Some TRICKS of using Residue

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Abstract

In this article, we mainly discuss four kinds of integration forms by using the **Residue**. The most important techniques are **ORDER** and **SURGERY** of the region. We shall show that the first one need the function to be "nice", or add some well factors such $e^{i\alpha z}(\alpha > 0)$ and $x^p(0 < p < 1)$ or $logz$. We call them **integration reducing factor** and **multivalue factor**, which can help to estimate the integration in a global region. Here we find some general properties of them and give some improvement of these factors so that we can solve more kinds of problems. Finally, we can claim that the surgery of region can be forecast by the form of integrands and the coordinate types.

Contents

Recall

Firstly, I want to recall the **Residue Formula**. Let a the m-order pole of a meromorphic function f, then we have the Residue formula:

$$Res_a f := \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \tag{1}$$

Particularly, when m=1, we have

$$Res_a f := \lim_{z \to a} (z-a) f(z)$$

Also one can find that take a closed curve γ around the unique[1] pole a and integrate along it. We can see

$$\int_\gamma f = 2\pi i Res_a f \tag{2}$$

Thus the main goal of this article is to show that how to transform calculation of some special integrals into the calculating residues. We need two fundamental tricks:**ORDER** and **SURGERY**.

1 $\int_{-\infty}^{\infty} f$ form integration

In this section, we will solve the integration like $\int_{-\infty}^{\infty}$, the two-side infinite interval form. Recall the **improper integral**, we didn't have some useful and elegant method to solve it just by the definition

$$\int_{-\infty}^{\infty} f = \lim_{R \to \infty} \int_{-R}^{R} f$$

But it actually the **first step** we need to try.

formula 2 inspires us to choose a proper curve which moves from $-R$ to R along the **Real** axis, and then return to $-R$ counterclockwise along an arc with the radius R at the center of the circle 0[2]. If we can calculate the value of upper semi-circle called

$$\int_{\gamma_R}$$

then let $R \to \infty$, we will get the Integration above. Maybe the integration of γ_R may difficult to calculate as $R \to \infty$, which is about another limitation $\lim_{R\to\infty} \int_{\gamma_R}$. But we can choose some **NICE** function f such that

$$\int_{\gamma_R} f \to 0 \qquad as \quad R \to \infty \tag{3}$$

[1]In this article, we use the unique pole to show our technique.

[2]Where we let $a = 0$.

1.1 NICE1:$\lim_{z\to\infty} zf(z) = 0$

The first NICE function is f such that$\lim_{z\to\infty} zf(z) = 0$. Using the technique of estimating the order we can easily know that

$$f \sim o(\frac{1}{z}) \qquad as \quad z \to \infty \tag{4}$$

A useful method to calculate $\int_{\gamma_R}$ is the **Polar coordinate**. We have

$$\left| \int_{\gamma_R} \right| = \left| \int_0^{\pi} Re^{i\theta} f(Re^{i\theta}) d\theta \right|$$

and let $R \to \infty$, immediately get $\left| \int_{\gamma_R} \right| \to 0$. So we get the first formula

$$\int_{-\infty}^{\infty} f = 2\pi i Res_a f \tag{5}$$

Thus we just need to count the number of poles in the upper plane.One may ask that how many examples are belong to the form above, indeed that we at least have the **Rational Functions** such that

$$\frac{P}{Q}, \quad degP \leq degQ - 2$$

1.2 NICE2:$e^{i\alpha z} f(z)$ and $\lim_{z\to\infty, Imz\geq 0} f = 0$

The second form is little hard to find. When I first see that, it's confused for me to understand why the factor $e^{i\alpha z}(\alpha > 0)$ taking place of the condition 4. We must recognize that the second condition $\lim_{z\to\infty, Imz\geq 0} f = 0$ just provide with a weaker estimation, i.e.

$$f \to O(\frac{1}{z^p}), \quad p > 0$$

which almost has no effect because we must need at least $p > 1$! However, the factor $e^{i\alpha z}$ improves the order "in the integration". As we can see

$$\left| \int_{\gamma_R} e^{i\alpha z} f \right| = R||f||_R \int_0^{\pi} e^{i\alpha R(\cos\theta + i\sin\theta)} d\theta = R||f||_R \int_0^{\pi} e^{-\alpha R\sin\theta} d\theta$$

where $||f||_R := \max_{z\in\gamma_R} f$. Observed that "Jordan inequality":

$$\sin x \geq \frac{2}{\pi} x, \quad 0 \leq x \leq \frac{\pi}{2}$$

By the symmetry we can estimate the equation above to be

$$\left| \int_{\gamma_R} e^{i\alpha z} f \right| \leq C||f||_R$$

One must observed that the factor $e^{-\alpha R\theta}$ provides a $\frac{1}{R}$ after being integrating! Therefore we can get a same limitation $\lim_{R\to\infty} \int_{\gamma_R} = 0$. Finally we have the similar consequence like 5

Here we call the factor $e^{i\alpha z}(z > 0)$ the **Integration reducing factor**. One may find more such factors $K(z)$, just observed that

$$\int_0^{\pi} K(Re^{i\theta})d\theta \sim o(\frac{1}{R}) \qquad as \quad R \to \infty$$

However, the special factor above has many the structures we're familiar with, because one can use the *Euler* Identity

$$e^{i\alpha z} = \cos\alpha z + i\sin\alpha z$$

Then we have

$$\int_{-\infty}^{\infty} \cos\alpha x f = Re\{Res_a e^{i\alpha z} f\} \tag{6}$$

Use these formulas we can calculate many classical integration with trigonometric functions.

example 1.1.

$$\int_{\infty}^{\infty} \frac{\cos ax}{x^2+b^2} \quad (a, b > 0)$$

Observed that f has pole ib in the upper plane, and calculate the residue

$$Res_{ib} f = \frac{e^{-ab}}{2bi}$$

then

$$\int_{\infty}^{\infty} \frac{\cos ax}{x^2+b^2} = \frac{\pi}{b}e^{-ab}$$

1.3 IMPROVE:Surgery

One may find the function has singularities on Real-axis, where we need using our second trick, **SURGERY**. We have shown the same method in the proof of the *Cauchy's* integral formula, where we dug a small circle γ_ϵ[3] near the singularity and it is bridged with a large curve γ to form a holomorphic simply connected region.

Here we just suppose that 0 is the unique singularity of such f. We let the curve travel along Real-axis from $-R$ to $-\epsilon$, and "jump" away from 0 by a $\epsilon-$circle, then travel from ϵ to R and then go back to starting along the large curve γ_R. One may find the track is a $\pi-$sector. We claim that: if we want to calculate the $\int_{-\infty}^{\infty}$, we need to estimate both $\int_{\gamma_\epsilon}$ and $\int_{\gamma_R}$.

We have got the γ_R in two subsections above, now we have to solve γ_ϵ. Observed that we will let $\epsilon \to 0$, so the order of f near the singularity 0 is important. Similarly we need the estimation $\lim_{z\to 0} zf(z) = A$, which is equal to

$$f \sim \frac{A}{z} \qquad as \quad z \to 0 \tag{7}$$

Then we have

$$\int_{\gamma_\epsilon} \to \int_0^{\pi} iAd\theta = iA\pi \qquad as \quad z \to 0$$

[3]with the negative orientation.

remark 1.1. *One may find that the angle π can be replaced with $\alpha(0 < \alpha < 2\pi)$, then the integration is equal to $iA\alpha$.*

Thus we can calculate the integration by

$$\int_{-\infty}^{\infty} = iA\pi \tag{8}$$

example 1.2.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = 1 \cdot \pi = \pi \tag{9}$$

Since the symmetry, we also have $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$.

2 $\int_0^\infty f$ form integration

2.1 multivalue factors

If we haven't the symmetry, such interval need another method, but we claim that the **surgery** is also key technique. Here we mainly discuss the form like

$$\int_0^\infty K(z)f$$

where K is some kind of factor[4]. And f satisfies the property like section above, just $\sim O(1/z)$ or $o(1/z)$ as $z \to \infty$, if has the singularity, we require it to have some property such as 7. Here we use the surgery to get a region like a "keyhole". For example, let the curve travel along a little higher than Real-axis from ϵ to R and return R(but lower than it) counterclockwise along the large circle γ_R, then travel along the "lower" Real-axis to ϵ and return ϵ(but higher than it) clockwise around the small circle γ_ϵ. Thus we eliminate 0 and get a simple connected region "keyhole".

However, one may find that the path from ϵ to R and the inverse counteract, so we need the f here be some kind of **multivalue function**, or so does K. But the f is always classical function, thus we usually require K to have such property, and call it **multivalue factor**. The usual multivalue functions are power functions and logarithmic functions.

So our first example is $K = z^{p-1}(0 < p < 1)$. Then K must be multivalue function. Here we choose the principle branch by construct the region $\mathbb{C} - [0, +\infty)$ and

$$K = e^{(p-1)logz}$$

where z on the higher side of Real-axis, and another is $logz = log|z| + 2\pi i$, thus

$$K = e^{2p\pi i}e^{(p-1)log|z|}$$

Then we write the different integration

$$\int_\epsilon^R [e^{(p-1)logz} - e^{2p\pi i}e^{(p-1)log|z|}]f = (1 - e^{2p\pi i})\int_\epsilon^R x^{p-1}f \tag{10}$$

[4]generally not be the integration reducing factor.

Now we start to require f. One can observe that

$$\int_{\epsilon}^{R}+\int_{\gamma_R}+\int_{\gamma_\epsilon^-}=2\pi i Res_a z^{p-1}f$$

If f has some of poles in the region above. To simplify calculation, we require both $\int_{\gamma_R}$ and $\int_{\gamma_\epsilon^-}$ are equal to 0.

We use the polar coordinate again, let $z=re^{i\theta}(r=R,\epsilon)$. We have

$$\left|\int_{\gamma_R}\right|=R^p\int_0^{2\pi}f(Re^{i\theta})d\theta$$

Thus we need f to satisfies 4. Also, if consider

$$\left|\int_{\gamma_\epsilon}\right|=\epsilon^p\int_0^{2\pi}f(\epsilon e^{i\theta})d\theta$$

where ϵ^p provides a infinitely small, let f be **bounded** near 0.

Therefore, we require f to be:

1. $f\sim o(\frac{1}{z})\qquad as\quad z\to\infty$[5];
2. be **bounded** near 0;
3. has some poles.

Where $f=\frac{1}{z+1}$ can induce a classical integration.

example 2.1.

$$(1-e^{2p\pi i})\int_0^\infty\frac{x^{p-1}}{1+x}=2\pi i Res_{-1}\frac{z^{p-1}}{1+z}=-2\pi i e^{p\pi i}$$

Thus

$$\int_0^\infty\frac{x^{p-1}}{1+x}=\frac{\pi}{\sin p\pi},\quad 0<p<1 \tag{11}$$

2.2 More Surgeries

As we can see in the previous subsection, $K=z^{p-1}$ is a multivalue factor and we construct a "keyhole" as the surgery region. But we shall show that the region need to be chosen respect to K. Let us consider such factor $K=logz$. If we choose the region as above, the integration along Real-axis will be counteracted. Since $logz=log|z|+2\pi i$ after a period. Thus we may need to restrict f to be symmetry about the Imaginary-axis and choose the region like previous section.

example 2.2.

$$\int_0^\infty\frac{logx}{(1+x^2)^2}$$

[5] $O(1/z)$ is also suitable.

Then we can consider that

$$\int_{\epsilon}^{R} + \int_{\gamma_R} + \int_{-R}^{-\epsilon} + \int_{\gamma_\epsilon} = Res_i \frac{logz}{(1+z^2)^2}$$

It's clearly that $\int_{\gamma_R} \to 0$ as $R \to \infty$ and $\int_{\gamma_\epsilon} \to 0$ as $\epsilon \to 0$. And along $-R$ to $-\epsilon$, we have $Kf = \frac{log|x|+\pi i}{(1+x^2)^2}$. Thus

$$2\int_{0}^{\infty} \frac{logx}{(1+x^2)^2} = Res_i \frac{logz}{(1+z^2)^2}$$

Finally

$$\int_{0}^{\infty} \frac{logx}{(1+x^2)^2} = -\frac{\pi}{4} \tag{12}$$

3 $\int_0^\infty f$ form integration

Here we just discuss two kinds of integration.

3.1 Trigonometric rational function

Recall Calculus, we have a kind of integration, which must can be calculate, i.e. **trigonometric rational function**

$$\int_{0}^{2\pi} R(\cos\theta, \sin\theta)$$

where we use the **Universal formula**. Let $t = \tan\frac{\theta}{2}$, then

$$\int_{0}^{2\pi} R(\cos\theta, \sin\theta) = \int_{-\infty}^{\infty} R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{1}{1+t^2}dt$$

But, here we consider a function on the unit circle, let $z = e^{i\theta}$, so we have

$$\int_{0}^{2\pi} R(\cos\theta, \sin\theta) = \int_{C_1} R\left(\frac{1}{2i}\left(z-\frac{1}{z}\right), \frac{1}{2}\left(z+\frac{1}{z}\right)\right)\frac{1}{iz}dz$$

Similarly, we have

$$\int_{0}^{2\pi} R(\cos n\theta, \sin n\theta) = \int_{C_1} R\left(\frac{1}{2i}\left(z^n-\frac{1}{z^n}\right), \frac{1}{2}\left(z^n+\frac{1}{z^n}\right)\right)\frac{1}{iz}dz$$

One can just count the poles and calculate the Residue to get the integration.

3.2 Square root function

Now we talk about the integration

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$

where $-1 < p, q < 1$ and $p+q = -1, 0, 1$. We claim that if f has unique pole $a \notin [a,b]$, and

$$\lim_{z\to\infty} z^{p+q+1} f(z) = A \neq \infty$$

then

$$\int_a^b (x-a)^p (b-x)^q f(x) = -\frac{A\pi}{\sin qx} + \frac{\pi}{e^{-q\pi i}\sin qx} Re_a F(z)$$

where $F(z) = (z-a)^p (b-z)^q f(z)$.

First, we can estimate that if $\lim_{z\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_{\gamma_R} f = i\alpha A$$

where $\gamma_R : \{z = Re^{i\theta} | 0 \leq \theta \leq \alpha\}$.

Proof. We first prove that can choose a single value branch of F outside $[a,b]$. Indeed, Let $z-a = r_1 e^{i\theta}, z-b = r_2 e^{i\theta}$. When z along any simple closed curve containing $[a,b]$ travels a period, the argument of $z-a, z-b$ adding 2π i.e.

$$r_1^p r_2^q e^{i(p\theta_1 + q\theta_2) + 2\pi(p+q)i} = r_1^p r_2^q e^{i(p\theta_1+q\theta_2)}$$

Thus F doesn't change.

We choose sufficiently R, ϵ, such that the singularities all in this simple connected region. We have

$$\int_\Gamma + \int_{\gamma_1} + \int_{l_1} + \int_{\gamma_2} + \int_{l_2} = 2\pi i Res_a F(z)$$

By the similar method, we can estimate that the second and forth terms tend to 0, the third and last one are counteracted. So we have

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = -\frac{2\pi i e^{-q\pi i} A}{1-e^{-2q\pi i}} + \frac{2\pi i}{1-e^{-2q\pi i}} Res_a F(z)$$

Therefore

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = -\frac{\pi A}{\sin q\pi} + \frac{\pi}{e^{-q\pi i}\sin q\pi} Res_a F(z)$$

□

4 Two Important Integration

Although we have introduced some of method to calculate different kind of integration, we can't get the two forms as follows.

4.1 $Fresnel's$ integration

example 4.1.

$$\int_0^\infty \cos x^2 dx \quad and \quad \int_0^\infty \sin x^2 dx$$

Firstly, we need let $f(z) = e^{iz^2}$ immediately. And the interval of integration is form 0 to R, then let $R \to \infty$. But how to determine a closed curve such that can be used to solve this integration? Recall the "integration reducing factor" in the previous sections, we can find that e^{iz^2} can be regard as $e^{iz} \cdot e^z$. And the formal one is a such factor, which will offer a $\frac{1}{R}$ in the integration. So we can choose the classical large curve γ_R. Thus the integration will be complete if we find a computable "back-path". In other words, we need to find a nice angle of sector. In fact, the z^2 has hinted us to choose $\pi/4$, which be doubled to become a pure Imaginary!

Thus we choose the surgery region to be a $\frac{\pi}{4}-$sector. Three piecewise curves from 0 are denoted by $\gamma_1, \gamma_R, \gamma_2$. First we shall show that $\int_{\gamma_R} \to 0$. Observed that

$$\begin{aligned}
\left|\int_{\gamma_R}\right| &= \left|R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\right| \\
&\leq \left|R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{4}{\pi}\theta}\right| \\
&= \frac{C}{R} \to 0 \qquad as \quad R \to \infty
\end{aligned}$$

Second, let $z = re^{i\pi/4}(0 \leq r \leq R)$, we have

$$\int_{\gamma_2} = -e^{i\pi/4}\int_0^R e^{-r^2}dr \to -e^{i\pi/4}\frac{\sqrt{\pi}}{2} \qquad as \quad R \to \infty$$

Finally we get the integration above

$$\int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4} \quad and \quad \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4} \tag{13}$$

remark 4.1. *One can calculate the general form* $\int_0^\infty \cos x^n dx$ *by choose the* $\pi/2n-$*sector, and the value is just changed by* $e^{i\pi/2n}\frac{\sqrt{\pi}}{4}$, *i.e.*

$$\int_0^\infty \cos x^n dx = \cos\frac{\pi}{2n}\frac{\sqrt{\pi}}{2} \tag{14}$$

4.2 $Poisson's$ integration

example 4.2.

$$\int_0^\infty e^{-ax^2}\cos bx dx, \quad a > 0$$

Firstly, we choose $f(z) = e^{-az^2}$, there are two questions:

1. How to get the factor $\cos bx$?

2. What kind of surgery region to choose?

Indeed, We can see that if change the z to $x+yi$, then

$$e^{az^2} = e^{-a(x^2-y^2)+2axyi}$$

where x is the Real-axis and y^2 can be regard as a constant if we integrate along the Real-axis. Also observed that $e^{i\theta} = \cos\theta + i\sin\theta$, the factor $2axyi$ can be drop to trigonometric function. With the skew-symmetry, we only have cos. The second question will solved when consider that e^{-az^2} is a **Reducing factor**. In fact the region with $R \to \infty$ all can tend the integration to 0! But since the discussion above, we choose the **rectangular region**.

remark 4.2. *One may claim that:*

1. *Polar coordinate: Sector Region*
2. *Real-Imaginary: Rectangular Region*

But how to choose the height of rectangular? In fact we have solve this question in the constructing of cos. We find that $2axy = b$, so $y = \frac{b}{2a}$. Thus we get the surgery region and denote them by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$(denote them counterclockwise from 0). It's clearly that

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0$$

We shall show that the second and last one tend to 0, and the first one can be calculate. Observed that

$$\left|\int_{\gamma_2}\right| = \int_0^{\frac{b}{2a}} e^{-a(R+yi)^2} dy = \int_0^{\frac{b}{2a}} e^{-a(R^2-y^2)} dy \sim O(e^{-aR^2}) \to 0$$

so does γ_4. And it's clearly that $\int_{\gamma_1} \to \sqrt{\pi/a}$. Finally

$$\int_0^{\infty} e^{-ax^2} \cos bx dx = \frac{1}{2} e^{-b^2/4a} \sqrt{\pi/a} \tag{15}$$

5 Exercises

We give some exercises, one can **imitate** or **improve** the techniques above.

exercise 1.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

exercise 2.

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{(2n-1)!!}{(2n)!!} \cdot \pi$$

exercise 3.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \pi \frac{e^{-a}}{a}, \quad a > 0$$

exercise 4.

$$\int_0^\infty \frac{1+\cos x}{x^2}dx = \frac{\pi}{2}$$

exercise 5.

$$\int_{-\infty}^\infty \frac{e^{px}}{1+e^x}dx = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1$$

Recall example2.1.

exercise 6.

$$\int_0^{2\pi} \frac{d\theta}{a+b\theta} = \frac{2\pi}{\sqrt{a^2+b^2}}$$

exercise 7.

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{(1+x)^2(1-x)}}$$

exercise 8.

$$e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{-2\pi i x\xi}dx$$

[hint: Recall that

$$1 = \int_{-\infty}^\infty e^{-\pi x^2}dx$$

and construct a rectangle region.]