2) Let $G$ be finite Abelian,

By Thm 11.1,

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_k, \quad H_i \text{ is cyclic}, \ |H_i| = p_i^{n_i}$$

Since $H_i \approx \mathbb{Z}_{p_i^{n_i}}$,

$$G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$$

Let $|G| = 8 = 2^3$, then

$k = 3 = 2+1 = 1+1+1$,

$\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\square$

5) WTS $G$ is Abelian, $|G| = 45$, $\exists a \in G : |a| = 15$

Let $G$ be Abelian, $|G| = 45 = 9 \cdot 5 = 3^2 \cdot 5$

$3^2 : \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

$5 : \mathbb{Z}_5$

$\therefore$ The possible isomorphic classes of $G$ are

$$\mathbb{Z}_9 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{45}$$

$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_3 \oplus \mathbb{Z}_{15}$$

$3 \in \mathbb{Z}_{45}$ , $|3| = 15$

$(0,1) \in \mathbb{Z}_3 \oplus \mathbb{Z}_{15}$ , $|(0,1)| = 15$

$\therefore$ $G$ is Abelian, $|G| = 45$, $\exists a \in G : |a| = 15$

$x \in \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ , $|x| \neq 9$

$\therefore$ Not all Abelian group of order 45 have elements of order 9

$\square$

8) Let $G$ be Abelian, $|G| = 108$,

By Thm 11.1, $G \cong \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$ ,

$$
\begin{aligned}
|G| = 108 &= 2 \cdot 54 \\
&= 2^2 \cdot 27 \\
&= 2^2 \cdot 3^3
\end{aligned}
$$

List of finite Abelian groups of order $2^2$:

| | |
|---|---|
| $k = 2$ | $\mathbb{Z}_4$ |
| $= 1+1$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |

List of finite Abelian groups of order $3^3$:

| | |
|---|---|
| $k = 3$ | $\mathbb{Z}_{27}$ |
| $= 2+1$ | $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ |
| $= 1+1+1$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ |

$\therefore$ List of finite Abelian groups of order 108:

$\mathbb{Z}_4 \oplus \mathbb{Z}_{27} \approx \mathbb{Z}_{108}$

$\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_{36} \oplus \mathbb{Z}_3$

$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_{12} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \approx \mathbb{Z}_{54} \oplus \mathbb{Z}_2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_{18} \oplus \mathbb{Z}_6$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \approx \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_3$

Since $\mathbb{Z}_{108} = \langle 1 \rangle$, by Thm 4.3,

$|\langle 1 \rangle| = 108 \Rightarrow 3 \mid 108,\ 3 > 0,\ \exists! H \leq \langle 1 \rangle : |H| = 3,\ H = \langle 1(108/3) \rangle$
$= \langle 36 \rangle$

$\therefore H = \langle 36 \rangle \leq \langle 1 \rangle = \mathbb{Z}_{108}$ is the only subgroup of order 3.

By Corollary 4.2.3, $|a| = 3$,

(i) $\langle a \rangle = \langle a^j \rangle \Leftrightarrow \gcd(3, j) = 1$

(ii) $|a| = |a^j| \Leftrightarrow \gcd(3, j) = 1$

$\therefore j \in \{1, 2\}$

$\therefore |a| = |a^2|,\ \langle a \rangle = \langle a^2 \rangle$

$\therefore$ G has 13 subgroups of order 3 $\Rightarrow$ G has $13 \cdot 2 = 26$ elements of order 3.

By Thm 8.1, $|(g_1, \dots, g_n)| = \text{lcm}(|g_1|, \dots, |g_n|)$

$\therefore \text{lcm}(|g_1|, \dots, |g_n|) = 3 \Rightarrow |g_i| \in \{1, 3\}$

For $\mathbb{Z}_{36} \oplus \mathbb{Z}_3$,

$\mathbb{Z}_{36}$ : $|0| = 1$, $|12| = |24| = 3$

$\mathbb{Z}_3$ : $|0| = 1$, $|1| = |2| = 3$

Since $|(0, 0)| = 1$,

$\therefore$ There are $3^2 - 1 = 8$ elements of order 3 //

For $\mathbb{Z}_{12} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$,

$\mathbb{Z}_{12}$ : $|0| = 1$, $|4| = |8| = 3$

$\mathbb{Z}_3$ : $|0| = 1$, $|1| = |2| = 3$

$\therefore$ There are $3^3 - 1 = 26$ elements of order 3 //

For $\mathbb{Z}_{54} \oplus \mathbb{Z}_2$,

$\mathbb{Z}_{54}$ : $|0| = 1$, $|18| = |36| = 3$

$\mathbb{Z}_2$ : $|0| = 1$

$\therefore$ There are $3 - 1 = 2$ elements of order 3 //

For $\mathbb{Z}_{18} \oplus \mathbb{Z}_6$,

$\mathbb{Z}_{18}$ : $|0|=1$, $|6|=|12|=3$

$\mathbb{Z}_6$ : $|0|=1$, $|2|=|4|=3$

$\therefore$ There are $3^2-1=8$ elements of order $3$ //

For $\mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_3$,

$\mathbb{Z}_6$ : $|0|=1$, $|2|=|4|=3$

$\mathbb{Z}_3$ : $|0|=1$, $|1|=|2|=3$

$\therefore$ There are $3^3-1=26$ elements of order $3$ //

Since $\mathbb{Z}_{12} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_3$ have 26 elements of order 3, they have 13 subgroups of order 3 $\square$

9) Let $G$ be Abelian, $|G|=120$, $G$ has 3 elements of order 2.

$$\begin{aligned}|G| = 120 &= 2 \cdot 60 \\ &= 2^2 \cdot 30 \\ &= 2^3 \cdot 15 \\ &= 2^3 \cdot 3 \cdot 5\end{aligned}$$

The list of Abelian groups of order $2^3$ is

| | |
|---|---|
| $k=3$ | $\mathbb{Z}_8$ |
| $=2+1$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ |
| $=1+1+1$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ |

The Abelian groups of order 3 and 5 are $\mathbb{Z}_3$, $\mathbb{Z}_5$

The list of Abelian groups of order 120 is

$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{40} \oplus \mathbb{Z}_3$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{20} \oplus \mathbb{Z}_6$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{10} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$

By Thm 6.2.7, if $G \approx \bar{G}$, $|G| = n$, then $G, \bar{G}$ have the same number of elements of every order.

By Corollary 4.2.3.2, $a \in G$, $a = 2$,

$|a| = |a^j| \iff \gcd(2, j) = 1$

$\therefore j = 1$

By Thm 8.1, $|(g_1, \dots, g_n)| = \operatorname{lcm}(|g_1|, \dots, |g_n|)$

$\therefore \operatorname{lcm}(|g_1|, \dots, |g_n|) = 2 \Rightarrow |g_i| \in \{1, 2\}$

For $\mathbb{Z}_{40} \oplus \mathbb{Z}_3$,

$\mathbb{Z}_{40}$ : $|0| = 1$, $|20| = 2$

$\mathbb{Z}_3$ : $|0| = 1$

Since $|(0,0)| = 1$, $(20, 0)$ is the only element of order 2 //

For $\mathbb{Z}_{20} \oplus \mathbb{Z}_6$,

$\mathbb{Z}_{20}$ : $|0| = 1$, $|10| = 2$

$\mathbb{Z}_6$ : $|0| = 1$, $|3| = 2$

There are $2^2 - 1 = 3$ elements of order 2 //

For $\mathbb{Z}_{10} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$,

$\mathbb{Z}_{10}$ : $|0| = 1$, $|5| = 2$

$\mathbb{Z}_6$ : $|0| = 1$, $|3| = 2$

$\mathbb{Z}_2$ : $|0| = 1$, $|1| = 2$

There are $2^3 - 1 = 7$ elements of order 2 //

$\therefore G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ □

(b) Let $G$ be Abelian, $|G| = 360$,

By Thm 11.1,

$$G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$$

$$\begin{aligned} |G| = 360 &= 2 \cdot 180 \\ &= 2^2 \cdot 90 \\ &= 2^3 \cdot 45 \\ &= 2^3 \cdot 9 \cdot 5 \\ &= 2^3 \cdot 3^2 \cdot 5 \end{aligned}$$

The Abelian groups of order $2^3$ are

$\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

The Abelian groups of order $3^2$ are

$\mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

The Abelian group of order 5 is $\mathbb{Z}_5$

$\therefore$ The Abelian groups of order 360 are

$$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{45} \oplus \mathbb{Z}_8$$

$$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{24} \oplus \mathbb{Z}_{15}$$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{20} \oplus \mathbb{Z}_{18}$$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{15} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{18} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_2$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{10} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$$

$\square$

12) Let $G$ be finite Abelian, $10 \mid |G|$

WTS $G$ has a cyclic subgroup of order 10

By Corollary 11.1.1,

$10 \mid |G| \Rightarrow \exists H \leq G : |H| = 10$

Let $H \leq G$, $|H| = 10$,

WTS $H$ is cyclic

$G$ is finite Abelian, $H \leq G \Rightarrow H$ is finite Abelian

By Thm 11.1,

$$H \approx \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$$

$|H| = 10 = 2 \cdot 5$

The finite Abelian groups of order 2 and 5 are $\mathbb{Z}_2$, $\mathbb{Z}_5$

$\therefore$ The finite Abelian groups of order 10 is $\mathbb{Z}_5 \oplus \mathbb{Z}_2$

and $H \approx \mathbb{Z}_5 \oplus \mathbb{Z}_2$

$\therefore \mathbb{Z}_5 \oplus \mathbb{Z}_2 = \langle (1,1) \rangle \Rightarrow H \approx \mathbb{Z}_5 \oplus \mathbb{Z}_2$ is cyclic

$\therefore \exists H \leq G$, $|H| = 10$, $H$ is cyclic $\square$

25) $G = \{1, 7, 43, 49, 51, 57, 93, 99, 101, 107, 143, 149, 151, 157, 193, 199\}$ under multiplication modulo 200.

$|G| = 16 = 2^4$

The Abelian groups of order 16 are

| | |
|---|---|
| $k = 4$ | $\mathbb{Z}_{16}$ |
| $= 3+1$ | $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ |
| $= 2+1+1$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $= 1+1+1+1$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ |

By Lagrange's Thm, $a \in G \Rightarrow |a| = |\langle a \rangle| \mid |G|$

$\therefore a \in G$, $|a| \in \{1, 2, 4, 8, 16\}$

By Corollary 4.2.3.2, $a \in G$, $|a| = n$,

$$|a| = |a^j| \Leftrightarrow \gcd(n, j) = 1$$

$$\begin{aligned}
\therefore\ & \gcd(1, j) = 1 \Rightarrow j = 1 \\
& \gcd(2, j) = 1 \Rightarrow j = 1 \\
& \gcd(4, j) = 1 \Rightarrow j \in \{1, 3\} \\
& \gcd(8, j) = 1 \Rightarrow j \in \{1, 3, 5, 7\} \\
& \gcd(16, j) = 1 \Rightarrow j \in \{1, 3, 5, 7, 9, 11, 13, 15\}
\end{aligned}$$

$\therefore$ The order of elements of $G$ are

$|1| = 1$

$|7| = 4 = |7^3| = |143|$

$|43| = 4 = |43^3| = |107|$

$|49| = 2$

$|51| = 2$

$|57| = 4 = |57^3| = |193|$

$|93| = 4 = |93^3| = |157|$

$|99| = 2$

$|101| = 2$

$|149| = 2$

$|151| = 2$

$|199| = 2$

For $\mathbb{Z}_{16}$,

$1 \in \mathbb{Z}_{16}$, $|1| = 16 \Rightarrow G \not\cong \mathbb{Z}_{16}$

For $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $|(1,0)| = 8$

$\therefore G \not\approx \mathbb{Z}_8 \oplus \mathbb{Z}_2$

For $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,

By Thm 8.1, $|(g_1, \dots, g_n)| = \operatorname{lcm}(|g_1|, \dots, |g_n|)$

$\therefore \operatorname{lcm}(|g_1|, \dots, |g_n|) = 2 \Rightarrow |g_i| \in \{1, 2\}$

$\mathbb{Z}_4$ : $|0| = 1$, $|2| = 2$

$\mathbb{Z}_2$ : $|0| = 1$, $|1| = 2$

There are $2^3 - 1 = 7$ elements of order 2

For $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there is no elements of order 4

$\therefore G \not\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\therefore G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ //

$|G| = 16 = 2^4$,

let $7 \in G$, $|7| = 4$, $G_1 = \langle 7 \rangle = \{7, 49, 143, 1\} = \langle 143 \rangle$

let $a \in G$, $|a| = 2^2 = 4 \leq |G| / |G_1| = 16/4 = 4$,

$a, a^2 \notin G_1 = \langle 7 \rangle$

$43 \in G$, $|43| = 4$, $\langle 43 \rangle = \{43, 49, 107, 1\} = \langle 107 \rangle$

$57 \in G$, $|57| = 4$, $\langle 57 \rangle = \{57, 49, 193, 1\} = \langle 193 \rangle$

$93 \in G$, $|93| = 4$, $\langle 93 \rangle = \{93, 49, 157, 1\} = \langle 157 \rangle$

$51 \in G$, $|51| = 2$, $\langle 51 \rangle = \{51, 1\}$

Let $G_2 = G_1 \times \langle 51 \rangle$

$= \langle 7 \rangle \times \langle 51 \rangle$

$= \{1, 7, 49, 51, 93, 99, 143, 157\}$

$|G_2| = 8 \neq |G| = 16$

$|101| = 2$, $\langle 101 \rangle = \{101, 1\}$

$G_3 = G_2 \times \langle 101 \rangle$

$= \langle 7 \rangle \times \langle 51 \rangle \times \langle 101 \rangle$

$|G_3| = 16 = |G|$

$\therefore G = \langle 7 \rangle \times \langle 51 \rangle \times \langle 101 \rangle$ $\square$

29) $G$ is finite Abelian, $|G| = 16$.

$\exists a, b \in G : |a| = |b| = 4, \; a^2 \neq b^2$

By Thm 11.1, $G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$

$|G| = 16 = 2^4$

The finite Abelian groups of order 16 are

| | |
|---|---|
| $k = 4$ | $\mathbb{Z}_{16}$ |
| $= 3+1$ | $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ |
| $= 2+2$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ |
| $= 2+1+1$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $= 1+1+1+1$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ |

For $\mathbb{Z}_{16}$,

$4, 8 \in \mathbb{Z}_{16}, \; |4| = |12| = 4, \; 2(4) = 8 = 8 = 2(12)$

$\therefore G \not\approx \mathbb{Z}_{16}$ //

For $\mathbb{Z}_8 \oplus \mathbb{Z}_2$,

$\mathbb{Z}_8 : |0| = 1, \; |2| = |6| = 4,$

$\mathbb{Z}_2 : |0| = 1$

$\therefore |(2,0)| = |(6,0)| = 4$

$2(2,0) = (4,0) = (4,0) = 2(6,0)$

$\therefore G \not\approx \mathbb{Z}_8 \oplus \mathbb{Z}_2$ //

For $\mathbb{Z}_4 \oplus \mathbb{Z}_4$,

$\mathbb{Z}_4$ : $|0|=1$, $|1|=|3|=4$

$\therefore |(0,1)| = |(0,3)| = 4$

$|(1,0)| = |(1,1)| = |(1,3)| = 4$

$|(3,0)| = |(3,1)| = |(3,3)| = 4$

$2(0,1) = (0,2)$

$2(0,3) = (0,2)$

$2(1,0) = (2,0)$

$2(1,1) = (2,2)$

$2(1,3) = (2,2)$

$2(3,0) = (2,0)$

$2(3,1) = (2,2)$

$2(3,3) = (2,2)$

For $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,

$\mathbb{Z}_4$ : $|0|=1$, $|1| = |3| = 4$

$\mathbb{Z}_2$ : $|0|=1$

$\therefore |(1,0)| = |(3,0)| = 4$

$2(1,0) = (2,0) = (2,0) = 2(3,0)$

$\therefore G \not\approx \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ //

For $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there is no elements of order 4,

$\therefore G \not\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ //

$\therefore G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_4$ $\square$