端午假期学习&研究计划

复分析

Cauchy 积分理论、Weierstrass 级数理论与亚纯函数

Cauchy 积分定理：从历史观点看一般情形(积分定理)(任意单连通区域)；

回忆 de Rham 理论，将 C 视为流形

Corollary 5.3 *If f is holomorphic in the simply connected region Ω, then*

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in Ω.

如何被最终解决。

三个基本(完美到不“真实”)性质：

积分定理

正则性

解析延拓(刚性)

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1. **Cauchy,1825**

回忆局部 Poincare 引理

5. Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z)\,dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

[Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then

$$\int_T F\,dx + G\,dy = \int_{\text{Interior of } T} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dxdy.$$

For appropriate F and G, one can then use the Cauchy-Riemann equations.]

2. Goursat,1900

注意证明需要的技术：折线逼近、紧区间套定理；为什么选择三角形？回忆三角剖分

Theorem 1.1 *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\,dz = 0$$

whenever f is holomorphic in Ω.

3. Local existence of primitives and Cauchy's theorem in a disc

类似局部 Poincare 引理，我们需要什么样的领域；把问题归结到原函数上，回忆微分形式。

Theorem 2.1 *A holomorphic function in an open disc has a primitive in that disc.*

注：证明这个定理需要如何构造原函数？导数的定义

Theorem 2.2 (Cauchy's theorem for a disc) *If f is holomorphic in a disc, then*

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in that disc.

有了 disc 以后我们可以对一些区域做 **surgery**，几个重要的积分的例子！（注意与分析中常

用的“挖洞”技巧类比），这直接导致了 **Cauchy 积分公式**的出现。

4. Cauchy 积分公式

实现定量

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \textit{for any point } z \in D.$$

基本的递推：

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

注意，这个事实已经刻画了全纯函数的正则性！由此带来的估计：

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

于是，Taylor 级数展开是必然的结果，并且每一项可以被函数本身(非导数)量化：

Theorem 4.4 *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

至此，我们建立了最基本的工具！以下是可以自行独立证明的：

5. 利用积分理论的工具解决问题！

a. Liouville 定理→代数基本定理(回忆代数拓扑—S^1 基本群，对比二者)；回忆 PDE 与几何中的 Liouville 定理

b. 刚性定理(→直接导致零点孤立，*注意与微分拓扑中的正则值对比，局部的 1-1 甚至微分同胚本质上依赖于什么*)→允许解析延拓→基本的延拓：Schwarz reflection

c. 非常有用的工具(往往使用充分性)：Morera 定理

d. 全纯序列，导数完美的敛散性(注意与实变版本对比)→含参变量积分(Riemann 和实现转化)

e. Runge 逼近：引理使用了什么技术？与实变版本(Weierstrass)有什么区别？注意，极点处的讨论自然引出 Laurent 级数

计算是很无聊的事情 QAQ，参见[2]即可。

6. 一般情形最终被解决(引入代数拓扑的工具——同伦)

Theorem 5.1 *If f is holomorphic in Ω, then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

亚纯函数理论

对奇点性质的理解是核心。Local 性质逐渐向 Global 推广，一些"坏点"处应该怎么处理，绕过去以后拓扑性质发生了什么变化？

1. 零点阶数的讨论(对偶于极点)

Theorem 1.1 *Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \quad \text{for all } z \in U.$$

Theorem 1.2 *If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that*

$$f(z) = (z - z_0)^{-n} h(z).$$

Theorem 1.3 *If f has a pole of order n at z_0, then*

$$(1) \qquad f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z),$$

where G is a holomorphic function in a neighborhood of z_0.

2. 工具：留数

本质上为映射 1/z(映射 logz 的导函数)对曲线拓扑量(外蕴的旋绕数)的影响，与之对偶的是辐角原理。

怎么算?

Theorem 1.4 *If f has a pole of order n at z_0, then*

$$\text{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

留数公式

Theorem 2.1 *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then*

$$\int_C f(z)\, dz = 2\pi i \ \text{res}_{z_0} f.$$

Corollary 2.3 *Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points $z_1, \ldots, z_N$ inside γ. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{N} \text{res}_{z_k} f.$$

几个例子!

3. 三类奇点

可去奇点（和台湾一样，拼上了才完整）

Theorem 3.1 (Riemann's theorem on removable singularities) *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

极点(blum up，对偶)

Corollary 3.2 *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

本性奇点(不稳定性)

Theorem 3.3 (Casorati-Weierstrass) *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

更强的还有 Picard 定理。

紧化复平面上的亚纯函数(global 性质，某种刚性)

Theorem 3.4 *The meromorphic functions in the extended complex plane are the rational functions.*

4. **曲线拓扑与辐角原理**(注意多值函数 log 与 Riemann 曲面的思路来源)

需要注意 f'/f 本质上是什么结构

Theorem 4.1 (Argument principle) *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{(number of zeros of } f \text{ inside } C\text{) minus}$$
$$\text{(number of poles of } f \text{ inside } C\text{)},$$

where the zeros and poles are counted with their multiplicities.

Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

注意到函数的 small part 对零点而言是不重要的。

开映射定理

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

最大模原理

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

与调和函数对比

5. 多值函数与 log 体现的拓扑信息

Theorem 6.1 *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

(i) *F is holomorphic in Ω,*

(ii) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(iii) *$F(r) = \log r$ whenever r is a real number and near 1.*

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

6. 与 Fourier 级数的关系——负项全部消失!

这为我们将复分析问题转化到调和分析提供范例

Theorem 7.1 *The coefficients of the power series expansion of f are given by*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta$$

for all $n \geq 0$ and $0 < r < R$. Moreover,

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta$$

whenever $n < 0$.

Corollary 7.3 *If f is holomorphic in a disc $D_R(z_0)$, and $u = \mathrm{Re}(f)$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta, \qquad \textit{for any } 0 < r < R.$$

参考文献

[1]Stein. Complex Analysis

[2]史济怀，复变函数

[3]龚昇，简明复分析

整函数理论与 Hadamard 因子分解

Riemann 几何理论与共形映射

*额外的专题：Gamma 函数与 zeta 函数、素数定理；椭圆函数简介与 Theta 函数

整体微分几何

曲线理论

等周不等式

旋转指标定理：与辐角原理对比(内蕴外蕴)

*额外的选题：空间曲线理论：Crofton 公式、全曲率与 Fenchel 定理、Fary-Milnor 定理、全挠率

曲面理论

球面的刚性：Liebmann 定理（特别关注证明使用的技术：椭圆点的存在性(外蕴直观与内蕴证明)、Hilbert 定理怎么证明，怎么理解？自然标架、幺正标架外微分都去尝试一下，注意计算上的区别）

刚性条件的弱化：凸曲面与 Hadamard 定理、Cohn-Vossen 定理、Minkowski 积分公式

全平均曲率与 Willmore 猜想：球面的特征（怎么证明？），环面的特征，管状曲面的特征

额外的选题：参考刘世平教授的讲义

注意收集整体微分几何需要的技术：Stokes 公式、极值原理(紧致性)、重要的拓扑工具、外蕴与内蕴技术的对比

参考文献

[1]沈一兵，整体微分几何初步
[2]刘世平，整体微分几何(讲义)

常微分方程定性理论

看论文。。。