1) c, d

3) None

5) (a) 13 in $\mathbb{Z}_{20}$

$\mathbb{Z}_{20} = \{0, 1, \dots, 19\}$, $e = 0$

$(13 + 7) \bmod 20 = 0$

$\therefore 13^{-1} = 7$ //

(b) 13 in $U(14)$

$U(14) = \{1, 3, 5, 9, 11, 13\}$, $e = 1$

$(13 \cdot 13) \bmod 14 = 1 \Rightarrow 13^{-1} = 13$ //

(c) $n-1$ in $U(n)$, $n > 2$, $e = 1$

$[(n-1)(n-1)] \bmod n$

$= (n^2 - 2n + 1) \bmod n$

$= 1$

$\therefore (n-1)^{-1} = n-1$ //

(d) $3 - 2i$ in $\mathbb{C}^*$, $e = 1$

$(3-2i)\left(\frac{3}{13} + \frac{2}{13}i\right)$

$= \frac{9}{13} + \frac{6}{13}i - \frac{6}{13}i + \frac{4}{13}$

$= \frac{13}{13} = 1 \Rightarrow (3-2i)^{-1} = \frac{3}{13} + \frac{2}{13}i$ //

7) $\{\ldots, -3, -1, 1, 3, \ldots\} = S$

(i) $a, -a \in S$, $a + (-a) = 0 \notin S$

(ii) $e \notin S$

9) $\{1, 2, 3\}$ under $\cdot$ modulo 4 not a group

$(2 \cdot 2) \bmod 4 = 0 \notin \{1, 2, 3\}$

$\{1, 2, 3, 4\}$ under $\cdot$ modulo 5 is a group

(i) closed

(ii) associative

(iii) $e = 1$

(iv) $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$

11) $a \in G$, $a^{12} = e$

$a^{-1} = a^{11}$

$(a^6)^{-1} = a^6$

$(a^8)^{-1} = a^4$

$(a^{11})^{-1} = a$

13) (a) $a^2 b^3$: $2a + 3b$

(b) $a^{-2}(b^{-1}c)^2$: $-2a + 2(-b+c)$

$= 2(-a-b+c)$

(c) $(ab^2)^{-3}c^2 = e$

$$a^{-3}b^{-6}c^2 = e$$

$$-3a - 6b + 2c = 0$$

15) $a, b \in G$

$a^5 = e,\ b^7 = e$

$$a^{-2} = (a^2)^{-1} = a^3$$

$$b^{-4} = (b^4)^{-1} = b^3$$

$$\therefore a^{-2}b^{-4} = a^3b^3$$

$$(a^2b^4)^{-2} = \left[(a^2b^4)^{-1}\right]^2$$

$$= (b^{-4}a^{-2})^2$$

$$= (b^3a^3)^2$$

$$= b^3a^3b^3a^3$$

17) $H = \{x^{-1} : x \in G\}$, WTS $G = H$

Let $x \in G$, $x^{-1} \in G \Rightarrow x^{-1} \in H$

$\therefore G \subseteq H$

let $b \in H$, $b = x^{-1} : x \in G \Rightarrow b \in G$

$\therefore H \subseteq G$

$\therefore H = G$ //

21) 29

23) $(ab)^n = \underbrace{ab\,ab \ldots ab}_{2n}$

$= aa \ldots ab \ldots b$

$= a^n b^n$

25) G is Abelian $\Leftrightarrow$ $(ab)^{-1} = a^{-1}b^{-1}$, $a, b \in G$

($\Rightarrow$) G is Abelian.

$a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$ //

($\Leftarrow$) $a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$

$(ab)(a^{-1}b^{-1}) = e$

$aba^{-1}b^{-1} = e$

$ab = ba$ //

27) $(a^{-1}ba)^n = a^{-1}b^n a$

$(a^{-1}ba)^n = (a^{-1}ba)(a^{-1}ba) \ldots (a^{-1}ba)$

$$= a^{-1}baa^{-1}ba \ldots a^{-1}ba$$
$$= a^{-1}bb \ldots ba$$
$$= a^{-1}b^n a$$ //

29) $e = 1$ ✓

$(5 \cdot 15) \bmod 56 = 19$ ✓

$(5 \cdot 19) \bmod 56 = 39$ ✓

$(5 \cdot 39) \bmod 56 = 195 \bmod 56 = 27$ ✓

$(5 \cdot 27) \bmod 56 = 135 \bmod 56 = 23$ ✓

$(5 \cdot 23) \bmod 56 = 115 \bmod 56 = 3$ ✓

$(5 \cdot 3) \bmod 56 = 15$ ✓

$(15 \cdot 19) \bmod 56 = 285 \bmod 56 = 5$ ✓

$(15 \cdot 39) \bmod 56 = 585 \bmod 56 = 25$ ✓

$(15 \cdot 25) \bmod 56 = 375 \bmod 56 = 39$

$(16 \cdot 27) \bmod 56 = 405 \bmod 56 = 13$ ✓

$(15 \cdot 13) \bmod 56 = 195 \bmod 56 = 27$

$(15 \cdot 23) \bmod 56 = 345 \bmod 56 = 9$ ✓

$(15 \cdot 9) \bmod 56 = 135 \bmod 56 = 23$

$(15 \cdot 37) \bmod 56 = 45$ ✓

$(15 \cdot 45) \bmod 56 = 675 \bmod 56 = 3$

33)

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

$aa = b$

$ad = e$

$ba = c$

$bb = d$

$bc = e$

$ca = d$

$cc = a$

$cd = b$

$ab = aaa$
$= ba$
$= c$

35) $a, b, c \in G$

$$axb = c$$

$$x = a^{-1}cb^{-1} \mathbin{/\!/}$$

$$a^{-1}xa = c$$

$$x = aca^{-1} /$$

39) $a, b, c, d, x \in G$

$$axb = cxd \Rightarrow ab = cd$$

WTS G is Abelian.

$$ab = ba$$

$$axb = bxa$$

$$b^{-1}axba^{-1} = x$$

$$b^{-1}a = e, \quad ba^{-1} = e$$

$(b^{-1})^{-1} = b = a$

$\therefore\ axb = bxa \Rightarrow ab = ba$ //

47) $x \in G,\ x^2 = e$

WTS G is Abelian

$a, b \in G \Rightarrow ab = ba$

If $a = b$, $ab = aa = e = bb = ba$

If $a \neq b$, $ab = eabe$

$= bbabaa$

$= b(ba)^2 a$

$= bea$

$= ba$ //

49) $GL(2, \mathbb{Z}_2)$

$\mathbb{Z}_2 = \{0, 1\}$

$$\overset{\times}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}},$$

$$\overset{\times}{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}},$$

$$\overset{\checkmark}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \overset{\checkmark}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}},$$

$$\overset{\checkmark}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{\times}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{\checkmark}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}},$$

$$\overset{\times}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{\checkmark}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \overset{\checkmark}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}},$$

$$\overset{\times}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

//

51) $\exists e \in G : \forall a \in G,\ ae = a$

WTS $\forall a \in G,\ ea = a$

$\forall a \in G,\ ae = a$

$a^{-1}(ea) = a^{-1}a = e$

$\therefore (a^{-1})^{-1} = a = ea$ //