1) $H = \{(1), (12)(34), (13)(24), (14)(23)\}$

$\alpha_1 H = (1)H = H$

$\alpha_2 H = (12)(34)H$

$= \{(12)(34)(1), (12)(34)(12)(34),$
$(12)(34)(13)(24), (12)(34)(14)(23)\}$

$= \{(12)(34), (1), (14)(23), (13)(24)\}$

$\alpha_3 H = \{(13)(24), (14)(23), (1), (12)(34)\}$

$\alpha_4 H = \{\alpha_4, \alpha_3, \alpha_2, \alpha_1\}$

$\alpha_5 H = \{\alpha_5, \alpha_8, \alpha_6, \alpha_7\}$

$\alpha_6 H = \{\alpha_6, \alpha_7, \alpha_5, \alpha_8\}$

$\alpha_7 H = \{\alpha_7, \alpha_6, \alpha_8, \alpha_5\}$

$\alpha_8 H = \{\alpha_8, \alpha_5, \alpha_7, \alpha_6\}$

$\alpha_9 H = \{\alpha_9, \alpha_{11}, \alpha_{12}, \alpha_{10}\}$

$\alpha_{10} H = \{\alpha_{10}, \alpha_{12}, \alpha_{11}, \alpha_9\}$

$\alpha_{11} H = \{\alpha_{11}, \alpha_9, \alpha_{10}, \alpha_{12}\}$

$\alpha_{12} H = \{\alpha_{12}, \alpha_{10}, \alpha_9, \alpha_{11}\}$

2) $4! = 24$

6) $n \in \mathbb{Z}^+$, $H = \{0, \pm n, \pm 2n, \pm 3n, \cdots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

let $a+H = b+H$, by lemma 7.1 (v),

$a+H = b+H \Leftrightarrow b-a \in H$

$\therefore n \mid b-a \Rightarrow a \equiv b \mod n$

$\therefore a+H = b+H \Rightarrow a \equiv b \mod n$

$\therefore$ The left cosets of H in $\mathbb{Z}$ are

$0+H, 1+H, \dots, (n-1)+H$

$\therefore$ # left cosets of H in $\mathbb{Z}$ $= n$

12) $H = \{a+bi \in \mathbb{C}^* : a^2+b^2=1\}$

$(3+4i)H$

$(3+4i)(a+bi) = 3a + 3bi + 4ai - 4b$

$= 3a - 4b + (4a+3b)i$

$(3a-4b)^2 + (4a+3b)^2$

$= 9a^2 - 24ab + 16b^2 + 16a^2 + 24ab + 9b^2$

$= 9a^2 + 16a^2 + 16b^2 + 9b^2$

$= 25a^2 + 25b^2$

$= 25(a^2+b^2)$

$= 25$

$(3+4i)H$ is a set of points on a circle with radius 25.

$(ca-db)^2 + (da+cb)^2$

$= c^2a^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2$

$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$

$= a^2(c^2+d^2) + b^2(c^2+d^2)$

$= (c^2+d^2)(a^2+b^2)$

$= c^2+d^2$

$(c+di)H$ is a set of points on a circle with radius c^2+d^2. //

14) $K < H$, $H < G$

$|K| = 42$, $|G| = 420$

By Lagrange's Thm,

$H \leq G \Rightarrow |H| \mid |G|$

$K < H$, $|K| = 42 \Rightarrow 42 \mid |H|$

$H < G$, $|G| = 420 \Rightarrow |H| \mid 420$

$\therefore |H| = 42s$, $420 = |H|t$

$|H| \in \{0, 42, 84, 126, 168, 210, 252, 294,$

$336, 378, 420, \ldots\}$

$|H| \mid 420 \Rightarrow |H| \in \{42, 84, 210, 420\}$ //

16) WTS $\gcd(a,n) = 1 \Rightarrow a^{\phi(n)} \bmod n = 1$

$\phi(n)$ is the number of positive integers less than n and relatively prime to n

Let $a \in \mathbb{Z}$, $\gcd(a,n) = 1$

$\therefore a \in U(n)$

By Corollary 7.1.4,

$|G| = n,\ a \in G \Rightarrow a^{|G|} = e$

$|U(n)| < n,\ a \in U(n) \Rightarrow a^{|U(n)|} = 1$

$\phi(n) = |U(n)| \Rightarrow a^{\phi(n)} \bmod n = a^{|U(n)|} \bmod n$

$= 1 \bmod n$

$= 1$ //

17) By Fermat's Little Thm,

$$a^p \bmod p = a \bmod p, \quad p \text{ is prime}$$

$$5^{15} \bmod 7 = 5^7 5^7 5 \bmod 7$$
$$= 5 \cdot 5 \cdot 5 \bmod 7$$
$$= 125 \bmod 7$$
$$= 6 \quad //$$

$$7^{13} \bmod 11 = 7^{11} 7^2 \bmod 11$$
$$= 7 \cdot 5 \bmod 11$$
$$= 2 \quad //$$

18) WTS $n > 2 \Rightarrow |U(n)|$ is even

By Corollary 7.1.2, $|G| = n \Rightarrow a \in G, |a| \mid |G|$

Let $n > 2$,

$|U(n)| < n \Rightarrow a \in U(n), |a| \mid |U(n)|$

$U(3) = \{1, 2\} = \langle 2 \rangle$

$U(4) = \{1, 3\} = \langle 3 \rangle$

$U(5) = \{1, 2, 3, 4\} = \langle 2 \rangle = \langle 3 \rangle, \langle 4 \rangle = \{1, 4\}$

$U(6) = \{1, 5\} = \langle 5 \rangle$

$U(7) = \{1, 2, 3, 4, 5, 6\} = \langle 3 \rangle = \langle 5 \rangle$

$\langle 2 \rangle = \{2, 4, 1\}$

$\langle 4 \rangle = \{4, 2, 1\}$

$\langle 6 \rangle = \{6, 1\}$

$U(8) = \{1, 3, 5, 7\}$

$\langle 3 \rangle = \{3, 1\}$

$\langle 5 \rangle = \{5, 1\}$

$\langle 7 \rangle = \{7, 1\}$

Since $(n-1)^2 \bmod n = n^2 - 2n + 1 \bmod n$

$= 0 - 0 + 1 \bmod n$

$= 1 \bmod n$

$\therefore\ n-1 \in U(n),\ |n-1| = 2$

$\therefore\ 2 \mid |U(n)| \Rightarrow |U(n)|$ is even //

21) $H \leq S_4,\ (12), (234) \in H$

WTS $H = S_4$

$|S_4| = 4! = 24$

$(12)(234) = (1234)$

$|\alpha| = \text{lcm}(\text{disjoint cycle lengths})$

$|(12)| = 2,\ |(234)| = 3,\ |(1234)| = 4$

$2, 3, 4 \mid |H|,\ |H| \mid |S_4| = 24$

$\therefore\ |H| \in \{12, 24\}$

If $|H| = 12$, by Thm 5.7,

there are 6 even permutations in H

and they form a subgroup of order 6 of

A_4, which by example 7.5 doesn't exist.

$\therefore |H| = 24$ //

29) Corollary 4.4.1, $|G| = n \Rightarrow \# a \in G : |a| = d$

is $\phi(d)k$

$|G| = 55$, $a \in G : |a| = 11$

$\phi(11) = 10$

$\therefore \# a \in G : |a| = 11$ is $10k$

By Lagrange's Thm,

$|G| = n$, $H \leq G \Rightarrow |H| \mid |G|$

Let $a \in G$, $|a| = 11$,

$\langle a \rangle \leq G \Rightarrow |\langle a \rangle| = |a| \mid |G|$

$\therefore |a| \mid 55 \Rightarrow |a| \in \{1, 5, 11, 55\}$

31) $|G| = n$

$H, K \leq G$

$H \subseteq K \subseteq G$

WTS $|G:H| = |G:K|\,|K:H|$

$H, K \leq G \Rightarrow |H| \mid |G|,\ |K| \mid |G|$

$|G:H| = |G|/|H|$

$|G:K| = |G|/|K| \Rightarrow |G| = |G:K|\,|K|$

$|K:H| = |K|/|H| \Rightarrow |H| = |K|/|K:H|$

$\therefore |G:H| = |G|/|H|$

$= |G:K|\,|K|/(|K|/|K:H|)$

$= |G:K|\,|K|\,|K:H|/|K|$

$= |G:K|\,|K:H|$ //

36) $|G| = p^n$, p is prime

WTS $|Z(G)| \neq p^{n-1}$

$Z(G) = \{a \in G : \forall x \in G, ax = xa\}$

FSOC, let $|Z(G)| = p^{n-1}$,

$|Z(G)| \neq |G| \Rightarrow Z(G) \neq G$

$\therefore G$ is non-abelian

let $x \in G$, $x \notin Z(G)$,

let $x^k \in Z(G)$, $2 \leq k \leq p-1$,

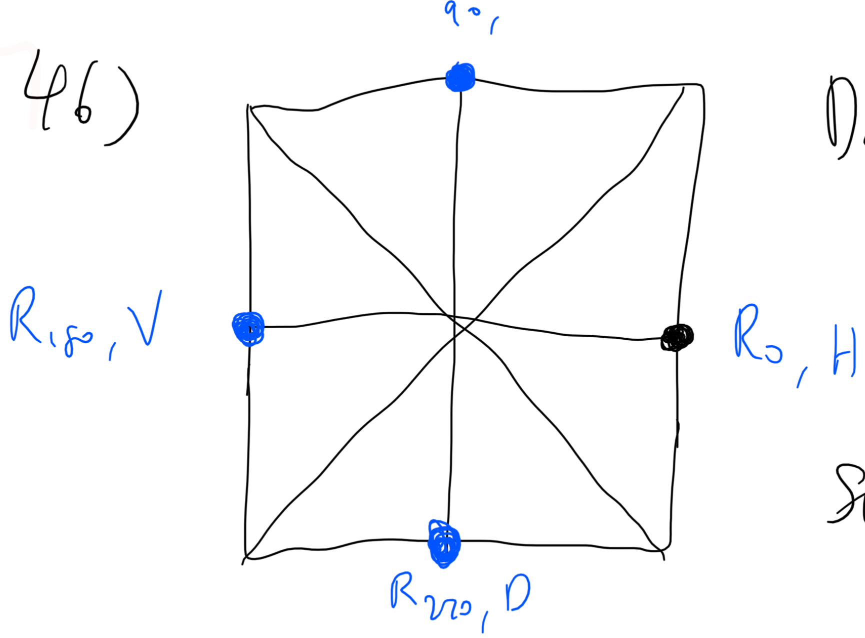
$|Z(G)| = p^{n-1} \Rightarrow (x^k)^{p^{n-1}} = e$

$\therefore x^{p^{n-1}k} = e \Rightarrow |x| \mid p^{n-1}k$

$|x| \mid |G| = p^n \Rightarrow k = p^t$ $(\Rightarrow\Leftarrow)$

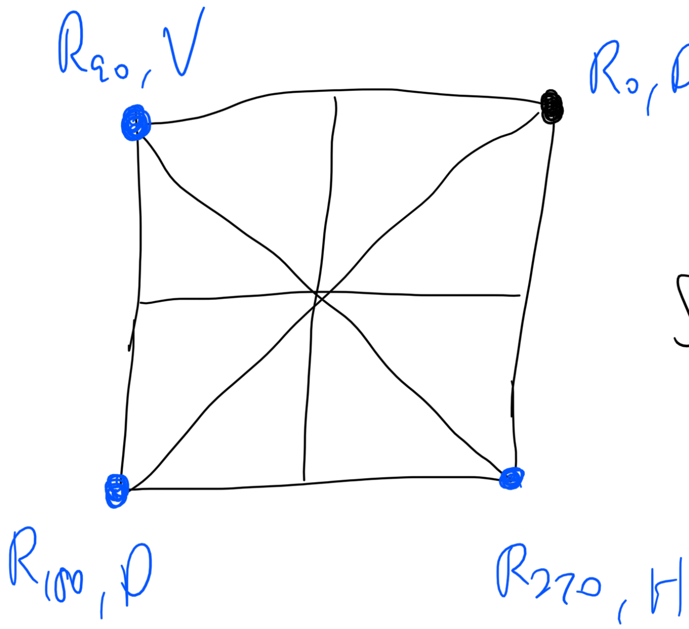
$\therefore x^k \notin Z(G)$, $2 \leq k \leq p-1$

4b)



$D_4 = \{R_{90}, R_{180}, R_{270}, R_0, H, V, D, D'\}$

$Stab_{D_4}(i) = \{R_0, H\}$



$Stab_{D_4}(i) = \{R_0, D'\}$

4a) $|G| = n < 100$

G has subgroups of orders 10, 25

By Lagrange's Thm,

$10 \mid |G|, \quad 25 \mid |G|$

$\therefore |G| = 50$ //