2) $\{0, 2, 4, 6, 8\}$ is a ring under addition and multiplication modulo 10.

Since $0(6) = 0$

$2(6) = 2$

$4(6) = 4$

$6(6) = 6$

$8(6) = 8$

6 is the unity. $\square$

5) Thm 12.2, Let R be a ring, then

1) R has a unique unity

2) $a \in R$, $a^{-1} \in R \Rightarrow a^{-1}$ is unique

Pf

Let R be a ring.

1) Assume that $e_1, e_2 \in R$: $\forall a \in R$, $e_1 a = a e_1 = a$,
$e_2 a = a e_2 = a$

$\therefore e_1 e_2 = e_2 e_1 = e_1$
$e_2 e_1 = e_1 e_2 = e_2$

$\therefore e_1 = e_1 e_2 = e_2 e_1 = e_2$ //

2) Assume that $a \in R$, $\exists a_1, a_2 \in R$: $a_1 a = a a_1 = e$,

$a_2 a = a a_2 = e$

$\therefore a_1 a = a_2 a$

$a_1 a a_1 = a_2 a a_1$

$a_1 e = a_2 e$

$a_1 = a_2$

$a a_1 = a a_2$

$a_1 a a_1 = a_1 a a_2$

$e a_1 = e a_2$

$a_1 = a_2$ $\square$

8) Let R be a ring, $a, b, c \in R$,

WTS $(ab = ca, a \neq 0 \Rightarrow b = c) \Rightarrow R$ is commutative

Assume that $a, b, c \in R$, $a \neq 0$, $ab = ca \Rightarrow b = c$,

Let $x, y \in R$, $a = x$, $b = yx$, $c = xy$,

$\therefore ab = x(yx) = (xy)x = ca \Rightarrow b = c$

$\therefore b = c \Rightarrow xy = yx$

Hence R is commutative. $\square$

13) Let $\mathbb{Z}$ be a ring.

$\emptyset, \mathbb{Z}$ are subrings of $\mathbb{Z}$.

By Thm 12.3, $S \subseteq \mathbb{Z}$, $S \neq \emptyset$, then

$a, b \in S$, $a-b$, $ab \in S \Rightarrow S$ is a subring of R.

$\therefore 2\mathbb{Z} = \{0, \pm 2, \pm 4, \ldots\}$

$4\mathbb{Z} = \{0, \pm 4, \pm 8, \ldots\}$

$3\mathbb{Z} = \{0, \pm 3, \pm 6, \dots\}$

$5\mathbb{Z} = \{0, \pm 5, \pm 10, \dots\}$

$\vdots$

are subrings of $\mathbb{Z}$.

$\therefore n\mathbb{Z}$, $n \in \mathbb{Z}$, is a subring of $\mathbb{Z}$

18) let $a \in R$,

let $S = \{x \in R : ax = 0\}$

WTS S is a subring of R

By Thm 12.3, $S \subseteq \mathbb{Z}$, $S \neq \emptyset$, then

$a, b \in S$, $a-b$, $ab \in S \Rightarrow S$ is a subring of R.

let $x, y \in S$,

$ax = 0$, $ay = 0$

$\therefore ax = ay$

$ax - ay = 0$

$a(x-y) = 0 \Rightarrow x - y \in S$ //

$\therefore ax = ay$

$axy = ayy$

$a(xy) = (ay)y = 0y = 0 \Rightarrow xy \in S$ //

$\therefore$ S is a subring of R $\square$

22) Let R be a commutative ring with unity.

Let $U(R)$ be the set of units of R

WTS $U(R)$ is a group under the multiplication of R.

Let $1 \in R$ be the unity,

$1(1) = 1 \Rightarrow 1 \in U(R)$ //

Let $a, b \in U(R)$, $a^{-1}, b^{-1} \in R$,

$a, b \in R \Rightarrow ab \in R$

$a^{-1}, b^{-1} \in R \Rightarrow (ab)^{-1} = b^{-1}a^{-1} \in R$,

$\therefore (ab)(ab)^{-1} = 1 \Rightarrow ab \in U(R)$

$\therefore U(R)$ is closed under multiplication //

Let $a \in U(R)$, $a^{-1} \in R$,

$a^{-1}a = 1 \Rightarrow a^{-1} \in U(R)$

$\therefore \forall a \in U(R), a^{-1} \in U(R)$ //

$\therefore U(R)$ is a group under multiplication $\square$

28) WTS In $\mathbb{Z}_6$, $4|2$

$\mathbb{Z}_8$, $3|7$

$\mathbb{Z}_{15}$, $9|12$

$4 \in \mathbb{Z}_6$, $4(2) = 8 \bmod 6 = 2 \Rightarrow 4|2$

$3 \in \mathbb{Z}_8$, $3(5) = 15 \bmod 8 = 7 \Rightarrow 3|7$

$9 \in \mathbb{Z}_{15}$, $9(8) = 72 \bmod 15 = 12 \Rightarrow 9|12$ $\square$