Order and surgical technique using Residue Theorem

Qian Zhenye

June 17, 2024

Contents

Recall

Firstly, I want to recall the **Residue Formula**. Let a the m-order pole of a meromorphic function f, then we have the Residue formula:

$$Res_a f := \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \tag{1}$$

Particularly, when m=1, we have

$$Res_a f := \lim_{z \to a} (z-a) f(z)$$

Also one can find that take a closed curve γ around the unique[1] pole a and integrate along it. We can see

$$\int_\gamma f = 2\pi i Res_a f \tag{2}$$

Thus the main goal of this article is to show that how to transform calculation of some special integrals into the calculating residues. We need two fundamental tricks:**ORDER** and **SURGERY**.

1 $\int_{-\infty}^{\infty} f$ form integration

In this section, we will solve the integration like $\int_{-\infty}^{\infty}$, the two-side infinite interval form. Recall the **improper integral**, we didn't have some useful and elegant method to solve it just by the definition

$$\int_{-\infty}^{\infty} f = \lim_{R \to \infty} \int_{-R}^{R} f$$

But it actually the **first step** we need to try.

formula 2 inspires us to choose a proper curve which moves from $-R$ to R along the **Real** axis, and then return to $-R$ counterclockwise along an arc with the radius R at the center of the circle 0[2]. If we can calculate the value of upper semi-circle called

$$\int_{\gamma_R}$$

then let $R \to \infty$, we will get the Integration above. Maybe the integration of γ_R may difficult to calculate as $R \to \infty$, which is about another limitation $\lim_{R\to\infty} \int_{\gamma_R}$. But we can choose some **NICE** function f such that

$$\int_{\gamma_R} f \to 0 \qquad as \quad R \to \infty \tag{3}$$

[1]In this article, we use the unique pole to show our technique.

[2]Where we let $a = 0$.

1.1 NICE1:$\lim_{z\to\infty} zf(z) = 0$

The first NICE function is f such that$\lim_{z\to\infty} zf(z) = 0$. Using the technique of estimating the order we can easily know that

$$f \sim o(\frac{1}{z}) \qquad as \quad z \to \infty \tag{4}$$

A useful method to calculate $\int_{\gamma_R}$ is the **Polar coordinate**. We have

$$\left|\int_{\gamma_R}\right| = \left|\int_0^\pi Re^{i\theta} f(Re^{i\theta}) d\theta\right|$$

and let $R \to \infty$, immediately get $\left|\int_{\gamma_R}\right| \to 0$. So we get the first formula

$$\int_{-\infty}^{\infty} f = 2\pi i Res_a f \tag{5}$$

Thus we just need to count the number of poles in the upper plane.One may ask that how many examples are belong to the form above, indeed that we at least have the **Rational Functions** such that

$$\frac{P}{Q}, \quad degP \leq degQ - 2$$

1.2 NICE2:$e^{i\alpha z} f(z)$ and $\lim_{z\to\infty, Imz\geq 0} f = 0$

The second form is little hard to find. When I first see that, it's confused for me to understand why the factor $e^{i\alpha z}(\alpha > 0)$ taking place of the condition 4. We must recognize that the second condition $\lim_{z\to\infty, Imz\geq 0} f = 0$ just provide with a weaker estimation, i.e.

$$f \to O(\frac{1}{z^p}), \quad p > 0$$

which almost has no effect because we must need at least $p > 1$! However, the factor $e^{i\alpha z}$ improves the order "in the integration". As we can see

$$\left|\int_{\gamma_R} e^{i\alpha z} f\right| = R||f||_R \int_0^\pi e^{i\alpha R(\cos\theta + i\sin\theta)} d\theta = R||f||_R \int_0^\pi e^{-\alpha R\sin\theta} d\theta$$

where $||f||_R := \max_{z\in\gamma_R} f$. Observed that "Jordan inequality":

$$\sin x \leq \frac{2}{\pi} x, \quad 0 \leq x \leq \frac{\pi}{2}$$

By the symmetry we can estimate the equation above to be

$$\left|\int_{\gamma_R} e^{i\alpha z} f\right| \leq C||f||_R$$

One must observed that the factor $e^{-\alpha R\theta}$ provides a $\frac{1}{R}$ after being integrating! Therefore we can get a same limitation $\lim_{R\to\infty} \int_{\gamma_R} = 0$. Finally we have the similar consequence like 5

Here we call the factor$e^{i\alpha z}(z>0)$ the **Integration reducing factor**. One may find more such factors $K(z)$, just observed that

$$\int_0^{\pi} K(Re^{i\theta})d\theta \sim o(\frac{1}{R}) \qquad as \quad R\to\infty$$

However, the special factor above has many the structures we're familiar with, because one can use the *Euler* Identity

$$e^{i\alpha z} = \cos\alpha z + i\sin\alpha z$$

Then we have

$$\int_{-\infty}^{\infty} \cos\alpha x f = Re\{Res_a e^{i\alpha z} f\} \tag{6}$$

Use these formulas we can calculate many classical integration with trigonometric functions.

example 1.1.

$$\int_{\infty}^{\infty} \frac{\cos ax}{x^2+b^2} \quad (a,b>0)$$

Observed that f has pole ib in the upper plane, and calculate the residue

$$Res_{ib} f = \frac{e^{-ab}}{2bi}$$

then

$$\int_{\infty}^{\infty} \frac{\cos ax}{x^2+b^2} = \frac{\pi}{b} e^{-ab}$$

1.3 IMPROVE:Surgery

One may find the function has singularities on Real-axis, where we need using our second trick, **SURGERY**. We have shown the same method in the proof of the *Cauchy's* integral formula, where we dug a small circle γ_ϵ[3] near the singularity and it is bridged with a large curve γ to form a holomorphic simply connected region.

Here we just suppose that 0 is the unique singularity of such f. We let the curve travel along Real-axis from $-R$ to $-\epsilon$, and "jump" away from 0 by a $\epsilon-$circle, then travel from ϵ to R and then go back to starting along the large curve γ_R. One may find the track is a $\pi-$sector. We claim that: if we want to calculate the $\int_{-\infty}^{\infty}$, we need to estimate both $\int_{\gamma_\epsilon}$ and $\int_{\gamma_R}$.

We have got the γ_R in two subsections above, now we have to solve γ_ϵ. Observed that we will let $\epsilon\to 0$, so the order of f near the singularity 0 is important. Similarly we need the estimation $\lim_{z\to 0} zf(z) = A$, which is equal to

$$f \sim \frac{A}{z} \qquad as \quad z\to 0 \tag{7}$$

Then we have

$$\int_{\gamma_\epsilon} \to \int_0^{\pi} iAd\theta = iA\pi \qquad as \quad z\to 0$$

[3]with the negative orientation.

remark 1.1. *One may find that the angle π can be replaced with $\alpha(0 < \alpha < 2\pi)$, then the integration is equal to $iA\alpha$.*

Thus we can calculate the integration by

$$\int_{-\infty}^{\infty} = iA\pi \tag{8}$$

example 1.2.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = 1 \cdot \pi = \pi \tag{9}$$