linearization

- Perturbation of the input $\eta = \eta_0 + \delta\eta$

$$\mathcal{F}^\omega[\eta] = \mathcal{F}^\omega[\eta_0 + \delta\eta] \approx \mathcal{F}^\omega[\eta_0] + F^\omega \delta\eta$$
$$\Lambda^\omega = \mathcal{F}^\omega[\eta] \approx F^\omega \eta$$

- Born approximation

$$(F^\omega \eta)(\mathbf{s}, \mathbf{r}) = C_{norm} \int_\Omega e^{-i\omega(\mathbf{r}-\mathbf{s})\cdot\mathbf{y}} \eta(\mathbf{y}) d\mathbf{y} \qquad (4)$$

- Involves very oscillatory integrals
- In this setting, the data Λ^ω can be viewed as a first-type Fredholm integration over η

Filtered back-projection

- Now, for the inverse problem we search for a solution η^* to the following optimization problem

$$\min_{\eta} \|\Lambda^\omega - F^\omega \eta\|^2 + \epsilon \|\eta\|^2 \tag{5}$$

where

$$\|\Lambda^\omega - F^\omega \eta\|^2 = \int_{\mathbb{S}^1 \times \mathbb{S}^1} |\Lambda^\omega(\boldsymbol{s}, \boldsymbol{r}) - (F^\omega \eta)(\boldsymbol{s}, \boldsymbol{r})|^2 \, d\boldsymbol{r} d\boldsymbol{s} \tag{6}$$

- The solution is explicitly given by

$$\eta^* = ((F^\omega)^* F^\omega + \epsilon I)^{-1} (F^\omega)^* \Lambda^\omega \tag{7}$$

- This formula isn't valid when real η is significant, and only serves as a guidance for the actual inversion.

Structure of input and output data

- n_{freq} frequencies. Again, in theory, one should be enough if we had complete perfect info.
- Each has n_{obs} source directions and observation points
- Probe with n_{obs} directions s, and then observe $u^{sc}(R\mathbf{r}; \mathbf{s})$ at n points r on the circle.

Equivariance (1 / 2)

- Remember filtered back-projection

$$\eta^* = ((F^\omega)^* F^\omega + \epsilon I)^{-1} (F^\omega)^* \Lambda^\omega \tag{8}$$

- Back-scattering operator $(F^\omega)^*$ is rotation equivariant, and naturally represented on the polar coordinates
- Filtering operator $((F^\omega)^* F^\omega + \epsilon I)^{-1}$ is translation equivariant, and naturally represented on a Cartesian grid
- There is a function $p(\mathbf{x})$ s.t.

$$(F^\omega)^* F^\omega(\eta) = p * \eta \tag{9}$$

- The proposed strategy is to encode this into a convolutional neural network, including coordinate transformations

Equivariance (2 / 2)

Rotational equivariance

$$((F^\omega)^*\Lambda^\omega(r-a,s-a))\,(\theta,\rho) = [(F^\omega)^*\Lambda^\omega](\theta-a,\rho)$$

Translational equivariance

$$\begin{aligned}(F^\omega)^*F^\omega[\eta](y) &= \int_{[0,2\pi]^2} e^{i\omega(r-s)\cdot y}\left(\int_\Omega e^{-i\omega(r-s)\cdot x}\,\eta(x)\,dx\right)\,ds\,dr\,,\\ &= \int_{[0,2\pi]^2}\int_{\mathbb{R}^2} e^{i\omega(r-s)\cdot y}e^{-i\omega(r-s)\cdot x}\,\eta(x)\,dx\,\,ds\,dr\,,\\ &= \int_{\mathbb{R}^2}\left(\int_{[0,2\pi]^2} e^{i\omega(r-s)\cdot(y-x)}\,ds\,dr\right)\,\eta(x)\,dx\,,\\ &= \int p(y-x)\eta(x)dx = p*\eta(y).\end{aligned} \tag{10}$$

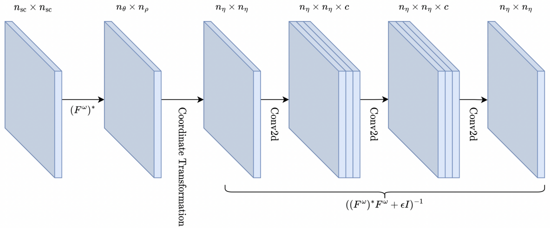
Application of rotation equivariance computing backscattering...

$$\mathsf{K}^{\omega} \in^{n_\times n_\rho}, \quad \text{with} \quad \mathsf{K}^{\omega}_{mn} = e^{-i\omega\rho_n \cos(t_m)} = K^{\omega}(\rho_n, t_m)\,.$$

$$\begin{aligned}\alpha^{\omega}(\theta_j, \cdot) &= ((\mathsf{F}^{\omega})^{*}\Lambda^{\omega})(\theta_{\mathrm{j}}, \cdot) \\ &= \mathrm{ones}(1, n_{source}) \cdot [\mathsf{K}^{\omega} \odot (\Lambda^{\omega}_{\theta_{\mathrm{j}}} \cdot \mathsf{K}^{\omega})] \\ &= \mathrm{diag}[(\mathsf{K}^{\omega})^{*} \cdot \Lambda^{\omega}_{\theta_{\mathrm{j}}} \cdot \mathsf{K}^{\omega}]\end{aligned}$$

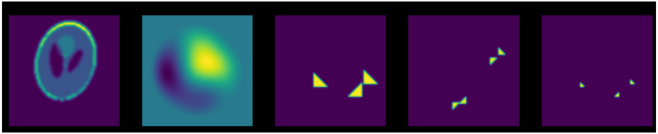
Wide-Band Equivariant Network

1. Compute the discretized back-scattering $\alpha^\omega(\theta,\rho) := (F^\omega)^*\Lambda^\omega$ for each frequency $\omega \in \bar{\Omega}$,
2. Coordinate transformation $\alpha^\omega(\theta,\rho)$ to $\alpha^\omega(x,y)$ using polynomial interpolation
3. Feedforward to the CNN that approximates the filtering operator $((F^\omega)^* F^\omega + \epsilon I)^{-1}$



Training dataset

- Media + corresponding wide-band far-field patterns at three different frequencies $2.5, 5$ and 10 Hz, computed using finite-differences
- Picked $n_\eta = 80$, so the resolution of the media is 80×80
- $n_{\mathrm{sc}} = 80$ equiangular recievers and sources
- Model was trained for each of 5 categories of media
  - Shepp-Logan phantom, representing a human head
  - Random smooth perturbations
  - Triangles of sizes 10, 5, 3, randomly located and oriented



Training procedure

- Let's denote our network as

$$\eta = \Phi_\Theta(\{\Lambda^\omega\}_{\omega \in \bar{\Omega}}) \tag{11}$$

- Then, we train it by minimizing the MSE

$$\min_\Theta \frac{1}{N_s} \sum_{s=1}^{N_s} \left\| \Phi_\Theta(\{\Lambda^{\omega,[s]}\}_{\omega \in \bar{\Omega}}) - \eta^{[s]} \right\|^2 \tag{12}$$

- Adam optimizer with learning rate 3×10^{-4}
- Batch size 16
- Exponential scheduler
- 100 epochs

Performance (1 / 2)

- The model was compared by relative RMSE with Wide-band Butterfly Network (WBBN) and Fourier Neural Operator (FNO) models

Media \ Model	Uncompressed	Compressed	WBBN	FNO
#parameters	88,186	73,210	1,914,061	1,188,385
Shepp-Logan	5.124 %	8.306 %	9.819 %	7.981 %
Random smooth	3.957 %	6.866 %	8.268 %	4.289 %
10h triangles	7.068 %	15.751 %	42.545 %	42.633 %
5h triangles	6.061 %	16.144 %	Untrainable	47.692 %
3h triangles	5.902 %	38.651 %	Untrainable	49.348 %

Performance (2 / 2)