5) let $\mathbb{Z} \oplus \mathbb{Z}$ be cyclic,

$\therefore \mathbb{Z} \oplus \mathbb{Z} = \langle (a,b) \rangle$, $a, b \in \mathbb{Z}$

$\langle (a,b) \rangle = \{ (a,b)^i \}$, $i \in \mathbb{Z}$,

But $(a, b+1) \notin \langle (a,b) \rangle$ $(\Rightarrow \Leftarrow)$

$\therefore \mathbb{Z} \oplus \mathbb{Z}$ is not cyclic //

7) WTS $G_1 \oplus G_2 \approx G_2 \oplus G_1$

let $\phi : G_1 \oplus G_2 \to G_2 \oplus G_1$

$\phi [(a,b)] = (b,a)$, $a \in G_1$, $b \in G_2$

let $\phi [(a,b)] = \phi [(a',b')]$

$(b,a) = (b',a')$

$\therefore a = a'$, $b = b'$

$\therefore (a,b) = (a',b')$

$\therefore \phi$ is one-to-one //

let $(b,a) \in G_2 \oplus G_1$,,

$\exists (a,b) \in G_1 \oplus G_2 : \phi[(a,b)] = (b,a)$

$\therefore \phi$ is onto //

$$\phi[(a_1,b_1)(a_2,b_2)] = \phi[(a_1a_2, b_1b_2)]$$
$$= (b_1b_2, a_1a_2)$$
$$= (b_1,a_1)(b_2,a_2)$$
$$= \phi[(a_1,b_1)]\phi[(a_2,b_2)]$$
//

$\therefore \phi$ is OP //

$\therefore G_1 \oplus G_2 \approx G_2 \oplus G_1$

11) By Thm 8.1,

$$|(g_1, \ldots, g_n)| = \operatorname{lcm}(|g_1|, \ldots, |g_n|)$$

$\therefore$ The elements of order 4 in $Z_4 \oplus Z_4$ would have the property

$$4 = |(a,b)| = \operatorname{lcm}(|a|,|b|)$$

$\therefore |a| = 1, |b| = 4$ $\quad |a| = 2, |b| = 4$

$|a| = 4, |b| = 4$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

Case 1 : $|a| = 1, |b| = 4$

$a = 0, b \in \{1, 2, 3\}$

$\therefore$ There are $1 \times 3 = 3$ elements

Case 2 : $|a| = 2, |b| = 4$

$a = 2, b \in \{1, 2, 3\}$

$\therefore$ There are $1 \times 3 = 3$ elements

Case 3 : $|a| = 4, |b| = 4$

$a = 3, b \in \{1, 2, 3\}$

$\therefore$ There are $1 \times 3 = 3$ elements

$\therefore$ There are 9 elements of order 4 in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

16) $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$

By Thm 8.1,

$$12 = |(a,b)| = \text{lcm}(|a|, |b|)$$

$\therefore |a| = 12, \ |b| \in \{1, 2, 3, 4, 6, 12\}$

$a \in \{20, 30, 40\}, \ b \in \{0, 15, 10, 20, \ldots\}$

$|(20, 0)| = |\langle(20, 0)\rangle| = 12$

$|(30, 15)| = |\langle(30, 15)\rangle| = 12$ //

24) $S_3 \oplus \mathbb{Z}_2$

(123) (213) ~~(312)~~

~~(132)~~ ~~(231)~~ ~~(321)~~

(23) (12) (13)

~~(32)~~ ~~(21)~~ ~~(31)~~

(1)

$|((1), 0)| = 1$

$|((1), 1)| = 2$

$|((12),0)| = |((13),0)| = |((23),0)| = 2$

$|((12),1)| = |((13),1)| = |((23),1)| = 2$

$|((123),0)| = |((213),0)| = 3$

$|((123),1)| = |((213),1)| = 6$

$1 \in \mathbb{Z}_{12}, \quad |1| = 12$

By Thm 6.2 (v),

$G \approx \overline{G} \Rightarrow |a| = |\phi(a)|$

$\neg(\exists a \in S_3 \oplus \mathbb{Z}_2 : |a| = 12) \Rightarrow S_3 \oplus \mathbb{Z}_{12} \not\approx \mathbb{Z}_{12}$

$|0| = 1$ $\qquad$ $|0| = 1$

$|1| = 6$ $\qquad$ $|1| = 2$

$|2| = 3$ $\qquad$ $1, 6, 3, 2, 3$

$|3| = 2$ $\qquad$ $2, 6, 6, 2, 6, 6$

$|4| = 3$

$|S| = 6$

$(0,1), (3,1), (3,0) \in \mathbb{Z}_6 \oplus \mathbb{Z}_6$

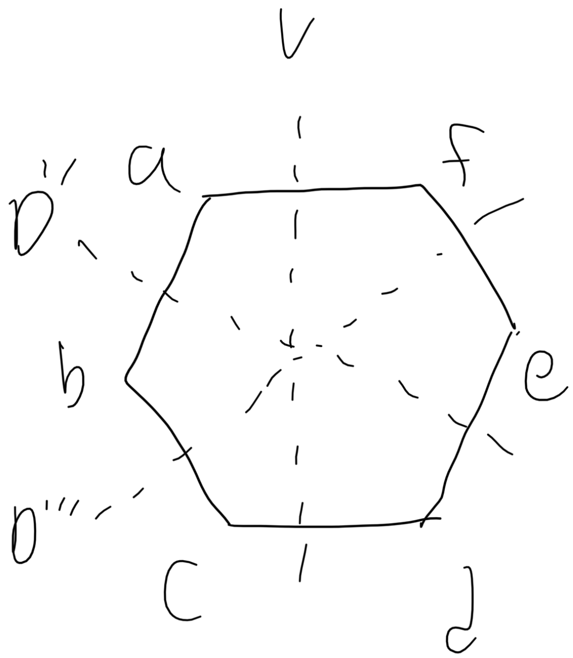
$|(0,1)| = |(3,1)| = |(3,0)| = 2$

$S_3 \oplus \mathbb{Z}_2$ has 7 elements of order 2,

$\mathbb{Z}_6 \oplus \mathbb{Z}_2$ has 3 elements of order 2,

By Thm 6.1 (V),

$S_3 \oplus \mathbb{Z}_2 \not\approx \mathbb{Z}_6 \oplus \mathbb{Z}_2$



$|R_0| = 1$ $\quad |R_{300}| = 6$

$|R_{60}| = 6$ $\quad |H| = 2$

$|R_{120}| = 3$ $\quad |V| = 2$

$|R_{180}| = 2$ $\quad |D| = 2$ $\quad |D''| = 2$

$|R_{240}| = 3$ $\quad |D'| = 2$ $\quad |D'''| = 2$

By Table 5.1,

$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in A_4$

$|\alpha_1| = |\alpha_2| = |\alpha_3| = |\alpha_4| = 1$

$\therefore S_3 \oplus \mathbb{Z}_2 \not\approx A_4$

$\therefore S_3 \oplus \mathbb{Z}_2 \approx D_6$ //

26) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$

$= \{(0,0), (1,0), (2,0), (3,0),$
$(0,1), (1,1), (2,1), (3,1)\}$

$\langle(1,1)\rangle = \{(1,1), (2,0), (3,1), (0,0)\}$

$\langle(1,1)\rangle \leq \mathbb{Z}_4 \oplus \mathbb{Z}_2$

$|(1,1)| = 4 \Rightarrow H = \mathbb{Z}_4$

$\mathbb{Z}_2$ has 2 subgroups, $K = \{0\}$ and

$K = \{0,1\}$

$(1,1), (3,1) \in \langle (1,1) \rangle \Rightarrow K \neq \{0\}$

If $K = \{(0,1)\}$, $H \oplus K = Z_4 \oplus Z_2$

But $\langle (1,1) \rangle \neq Z_4 \oplus Z_2$ $(\Rightarrow)(\Leftarrow)$

$\therefore \langle (1,1) \rangle \leq Z_4 \oplus Z_2$, $\langle (1,1) \rangle \neq H \oplus K$

35) $G = \{3^m 6^n : m, n \in \mathbb{Z}\}$ under multiplication.

WTS $G \approx \mathbb{Z} \oplus \mathbb{Z}$

Let $\phi : G \to \mathbb{Z} \oplus \mathbb{Z}$

$\phi(3^m 6^n) = (m, n)$

Let $\phi(3^a 6^b) = \phi(3^c 6^d)$

$(a, b) = (c, d)$

$\therefore a = c, b = d$

$\therefore 3^a 6^b = 3^c 6^d$

$\therefore \phi$ is one-to-one

Let $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$,

$m, n \in \mathbb{Z} \Rightarrow \exists\, 3^m 6^n \in G : \phi(3^m 6^n) = (m,n)$

$\therefore \phi$ is onto

$$\begin{aligned}
\phi(3^a 6^b \cdot 3^c 6^d) &= \phi(3^{a+c} 6^{b+d}) \\
&= (a+c, b+6) \\
&= (a,b) + (c,d) \\
&= \phi(3^a 6^b) + \phi(3^c 6^d)
\end{aligned}$$

$\therefore \phi$ is OP.

$\therefore G \approx \mathbb{Z} \oplus \mathbb{Z}$ //

No, since $3^0 9^1 = 3^2 9^0$.

$\phi(3^0 9^1) = (0,1) \neq (2,0) = \phi(3^2 9^0)$

ϕ is not well-defined //

39) G has 24 elements of order 6

$a \in G$, $|a| = 6$,

By Thm 4.4, $\langle a \rangle$ is cyclic, $6|6$,

the number of elements in $\langle a \rangle$ of order 6 is $\phi(6) = 2$, namely a, a^5

Since $a^5 \in \langle a \rangle$, $|a^5| = 6$,

$$\langle a^5 \rangle = \langle a \rangle$$

$\therefore$ G has $24/2 = 12$ cyclic subgroups of order 6 //

42) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

$a \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $(0,0)a = a(0,0) = (0,0)$

$\therefore (0,0) = e$

$a \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $a^{-1} = a$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is abelian

Let $\phi \in Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$,

By Thm 6.2 (i),

$\phi[(0,0)] = (0,0)$

By Thm 6.2 (v),

$|(1,1)| = 2 = \phi[(1,1)]$

$|(0,1)| = 2 = \phi[(0,1)]$

$|(1,0)| = 2 = \phi[(1,0)]$

$\therefore \phi \in Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$,

$$|\phi[(1,1)]| = |\phi[(0,1)]| = |\phi[(1,0)]| = 2$$

Let $\phi_1[(1,1)] = (1,1)$
$\phi_1[(0,1)] = (0,1)$
$\phi_1[(1,0)] = (1,0)$

$\phi_2[(1,1)] = (1,1)$
$\phi_2[(0,1)] = (1,0)$
$\phi_2[(1,0)] = (0,1)$

$\phi_3[(1,1)] = (0,1)$
$\phi_3[(0,1)] = (1,1)$
$\phi_3((1,0)) = (1,0)$

$\phi_4[(1,1)] = (0,1)$
$\phi_4((0,1)) = (1,0)$
$\phi_4((1,0)) = (1,1)$

$\phi_5[(1,1)] = (1,0)$
$\phi_5((0,1)) = (0,1)$
$\phi_5((1,0)) = (1,1)$

$\phi_6[(1,1)] = (1,0)$
$\phi_6((0,1)) = (1,1)$
$\phi_6((1,0)) = (0,1)$

$\phi \in Aut(Z_2 \oplus Z_2)$,

$\phi_1 \phi = \phi$, $\phi \phi_1 = \phi$

$\therefore \phi_1$ is the identity element

$\phi^{-1} = \phi$
//

48) $\phi : Z_3 \oplus Z_5 \to Z_{15}$

$\phi(2,3) = 2$

By Thm 6.2 (iv), $G \approx \overline{G}$,

$G = \langle a \rangle \Longleftrightarrow \overline{G} = \langle \phi(a) \rangle$

$\therefore Z_{15} = \langle 1 \rangle \Longleftrightarrow Z_3 \oplus Z_5 = \langle a \rangle$

By Thm 8.2, $Z_3 = \langle 1 \rangle$, $Z_5 = \langle 1 \rangle$,

$|Z_3| = 3$, $|Z_5| = 5$,

$\mathbb{Z}_3 \oplus \mathbb{Z}_5$ is cyclic $\Leftrightarrow$ $\gcd(|\mathbb{Z}_3|, |\mathbb{Z}_5|)$

$\gcd(|\mathbb{Z}_3|, |\mathbb{Z}_5|) = \gcd(3, 5)$
$= 1$

$\therefore$ $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ is cyclic

$\langle (1,1) \rangle = \{(1,1), (2,2), (0,3), (1,4),$
$(2,0), (0,1), (1,2), (2,3),$
$(0,4), (1,0), (2,1), (0,2),$
$(1,3), (2,4), (0,0)\}$

$= \mathbb{Z}_3 \oplus \mathbb{Z}_5$

$\therefore$ $\phi(1,1) = 1$ //

53) p is a prime

WTS $\mathbb{Z}_p \oplus \mathbb{Z}_p$ has exactly $p+1$ subgroups of order p

$p=2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

$|\langle(0,1)\rangle| = |\langle(1,0)\rangle| = |\langle(1,1)\rangle| = 2$

$|(a,b)| = \text{lcm}(|a|, |b|) = p$

$|a| = 1, |b| = p$ and $|a| = p, |b| \in \{1, p\}$

Case 1: $|a| = 1, |b| = p$

By Thm 4.4, $\mathbb{Z}_p = \langle 1 \rangle$, $1 | p$, there are

$\phi(1) = 1$ element $a \in \mathbb{Z}_p : |a| = 1$.

$p | p$, there are $\phi(p) = p-1$ elements

$b \in \mathbb{Z}_p : |b| = p$

$\therefore$ There are $1(p-1) = p-1$ elements

$(a,b) \in \mathbb{Z}_p \oplus \mathbb{Z}_p : |(a,b)| = p$

Case 2: $|a|=p$, $|b|\in\{1,p\}$

By Thm 4.4, $\mathbb{Z}_p=\langle 1\rangle$, $p|p$, there are

$\phi(p)=p-1$ elements $a\in\mathbb{Z}_p : |a|=p$

$1|p$, there are $\phi(1)=1$ element $b\in\mathbb{Z}_p : |b|=1$

$p|p$, there are $\phi(p)=p-1$ elements $b\in\mathbb{Z}_p : |b|=p$

$\therefore$ There are $(p-1)(1+p-1)=(p-1)(p)=p^2-p$ elements $(a,b)\in\mathbb{Z}_2\oplus\mathbb{Z}_2 : |(a,b)|=p$

$\therefore$ There are $p-1+p^2-p=p^2-1$ elements $(a,b)\in\mathbb{Z}\oplus\mathbb{Z} : |(a,b)|=p$

By Thm 4.4, $|\langle a,b\rangle|=p$, $p|p$, there are

$\phi(p)=p-1$ elements of order p in $\langle a,b\rangle$

$\therefore$ There are totally $\dfrac{p^2-1}{p-1}=\dfrac{(p+1)(p-1)}{p-1}$

$=p+1$

Elements $(a,b) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$: $|(a,b)| = p$

59) $\phi \in Aut(\mathbb{Z}_{720})$; $|\phi| = 6$

By Corollary 8.3.1,

$m = n_1 n_2 \cdots n_k$, $\gcd(n_i, n_j) = 1$, $i \neq j$.

$U(m) \approx U(n_1) \oplus \cdots \oplus U(n_k)$

By the facts

$U(2) \approx \{0\}$, $U(4) \approx \mathbb{Z}_2$, $U(2^n) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-2}}$, $n \geq 3$,

and

$U(p^n) \approx \mathbb{Z}_{p^n - p^{n-1}}$, p is an odd prime

$$\begin{aligned} U(720) &= U(16 \cdot 9 \cdot 5) \\ &\approx U(16) \oplus U(9) \oplus U(5) \\ &= U(2^4) \oplus U(3^2) \oplus U(5) \\ &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4 \end{aligned}$$

Since $Aut(\mathbb{Z}_{720}) \approx U(720)$, the number of elements of order 6 in $Aut(\mathbb{Z}_{720})$ is the same as in $U(720)$

By Thm 8.2, $(a,b,c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$,

$6 = |(a,b,c,d)| = \operatorname{lcm}(|a|, |b|, |c|, |d|)$

$a \in \mathbb{Z}_2$, $|a| \in \{1,2\}$

$b \in \mathbb{Z}_4$, $|b| \in \{1,2,4\}$

$c \in \mathbb{Z}_6$, $|c| \in \{1,2,3,6\}$

$d \in \mathbb{Z}_4$, $|d| \in \{1,2,4\}$

$\therefore$ $a = 1$, $b = 1$, $c = 6$, $d = 1$

$a = 1$, $b = 1$, $c = 3$, $d = 2$

$a = 2$, $b = 1$, $c = 3$, $d = 1$

$a = 1$, $b = 2$, $c = 3$, $d = 1$

Case 1 : $a=1, b=1, c=6, d=1$

By Thm 4.4,

There are $\phi(1)=1$ elements $a \in \mathbb{Z}_2 : |a|=1$

$\phi(1)=1$ elements $b \in \mathbb{Z}_4 : |b|=1$

$\phi(6)=2$ elements $c \in \mathbb{Z}_6 : |c|=6$

$\phi(1)=1$ elements $d \in \mathbb{Z}_4 : |d|=1$

$\therefore$ There are $1 \cdot 1 \cdot 2 \cdot 1 = 2$ elements

$(a,b,c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4 : |(a,b,c,d)| = 6$

Case 2 : $a=1, b=1, c=3, d=2$

There are $\phi(1)=1$ elements $a \in \mathbb{Z}_2 : |a|=1$

$\phi(1)=1$ elements $b \in \mathbb{Z}_4 : |b|=1$

$\phi(3)=2$ elements $c \in \mathbb{Z}_6 : |c|=3$

$\phi(2)=1$ elements $d \in \mathbb{Z}_4 : |d|=2$

$\therefore$ There are $1 \cdot 1 \cdot 2 \cdot 1 = 2$ elements

Similar to Case 3, Case 4,

$\therefore$ There are $2+2+2+2=8$ elements

$(a,b,c,d) \in \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4 : |(a,b,c,d)| = 6$ //