7) $\phi: G \to H$, $\sigma: H \to K$ are homomorphisms

WTS $\sigma\phi: G \to K$ is a homomorphism

Let $g_1, g_2 \in G$,

$$\sigma\phi(g_1 g_2) = \sigma(\phi(g_1)\phi(g_2))$$
$$= \sigma\phi(g_1)\sigma\phi(g_2)$$

$\therefore \sigma\phi: G \to K$ is a homomorphism //

Let $\ker\phi = \{g \in G : \phi(g) = e_H\}$,

$\ker\sigma\phi = \{g \in G : \sigma\phi(g) = e_K\}$

Let $g \in \ker\phi$. Then $\phi(g) = e_H$. By Thm 10.1.1,

$\sigma\phi(g) = \sigma(e_H) = e_K \Rightarrow g \in \ker\sigma\phi$

$\therefore \ker\phi \subseteq \ker\sigma\phi$

By Thm 10.1.4,

$\ker\phi \leq G$, $\ker\sigma\phi \leq G \Rightarrow \ker\phi \leq \ker\sigma\phi$

Let $x \in \ker\sigma\phi$, $g \in \ker\phi$,

WTS $x \ker\phi\, x^{-1} \subseteq \ker\phi$, $xgx^{-1} \in \ker\phi$

$$\phi(xgx^{-1}) = \phi(x)\phi(g)(\phi(x))^{-1}$$
$$= \phi(x)\, e_H\, (\phi(x))^{-1}$$
$$= \phi(x)(\phi(x))^{-1}$$

$= e_H$

$\therefore xgx^{-1} \in \ker \phi \Rightarrow x \ker \phi x^{-1} \subseteq \ker \phi$

By Thm 9.1,

$\ker \phi \lhd \ker \sigma\phi \Longleftrightarrow x \ker \phi x^{-1} \subseteq \ker \phi$

$\therefore \ker \phi \lhd \ker \sigma\phi$ //

Let ϕ, σ be onto, $|G| = n$,

Let $[\ker \sigma\phi : \ker \phi] = \{x \ker \phi, x \in \ker \sigma\phi\}$

By Lagrange's Thm, $\ker \phi \leq \ker \sigma\phi$,

$|\ker \sigma\phi : \ker \phi| = |\ker \sigma\phi| / |\ker \phi|$

By Thm 10.2.5,

$\phi : G \to H$ is an $|\ker \phi|$-to-1 mapping

$\sigma\phi : G \to K$ is an $|\ker \sigma\phi|$-to-1 mapping

Since ϕ, σ are onto, $\sigma\phi$ is onto,

$\therefore |G| = |H| |\ker \phi| \Rightarrow |\ker \phi| = |G| / |H|$

$|G| = |K| |\ker \sigma\phi| \Rightarrow |\ker \sigma\phi| = |G| / |K|$

$\therefore |\ker \sigma\phi : \ker \phi| = |\ker \sigma\phi| / |\ker \phi|$

$$= \frac{|G|/|K|}{|G|/|H|}$$

$$= |H|/|K|$$

8) Let G be a group of permutation.

$\sigma \in G$, $sgn(\sigma) = \begin{cases} +1, & \sigma \text{ is an even permutation} \\ -1, & \sigma \text{ is an odd permutation} \end{cases}$

WTS $Sgn: G \to \{+1,-1\}$ under multiplication is a homomorphism

Let $Sgn: G \to \{+1, -1\}$ under multiplication,

Let $a, b \in G$,

If a, b are both even/odd, ab = even,

$$Sgn(ab) = +1 = (+1)(+1) \text{ or } (-1)(-1)$$
$$= Sgn(a)Sgn(b)$$

If a is even, b is odd, ab = odd,

$$Sgn(ab) = -1 = (+1)(-1)$$
$$= Sgn(a)Sgn(b)$$

$\therefore$ Sgn is a homomorphism

Since $(+1)(+1) = +1$, $(+1)(-1) = -1 = (-1)(+1)$,

$+1$ is the identity in $\{+1, -1\}$,

$\therefore \ker Sgn = \{g \in G : Sgn(g) = +1\} = \{g \in G : g \text{ is even}\}$

WTS $A_n \triangleleft S_n$, $|S_n : A_n| = 2$

By Thm 5.6, $A_n \leq S_n$

WTS by Thm 9.1, $x \in S_n$, $x A_n x^{-1} \subseteq A_n \iff A_n \triangleleft S_n$

Let $x \in S_n$, $a \in A_n$,

$$\begin{aligned} \operatorname{sgn}(x a x^{-1}) &= \operatorname{sgn}(x)\operatorname{sgn}(a)(\operatorname{sgn}(x))^{-1} \\ &= \operatorname{sgn}(x)(+1)(\operatorname{sgn}(x))^{-1} \\ &= \operatorname{sgn}(x)(\operatorname{sgn}(x))^{-1} \\ &= +1 \end{aligned}$$

$\therefore x a x^{-1}$ is even $\Rightarrow x a x^{-1} \in A_n \Rightarrow x A_n x^{-1} \subseteq A_n$

$\therefore$ By Thm 9.1, $x A_n x^{-1} \subseteq A_n \iff A_n \triangleleft S_n$ //

By Thm 5.7, $|A_n| = n!/2$

By Lagrange's Thm,

$$\begin{aligned} |S_n : A_n| &= |S_n| / |A_n| \\ &= n!/(n!/2) \\ &= 2 \end{aligned}$$ //

Alternatively, let $\operatorname{sgn} : S_n \to \{+1, -1\}$,

Since $a \in A_n$, $\operatorname{sgn}(a) = +1$,

$\therefore \ker \operatorname{sgn} = A_n$,

By Corollary 10.2.1, $A_n = \ker \operatorname{sgn} \triangleleft S_n$ //

By the First Isomorphism Thm,

$S_n / A_n = \{xA_n : x \in S_n\} = S_n / \ker sgn \cong sgn(S_n) = \{+1, -1\}$

$\therefore |S_n / A_n| = |\{+1, -1\}| = 2$ //

9) Let $\phi : G \oplus H \to G$, $\phi((g,h)) = g$

$G \oplus H = \{(g,h), g \in G, h \in H\}$

Let $(g_1, h_1), (g_2, h_2) \in G \oplus H$,

$\phi((g_1, h_1)(g_2, h_2)) = \phi((g_1 g_2, h_1 h_2))$

$= g_1 g_2$

$= \phi((g_1, h_1)) \phi((g_2, h_2))$

Let $e_G \in G$,

$\ker \phi = \{(g,h) \in G \oplus H : \phi((g,h)) = e_G\}$

$\phi((g,h)) = g = e_G$

$\therefore \ker \phi = \{(e_G, h) \in G \oplus H, h \in H\}$ //

12) WTS $k | n \Rightarrow \mathbb{Z}_n / \langle k \rangle \cong \mathbb{Z}_k$

Let $k | n$, $k \leq n \Rightarrow k \in \mathbb{Z}_n$,

Let $\phi : \mathbb{Z}_n \to \mathbb{Z}_k$, $\phi(a) = a \bmod k$, $a \in \mathbb{Z}_n$, then

$$\phi(a+b) = (a+b) \bmod k$$
$$= (a \bmod k + b \bmod k) \bmod k$$
$$= (\phi(a) + \phi(b)) \bmod k$$

$\therefore \phi$ is a momorphism

$\text{Ker } \phi = \{x \in \mathbb{Z}_n : \phi(x) = 0\}$

$\phi(x) = x \bmod k = 0 \Rightarrow x = 0, k, 2k, \ldots$

$\therefore \text{Ker } \phi = \{0, k, 2k, \ldots\} = \langle k \rangle$

By the First Isomorphism Thm,

$\mathbb{Z}_n / \text{Ker } \phi = \mathbb{Z}_n / \langle k \rangle \approx \phi(\mathbb{Z}_n) = \mathbb{Z}_k$ //

14) WTS $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$, $\phi(x) = 3x$ is not a homomorphism

By Thm 10.1.3,

If $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ is a homomorphism, then

$x \in \mathbb{Z}_{12}$, $|x| = n \Rightarrow |\phi(x)| \mid |x|$

But $1 \in \mathbb{Z}_{12}$, $|\phi(1)| = |3| = 10 \nmid 12 = |1|$

$\therefore \phi$ is not a homomorphism //

21) $\phi : \mathbb{Z}_{30} \to G$ is a homomorphism, $|G| = 5$, ϕ is onto

$\ker\phi = \{x \in \mathbb{Z}_{30} : \phi(x) = e\}$

ϕ is onto $\Rightarrow |\phi(\mathbb{Z}_{30})| = 5$

By Thm 10.3,

$\mathbb{Z}_{30}/\ker\phi \approx \phi(\mathbb{Z}_{30})$

$\therefore |\phi(\mathbb{Z}_{30})| = |\mathbb{Z}_{30}/\ker\phi| = 5$

By Lagrange's Thm,

$|\mathbb{Z}_{30}/\ker\phi| = |\mathbb{Z}_{30}| / |\ker\phi|$

$\therefore |\ker\phi| = |\mathbb{Z}_{30}| / |\mathbb{Z}_{30}/\ker\phi|$

$= 30/5$

$= 6$

By Thm 4.3,

$\mathbb{Z}_{30} = \langle 1 \rangle$, $|\langle 1 \rangle| = 30$, $\ker\phi \leq \mathbb{Z}_{30}$, $|\ker\phi| = 6$

$\ker\phi = \langle 1(30/6) \rangle = \langle 5 \rangle = \{0, 5, 10, 15, 20, 25\}$

20) $\phi : Z_{20} \to Z_8$, ϕ is onto

ϕ is onto $\Rightarrow |\phi(Z_{20})| = |Z_8| = 8$

$Z_{20}/\ker\phi \approx \phi(Z_{20})$

$\therefore |Z_{20}/\ker\phi| = |Z_8| = 8$

$|Z_{20}/\ker\phi| = |Z_{20}| / |\ker\phi|$

$|\ker\phi| = |Z_{20}| / |Z_{20}/\ker\phi|$

$= 20/8$

$= 4$

$|\ker\phi| = 4 \Rightarrow \phi$ is a 4-to-1 mapping

$1 \in Z_{20}$, $|1| = 20 \Rightarrow |\phi(1)| \mid |1| = 20$

$|\phi(1)| \in \{1, 2, 4, 5, 10, 20\}$

$|Z_8| = 8 \Rightarrow |\phi(1)| \in \{1, 2, 4, 8\}$

$\therefore |\phi(1)| = \{1, 2, 4\}$

$\phi(1) \in \{0, 4, 2\}$

ϕ_0, ϕ_2, ϕ_4

24) $\phi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a homomorphism, $\phi(7) = 6$.

a) $\phi(7) = \phi(1 \cdot 7) = (\phi(1))\, 7 = 6$

$\therefore 3 \cdot 7 \bmod 15 = 6 \implies \phi(1) = 3$

$\therefore x \in \mathbb{Z}_{50}$, $\phi(x) = \phi(1 \cdot x)$

$= \phi(1)\, x$

$= 3x$ //

b) $\phi(1) = 3$

$\phi(2) = 3(2) = 6$

$\phi(3) = 9$

$\phi(4) = 12$

$\phi(5) = 0$

$\therefore \text{Img}\, \phi = \{0, 3, 6, 9, 12\}$ //

c) $\ker \phi = \{x \in \mathbb{Z}_{50} : \phi(x) = 0\}$

$\therefore \ker \phi = \{x \in \mathbb{Z}_{50} : \phi(x) \bmod 15 = 0\}$

$= \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$ //

d) By Thm 10.1.6,

$\phi(1) = 3 \implies \phi^{-1}(3) = \{x \in \mathbb{Z}_{50} : \phi(x) = 3\}$

$= 1 + \ker \phi$

$= \{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$ //

30) Let $\phi: G \to \mathbb{Z}_6 \oplus \mathbb{Z}_2$ be a homomorphism, ϕ is onto,

$|\text{Ker}\,\phi| = 5$

WTS $H \triangleleft G$, $|H| = 5, 10, 15, 20, 30, 60$

ϕ is onto $\Rightarrow |\phi(G)| = |\mathbb{Z}_6 \oplus \mathbb{Z}_2| = 12$

Let $H \triangleleft G$,

By Thm 10.2.4, $H \triangleleft G \Rightarrow \phi(H) \triangleleft \phi(G)$

By Lagrange's Thm,

$\phi(H) \triangleleft \phi(G) = \mathbb{Z}_6 \oplus \mathbb{Z}_2 \Rightarrow |\phi(H)| \mid |\phi(G)| = 12$

$\therefore |\phi(H)| \in \{1, 2, 3, 4, 6, 12\}$

By Thm 10.2.5, $|\text{Ker}\,\phi| = 5 \Rightarrow \phi$ is an 5-to-1 mapping

Since ϕ is an 5-to-1 mapping,

$|H| \in \{5, 10, 15, 20, 30, 60\}$ //

39) Let $K \leq G$, $N \trianglelefteq G$,

WTS $K/(K\cap N) \approx KN/N$

By Thm 10.3, $\phi: G \to \bar{G}$ is a homomorphism,

$K/\ker\phi \approx \phi(G)$

$K \cap N = \{g \in G : g \in K, g \in N\}$

$KN = \{kn : k \in K, n \in N\}$

$K/(K\cap N) = \{k(K\cap N) : k \in K\}$

By Lemma 7.1, $N \trianglelefteq G$, $k \in G$,

$n \in N \iff nN = N$

$\therefore KN/N = \{knN : k \in K, n \in N\}$

$= \{kN : k \in K\}$

Let $\phi : K \to KN/N$, $\phi(k) = keN = kN$

Let $a, b \in K$, $\phi(ab) = abN$

$= (aN)(bN)$

$= \phi(a)\phi(b)$

$\therefore \phi$ is an homomorphism

$K\cap N \in KN/N$, $(K\cap N)(N) = K\cap N = (N)(K\cap N)$

$\therefore N \in KN/N$ is the identity element

$\ker \phi = \{x \in K : \phi(x) = N\}$

WTS $\ker \phi = K \cap N$

Let $x \in \ker \phi$, $\phi(x) = N$

$\therefore \phi(x) = xN = N$

By lemma 7.1, $N \lhd G$, $x \in G$,

$xN = N \iff x \in N$

$\therefore x \in K, x \in N \implies x \in K \cap N$

$\therefore \ker \phi \subseteq K \cap N$ //

Let $x \in K \cap N$, $x \in K$, $x \in N$,

then $\phi(x) = xN$

By Lemma 7.1, $N \lhd G$, $x \in G$,

$x \in N \iff xN = N$

$\therefore \phi(x) = N \implies x \in \ker \phi$

$\therefore K \cap N \subseteq \ker \phi$ //

$\therefore \ker \phi = K \cap N$ //

By Thm 10.3, $\phi : K \to KN/N$ is a homomorphism, then

$K/\ker \phi = K/K \cap N \approx \phi(K) = KN/N$ $\square$

44) Let $N \trianglelefteq G$, $|G| = n$,

WTS $gN \in G/N$, $g \in G$, $|gN| \mid |g|$

Let $\phi : G \to G/N$, $\phi(g) = gN$

Let $a, b \in G$,

$$\begin{aligned}\phi(ab) &= abN \\ &= (aN)(bN) \\ &= \phi(a)\phi(b)\end{aligned}$$

$\therefore \phi$ is a homomorphism

By Lagrange's Thm, $g \in G$,

$|G| = n \Rightarrow |g| = |\langle g \rangle| \mid |G| = n$

$\therefore |g| m = n \Rightarrow |g| = n/m = p \in \mathbb{Z}$

By Thm 10.1.3,

$|g| = p \Rightarrow |\phi(g)| = |gN| \mid |g|$ $\square$