4) $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R},\ ad \neq 0 \right\}$

$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\right.$
$\left. ab - cd \neq 0 \right\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a^2 & ab + bd \\ ac & bc + d^2 \end{pmatrix} \right\}$

$H \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a^2 + bc & ab + bd \\ cd & d \end{pmatrix} \right\}$

$a^2 \neq a^2 + bc,\quad ac \neq cd,\quad bc + d^2 \neq d$

$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} H \neq H \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ //

7) WTS $|G:H| = 2 \Rightarrow H \triangleleft G$

Let $|G:H| = 2$,

If $g \in G$, $g \in H$, by lemma 7.1,

$gH = H = Hg$

If $g \notin H$, then since $|G:H| = 2$,

and the cosets of H partition G,

$gH = G \setminus H = Hg$

$\therefore gH = Hg$, $g \in G \Rightarrow H \triangleleft G$ //

9) $G = \mathbb{Z}_4 \oplus U(4)$

$H = \langle (2,3) \rangle$

$K = \langle (2,1) \rangle$

WTS $G/H \not\cong G/K$

By Thm 9.2,

$G/H = \{aH : a \in G\}$ is a group under

$(aH)(bH) = abH$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$U(4) = \{1, 3\}$

$$G = \mathbb{Z} \oplus U(4)$$
$$= \{(0,1), (0,3), (1,1), (1,3), (2,1), (2,3), (3,1), (3,3)\}$$

$$H = \langle (2,3) \rangle$$

$$= \{ (2,3), (0,1) \}$$

$$K = \langle (2,1) \rangle$$

$$= \{ (2,1), (0,1) \}$$

$$G/H = \{ H, \{(2,1), (0,3)\}, \{(3,3), (1,1)\}, \{(3,1), (1,3)\} \}$$

$$G/K = \{ K, \{(2,3), (0,3)\}, \{(3,1), (1,1)\}, \{(3,3), (1,3)\} \}$$

$$G/H = \{ (0,1)H, (0,3)H, (1,1)H, (1,3)H \}$$

$$G/K = \{ (0,1)K, (0,3)K, (1,1)K, (1,3)K \}$$

$|(0,1)H| = 1$ $\quad$ $|(0,1)K| = 1$

$|(0,3)H| = 2$ $\quad$ $|(0,3)K| = 2$

$|(1,1)H| = 4$ $\quad$ $|(1,1)K| = 4$

$|(1,3)H| = 4$ $\quad$ $|(1,3)K| = 4$

12) Let G be Abelian,

WTS G/H is Abelian

G is Abelian $\Rightarrow a,b \in G,\ ab=ba$

$G/H = \{aH,\ a \in G\}$, $(aH)(bH) = abH$

G/H is Abelian $\Rightarrow (aH)(bH) = (bH)(aH)$

let $aH, bH \in G/H$,

$(aH)(bH) = abH$

$(bH)(aH) = baH$

$ab = ba \Rightarrow abH = baH$

$\therefore (aH)(bH) = (bH)(aH)$ //

20) $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$U_5(20) = \{1, 11\}$

Let $H = U_5(20)$,

	1H	3H	7H	9H	11H	13H	17H	19H
1H	1H	3H	7H	9H	11H	13H	17H	19H
3H	3H	9H	1H	7H	13H	19H	11H	17H
7H	7H	1H	9H	3H	17H	11H	19H	13H
9H	9H	7H	3H	1H	19H	17H	13H	11H
11H	11H	13H	17H	19H	1H	3H	7H	9H
13H	13H	19H	11H	17H	3H	9H	1H	7H
17H	17H	11H	19H	13H	7H	1H	9H	3H
19H	19H	17H	13H	11H	9H	7H	3H	1H

//

21) WTS G is Abelian, $|G| = 33 \Rightarrow G = \langle g \rangle, g \in G$

Let G be Abelian, $|G| = 33$,

By Thm 9.5,

$3, 11 \mid 33$, $3, 11$ are primes,

$\therefore \exists a, b \in G : |a| = 3, |b| = 11$

$(ab)^2 = a^2 b^2$

$(ab)^3 = a^3 b^3 = e b^3 = b^3$

$(ab)^4 = a^4 b^4 = a b^4$

$\vdots$

$(ab)^{11} = a^{11} b^{11} = a^2 e = a^2$

$\vdots$

$(ab)^{33} = a^{33} b^{33} = e e = e$

$\therefore |ab| = 33 = |\langle ab \rangle|$

$\therefore G = \langle ab \rangle$ //

25) $G = U(32)$

$H = \{1, 31\}$

$\mathbb{Z}_8$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

G is Abelian $\Rightarrow$ G/H is Abelian

$$|G/H| = |G|/|H| = 16/2 = 8$$

$$U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$$

$$G/H = \{1H, 3H, 5H, 7H, 9H, 11H, 13H, 15H\}$$

$|1H| = 1$ $\qquad$ $|9H| = 4$

$|3H| = 8$ $\qquad$ $|11H| = 8$

$|5H| = 8$ $\qquad$ $|13H| = 8$

$|7H| = 4$ $\qquad$ $|15H| = 2$

By Thm .2 (V,

$G \approx \overline{G} \Rightarrow |a| = |\phi(a)|$

$\forall x \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ |x| = 2 \Rightarrow G/H \not\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1),$
$(2,0), (2,1), (3,0), (3,1)\}$

By Thm 8.1,

$|(0,0)| = \text{lcm}(|0|, |0|) = \text{lcm}(1,1) = 1$

$|(0,1)| = \text{lcm}(1,2) = 2$

$|(1,0)| = \text{lcm}(4,1) = 4$

$|(1,1)| = \text{lcm}(4,2) = 4$

$|(2,0)| = \text{lcm}(2,1) = 2$

$|(2,1)| = \text{lcm}(2,2) = 2$

$|(3,0)| = \text{lcm}(4,1) = 4$

$|(3,1)| = \text{lcm}(4,2) = 4$

$\therefore \forall x \in \mathbb{Z}_4 \oplus \mathbb{Z}_2,\ |x| \neq 8$

$\therefore G/H \not\approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$

$\therefore G/H \approx \mathbb{Z}_8$ //

30) $U(165) = \{1, 2, 4, 7, 8, 13, 14, 16, 17, 18, 19, 22, \ldots, 164\}$

$\langle 2 \rangle = \{2, 4, 8, 16, 32, 64, 128, 91, 17, 34, 68, 136, 107, 49, 98, 31, 62, 124, 83, 1\}$

$\langle 4 \rangle = \{4, 16, 64, 91, \ldots\} \subseteq \langle 2 \rangle$

$\langle 7 \rangle = \{7, 49, 13, 91, \ldots\}$

$\langle 8 \rangle = \{8, 64, 17, \ldots\} \subseteq \langle 2 \rangle$

$\langle 13 \rangle = \{13, 4, 52, 16, \ldots\}$

$\langle 16 \rangle = \{16, 128, \ldots\}$

$\langle 17 \rangle = \{17, 124, 128, \ldots\}$

$\langle 18 \rangle = \{18, 159, 57, 36, 153, 114, 72, 141,$
$63,$

38) $H \triangleleft G$, $a \in G$

$|aH| = 3$, $aH \in G/H$, $|H| = 10$

$G/H = \{aH, a \in G\}$, $(aH)(bH) = abH$

$|aH| = 3 \Rightarrow (aH)^3 = a^3H = H$

By Lemma 7.1 (ii),

$a^3H = H \iff a^3 \in H$

By Lagrange's Thm

$a^3 \in H$, $\langle a^3 \rangle \leq H \Rightarrow |a^3| = |\langle a^3 \rangle| \mid |H| = 10$

$\therefore |a^3| = 1, 2, 5$ or 10

By Thm 4.2 (ii), $|a| = n$,

$|a^3| = n/\gcd(n,3)$

$\gcd(n,3) = 1$ or 3

If $\gcd(n,3) = 1$,

$|a^3| = n/1 = n = |a|$

By Thm 4.2 (i),

$\langle a^3 \rangle = \langle a^{\gcd(n,3)} \rangle = \langle a \rangle$

$\therefore a^3 \in H \Rightarrow a \in H$ ($\Rightarrow \Leftarrow$)

$\therefore \gcd(n,3) \neq 1$

$\therefore \gcd(n,3) = 3$

$|a| = n = \gcd(n,3)\,|a^3|$

$= 3\,|a^3|$

$= 3, 1, 5$ or //

41) $aH, bH \in G/H$,

$aH = bH$, $|a| \neq |b|$

$G = \mathbb{Z}_{18}$, $H = \langle 6 \rangle = \{0, 6, 12\}$

$G/H = \{0+H, 1+H, 2+H, 3+H, 4+H, 5+H\}$

$3+H = \{3, 9, 15\}$

$9+H = \{9, 15, 3\} = 3+H$

$\langle 3 \rangle = \{3, 6, 9, 12, 15, 0\}$

$\langle 9 \rangle = \{9, 0\}$

or by Thm 4.2 (ii),

$|9| = |3 \cdot 3| = |3| / \gcd(|3|, 3)$

$= 6 / \gcd(6, 3)$

$= 6/3$

$= 2$

$\therefore |3| \neq |9|$ //

46) $|G : Z(G)| = 4$

WTS $G/Z(G) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$|G : Z(G)| = 4 \Rightarrow |G/Z(G)| = 4$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

By Thm 9.7, 2 is prime,

$|G/Z(G)| = 4 = 2^2 \Rightarrow G/Z(G) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$ //

52) $G = \{\pm 1, \pm i, \pm j, \pm k\}$

$i^2 = j^2 = k^2 = -1$

$-i = (-1)i$

$1^2 = (-1)^2 = 1$

$ij = -ji = k$

$jk = -kj = i$

$ki = -ik = j$

a)

G	-1	$-i$	$-j$	$-k$	1	i	j	k
-1	1	i	j	k	-1	$-i$	$-j$	$-k$
$-i$	i	-1	k	$-j$	$-i$	1	$-k$	j
$-j$	j	$-k$	-1	i	$-j$	k	1	$-i$
$-k$	k	j	$-i$	-1	$-k$	$-j$	i	1
1	-1	$-i$	$-j$	$-k$	1	i	j	k
i	$-i$	1	$-k$	j	i	-1	k	$-j$
j	$-j$	k	1	$-i$	j	$-k$	-1	i
k	$-k$	$-j$	i	1	k	j	$-i$	-1

//

b) WTS $H = \{1,-1\} \trianglelefteq G$

By Thm 9.1,

$$-1H1 = \{-1,1\}-1 = \{1,-1\} = H \subseteq H$$
$$= 1H-1$$
$$-iHi = \{i,-i\}i = \{-1,1\} = H \subseteq H$$
$$= -jHj$$
$$= -kHk$$

$= 1 - i$

$= jH - j$

$= kH - k$

$\therefore H \trianglelefteq G$ //

c) $G/H = \{aH : a \in G\}$

$a \in G$, $\pm aH = \{-a, a\}$

$\therefore G/H = \{H, iH, jH, kH\}$

G/H	H	iH	jH	kH
H	H	iH	jH	kH
iH	iH	H	kH	jH
jH	jH	kH	H	iH
kH	kH	jH	iH	H

If $G/H \cong \mathbb{Z}_4$, by Thm 6.2 (iv),

G/H is cyclic $\Leftrightarrow$ $\mathbb{Z}_4$ is cyclic

But G/H is not cyclic ($\Rightarrow\Leftarrow$)

$\therefore G/H \not\cong \mathbb{Z}_4$

$\therefore G/H \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(2) $H \leq G$, $|H| = n$

WTS $\bigcap_i H_i \triangleleft G$, $|H_i| = n$

Let $a \in G$, $h_1, h_2 \in H$,

$ah_1a^{-1}, ah_2a^{-1} \in aHa^{-1}$,

$$(ah_1a^{-1})(ah_2a^{-1})^{-1} = ah_1a^{-1}a\, h_2^{-1}a^{-1}$$
$$= ah_1h_2^{-1}a^{-1}$$

$h_1h_2^{-1} \in H \Rightarrow ah_1h_2^{-1}a^{-1} \in aHa^{-1}$

By Thm 3.1, $aHa^{-1} \leq G$, $|aHa^{-1}| = n$

Let $I = \bigcap_i H_i$, WTS $aIa^{-1} \subseteq I$

$i \in I \Rightarrow i \in a^{-1}Ha$

$\therefore i = a^{-1}ha$

$$aia^{-1} = a(a^{-1}ha)a^{-1} = h \in H$$

$$|H| = n,\ h \in H \Rightarrow h \in I$$

$$\therefore aia^{-1} \in I \Rightarrow aIa^{-1} \subseteq I$$ //