1) $\mathbb{Z}_{12} = \{0, 1, \dots, 11\}$

$|\mathbb{Z}_{12}| = 12$ //

$|0| = 1$ $|4| = 3$ $|8| = 3$

$|1| = 12$ $|5| = 12$ $|9| = 4$

$|2| = 6$ $|6| = 2$ $|10| = 6$

$|3| = 4$ $|7| = 12$ $|11| = 12$ //

$1, 2, 3, 4, 6, 12 \mid 12$

$U(10) = \{1, 3, 7, 9\}$

$|U(10)| = 4$ // $1, 4, 2$

$|1| = 1$, $|3| = 4$, $|7| = 4$, $|9| = 2$ //

$1, 2, 4 \mid 4$

$U(12) = \{1, 5, 7, 11\}$

$|U(12)| = 4$ //

$|1| = 1$, $|5| = 2$, $|7| = 2$, $|11| = 2$ //

$1, 2 \mid 4$

$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$|U(20)| = 8$ // $\quad 1, 8, 2, 4$

$|1| = 1$, $|3| = 4$, $|7| = 4$, $|9| = 2$,

$|11| = 2$, $|13| = 4$, $|17| = 4$, $|19| = 2$ //

$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

$|D_4| = 8$ //

$|R_0| = 1$, $|R_{90}| = 4$, $|R_{180}| = 2$, $|R_{270}| = 4$

$|H| = 2,\ |V| = 2,\ |D| = 2,\ |D'| = 2$

3) $\mathbb{Q}$ is the group of rational numbers under addition.

$\mathbb{Q}^*$ is the group of nonzero rational numbers under multiplication.

$\mathbb{Q}$: $e = 0$

$|0| = 1$, $a \in \mathbb{Q}$: $a \neq 0$, $|a| = \infty$

$\mathbb{Q}^*$: $e = 1$

$|1| = 1$, $|-1| = 2$

$a \in \mathbb{Q}^*$: $a \neq 1$, $|a| = \infty$

5) $\mathbb{Z}_{30} = \{0, 1, \dots, 29\}$

$\{2, 28\}$, $\{8, 22\}$

$(2+28) \bmod 30 = 0$, $(8+22) \bmod 30 = 0$

$-2 = 28$, $-28 = 2$, $-8 = 22$, $-22 = 8$

$U(15)$: $\{2, 8\}$, $\{7, 13\}$

$(2 \cdot 8) \bmod 15 = 1 \Rightarrow 2^{-1} = 8$, $8^{-1} = 2$

$(7 \cdot 13) \bmod 15 = 1 \Rightarrow 7^{-1} = 13$, $13^{-1} = 7$

7) $a, b, c \in G$

$|a| = 6$, $|b| = 7$

$(a^4 c^{-2} b^4)^{-1}$

$(a^4 c^{-2} b^4)(b^{-4} c^2 a^{-4}) = e$

$\therefore (a^4c^{-2}b^4)^{-1} = b^{-4}c^2a^{-4}$

$= (b^4)^{-1}c^2(a^4)^{-1}$

$b^4b^3 = b^7 = e \Rightarrow (b^4)^{-1} = b^3$

$a^4a^2 = a^6 = e \Rightarrow (a^4)^{-1} = a^2$

$\therefore (a^4c^{-2}b^4)^{-1} = b^3c^2a^2$

11) $\mathbb{R}^*$ under multiplication

$|1| = 1, \; |-1| = 2$

13) $a, x \in G$

$|xax^{-1}| = |a|$

$|a| = n$

$(xax^{-1})^n = (xax^{-1})(xax^{-1}) \cdots (xax^{-1})$

$= xaa\ldots ax^{-1}$

$= xa^nx^{-1}$

$= xx^{-1}$

$= e$

$|xax^{-1}| \leq n$

$i < n : (xax^{-1})^i = e$

$(xax^{-1})^i = x\underbrace{aa\ldots a}_{i}x^{-1} = e$

$|a| = n,\ i < n \Rightarrow a^i \neq e \ (\Rightarrow\Leftarrow)$

$\therefore |xax^{-1}| = n$ //

17) $k \mid n,\ k > 1$

$U_k(n) = \{x \in U(n) : x \bmod k = 1\}$

$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$U_4(20) = \{1, 9, 13, 17\}$ //

$U_5(20) = \{1, 11\}$ //

$U(30) = \{1, 7, 11, 13, 17, 19, 23, 27, 29\}$

$U_5(30) = \{1, 11\}$ //

$U_{10}(30) = \{1, 11\}$ //

19) $a \in G$, $|a| = \infty$

$m \neq n \Rightarrow a^m \neq a^n$

Let $m \neq n$, $a^m = a^n$

$a^{m-n} = e$

$|a| = \infty \Rightarrow m - n = 0 \Rightarrow m = n$ $(\Rightarrow\Leftarrow)$

//

21) $a \in G \Rightarrow |a| \leq |G|$

Let $|a| > |G|$,

$|G| = n$, $|a| = m$

$m > n$

$a \in G \Rightarrow a, a^2, \dots, a^n, a^{n+1}, \dots a^m \in G$

$\Rightarrow |G| > n$ ($\Rightarrow\Leftarrow$)

$\therefore |a| \leq |G|$

23) WTS $U(20) \neq \langle k \rangle$, $k \in U(20)$

$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$\langle 1 \rangle = \{1\}$

$\langle 3 \rangle = \{3, 9, 7, 1\}$

$\langle 7 \rangle = \{7, 9, 3, 1\} = \langle 3 \rangle$

$\langle 9 \rangle = \{9, 1\}$

$\langle 11 \rangle = \{11, 1\}$

$\langle 13 \rangle = \{13, 9, 7, 1\}$

$\langle 17 \rangle = \{17, 9, 13, 1\}$

$\langle 19 \rangle = \{19, 1\}$ //

25) $n = 2k,\ k \in \mathbb{N}$

$H \leq \mathbb{Z}_n$

WTS $x \in H,\ x = 2k$

Let $H \leq \mathbb{Z}_n,\ n = 2k,\ k \in \mathbb{N}$

$|H| = 2k+1$

$|\mathbb{Z}_n| = 2k$

$x \in H$, $(2k+1)x = e$

$(2k+1)x \mod 2k = 0$

$\therefore (2k+1)x$ is even $\Rightarrow$ x is even

31) $H < \mathbb{Z}$

$12, 30, 54 \in H$

$-12, -30, -54 \in H$

$0 \in H$

$12+30 = 42 \in H$

$12+54 = 66 \in H$

$12+(-30) = -18 \in H$

$12+(-54) = -42 \in H$

$30+54 = 84 \in H$

$30+(-12) = 18 \in H$

$30+(-54) = -24 \in H$

$54+(-12) = 42 \in H$

$54+(-30) = 24 \in H$

$12, 24, 36, \ldots, -12, -24, -36, \ldots \in H$

$30, 60, 90, \ldots, -30, -60, -90, \ldots \in H$

$54, 108, 162, \ldots, -54, -108, -162, \ldots \in H$

$18, 36, 54, \ldots, -18, -36, -54, \ldots \in H$

35) $Z(G) = \bigcap_{a \in G} C(a)$

$Z(G) = \{a \in G : \forall x \in G, ax = xa\}$

$C(a) = \{x \in G : ax = xa\}$

$a \in G$

Let $b \in Z(G)$.

$\forall x \in G, bx = xb$

$\therefore ba_1 = a_1 b \Rightarrow b \in C(a_1)$

$ba_2 = a_2 b \Rightarrow b \in C(a_2)$

$\vdots$

$\therefore b \in \bigcap_{a \in G} C(a)$

$\therefore Z(G) \subseteq \bigcap_{a \in G} C(a)$

let $x \in \bigcap_{a \in G} C(a)$,

$x \in C(a_1) \Rightarrow x a_1 = a_1 x$
$x \in C(a_2) \Rightarrow x a_2 = a_2 x$
$\vdots$

$\therefore \forall a \in G,\ xa = ax$

$\therefore x \in Z$

$\bigcap_{a \in G} C(a) \subseteq Z(G)$

$\therefore Z(G) = \bigcap_{a \in G} C(a)$ //

37) WTS $C(a) \subseteq C(a^k)$

$C(a) = \{x \in G : xa = ax\}$

$C(a^k) = \{x \in G : xa^k = a^k x\}$

Let $x \in C(a)$,

$xa = ax$

$$xa^k = x\underbrace{aa\ldots a}_{k}$$

$$= axa\ldots a$$

$$= aa\ldots ax$$

$$= a^k x$$

$\therefore C(a) \subseteq C(a^k)$

In a group, if $xa = ax$, then $xa^k = a^k x$.

45) $C(a) = \{x \in G : xa = ax\}$

No.

$Z(G) = \{a \in G : \forall x \in G, ax = xa\}$

Yes

47) G is Abelian

$n \in \mathbb{Z}$

$S = \{x \in G : x^n = e\}$

WTS $S \leq G$.

Let $a, b \in S$,

$a^n = e,\ b^n = e$

$(ab)^n = \underbrace{(ab)(ab)\ldots(ab)}_{n}$

$a, b \in G \Rightarrow ab = ba$

$\therefore (ab)^n = aa \ldots a\, bb \ldots b$

$= a^n b^n$

$= ee$

$= e$

$\therefore ab \in S$

Let $a \in S$,

$a^n = e$

$(a^{-1})^n = a^{-1}a^{-1} \ldots a^{-1}$

$= (a^n)^{-1}$ $\quad [(ab)^{-1} = b^{-1}a^{-1}]$

$= e^{-1}$

$= e$

$\therefore a^{-1} \in S$

By Two-step subgroup test,

$S \leq G$ //

5.1) $a \in G$, $|a| = n$

$d \mid n$, $d \in \mathbb{N}$

WTS $|a^d| = n/d$

$(a^d)^{n/d} = a^n = e$

$\therefore |a^d| \leq n/d$

Let $i < n/d$: $(a^d)^i = e$

$(a^d)^i = a^{di} = e$

$|a| = n \Rightarrow n \leq di \Rightarrow i \geq n/d$

$(\Rightarrow\Leftarrow)$

$\therefore |a^d| = n/d$ //

5a) $H \leq G$, $|G| = n$

$g \in G$

n is the smallest positive integer

s.t. $g^n \in H$.

WTS $n \mid |g|$

Let $|g| = k$,

$k = nq + r$, $0 \leq r < n$

$g^k = g^{nq+r} = g^{nq} g^r$

$g^r = g^{k-nq} = g^k g^{-nq}$

$= g^k (g^n)^{-q}$

$= (g^n)^{-q} \in H$

n is the smallest integer s.t. $g^n \in H$,

$0 \leq r < n \Rightarrow r = 0$

$\therefore k = nq \Rightarrow n \mid k = |g|$ //

6.1) $\mathbb{R}^*$ under multiplication

$H = \{x \in \mathbb{R}^* : x^2 \text{ is rational}\}$

WTS $H \leq \mathbb{R}^*$

Let $a, b \in H$,

$(ab)^2 = abab$

$= a^2 b^2 \in \mathbb{I}$

$\therefore ab \in H$

Let $a \in H$,

$$(a^{-1})^2 = a^{-1}a^{-1}$$
$$= (a^2)^{-1} \in H$$

$\therefore a^{-1} \in H$

By 2-step subgroup test, $H \leq \mathbb{R}^*$

65) $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$

under addition.

$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : a+b+c+d = 0 \right\}$

WTS $H \leq G$.

Let $A, B \in H$,

$$A+B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix}$$

$$(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)$$

$$= a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4$$

$$= 0+0$$

$$= 0$$

$\therefore A+B \in H$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$

$$-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$-a-b-c-d = -(a+b+c+d) = 0$$

$\therefore -A \in H$

By 2-step subgroup test, $H \leq G$. //

67) $H \leq \mathbb{R}$ under addition

$K = \{2^a : a \in H\}$

WTS $K \leq \mathbb{R}^*$ under multiplication.

Let $2^a, 2^b \in K$,

$(2^a 2^b) = 2^{a+b}$

$H \leq \mathbb{R} \Rightarrow a, b \in H,\ a+b \in H$

$\therefore (2^a 2^b) = 2^{a+b} \in K$

Let $2^a \in K$

$2^a 2^{-a} = 2^{a-a} = 2^0 = 1 \Rightarrow (2^a)^{-1} = 2^{-a}$

$H \leq \mathbb{R} \Rightarrow a \in H, -a \in H$

$\therefore (2^a)^{-1} = 2^{-a} \in K$

By 2-step subgroup test,

$K \leq \mathbb{R}^*$ //

69) $G = GL(2, \mathbb{R})$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{Z}^* \right\}$

Let $A, B \in H$,

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}$$

$a_1 b_1, a_2 b_2 \in \mathbb{Z}^* \Rightarrow AB \in H$

let $A \in H$,

$$A^{-1} = \frac{1}{ab}\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}$$

$1/a, 1/b \notin \mathbb{Z}^* \Rightarrow A^{-1} \notin H$.

$\therefore H \not\leq GL(2, \mathbb{R})$ //

7) $H = \{a+bi : a, b \in \mathbb{R}, a^2+b^2=1\}$

let $a+bi, c+di \in H$,

$(a+bi)(c+di) = ac + adi + cbi - bd$

$= ac - bd + (ad+cb)i$

$(ac-bd)^2 + (ad+cb)^2$

$= a^2c^2 - 2abcd + b^2d^2$
$+ a^2d^2 + 2abcd + c^2b^2$

$= a^2c^2 + a^2d^2 + b^2d^2 + b^2c^2$

$= a^2(c^2+d^2) + b^2(d^2+c^2)$

$= a^2+b^2$

$= 1$

$\therefore (a+bi)(c+di) \in H$

Let $a+bi \in H$,

$(a+bi)(a-bi) = a^2+b^2 = 1$

$\therefore (a+bi)^{-1} = (a-bi)$

$a^2+(-b)^2 = a^2+b^2 = 1$

$\therefore (a+bi)^{-1} \in H$

$\therefore H \leq C^*$ //

73) $H \leq G$

$HZ(G) = \{hz : h \in H, z \in Z(G)\}$

WTS $HZ(G) \leq G$

$Z(G) = \{a \in G : \forall x \in G, ax = xa\}$

Let $h_1 z_1, h_2 z_2 \in HZ(G)$,

$(h_1 z_1)(h_2 z_2) = h_1 z_1 h_2 z_2$
$= h_1 h_2 z_1 z_2$

$h_1, h_2 \in H$, $z_1, z_2 \in Z(G)$

$\Rightarrow h_1 h_2 z_1 z_2 \in HZ(G)$

$\therefore (h_1 z_1)(h_2 z_2) \in HZ(G)$

Let $hz \in HZ(G)$,

$$(hz)^{-1} = z^{-1}h^{-1}$$
$$= h^{-1}z^{-1}$$

$h^{-1} \in H$, $z^{-1} \in Z(G)$

$\therefore (hz)^{-1} \in HZ(G)$

$\therefore HZ(G) \leq G$ //