1) $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$

$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$

a) $\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ //

b) $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$ //

c) $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$ //

3) a) $(1235)(413)$

$(15)(234)$ //

b) $(13256)(23)(46512)$

$(124)(35)(6)$ //

c) $(12)(13)(23)(142)$

(1423) //

5) a) $(124)(357)$, $\text{lcm}(3,3) = 3$ // (Thm 5.3)

b) $(124)(3567)$, $\text{lcm}(3,4) = 12$ //

c) $(124)(35)$, $\text{lcm}(3,2) = 6$ //

d) $(124)(357869)$, $\text{lcm}(3,6) = 6$ //

e) $(1235)(24567)$

$= (124)(3567)$, $\text{lcm}(3,4) = 12$ //

f) $(345)(245)$

$= (25)(34)$, $\text{lcm}(2,2) = 2$ //

7) Thm 5.3, $\text{lcm}(4,6) = 12$

9) S_6: 1, 5, 4, 3, 6, 2

A_6: 1, 2, 3, 4, 5, 6

A_7 : 1, 2, 3, 4, 5, 6, 7

13) $\alpha : S \to S$

$\alpha(\alpha(x)) = x, \ \forall x \in S$

Let $a, b \in S$, $a = b$,

$\alpha(\alpha(a)) = a$, $\alpha(\alpha(b)) = b$

$\alpha(\alpha(a)) = \alpha(\alpha(b))$

$\alpha^{-1}(\alpha(\alpha(a))) = \alpha^{-1}(\alpha(\alpha(b)))$

$\alpha(a) = \alpha(b)$ // (1-1)

Let $a \in S$,

$\exists \alpha(a) \in S$; $\alpha(\alpha(a)) = a$ (onto) //

2) a) $(1235)(413)$

$= (15)(234)$

b) $(13256)(23)(46512)$

$= (124)(35)$

c) $(12)(13)(23)(142)$

$= (1423)$

3) d) By Thm 5.3, if α is a permutation in disjoint cycle form, then

$|\alpha| = \text{lcm}(\text{length of cycles})$

$\alpha = (124)(357869)$

$|\alpha| = \text{lcm}(3,6) = 6$ //

e) $\alpha = (1235)(24567)$

$= (124)(3567)$

$|\alpha| = \text{lcm}(3,4) = 12$ //

f) $\alpha = (345)(245)$

$= (25)(34)$

$|\alpha| = \text{lcm}(2,2) = 2$ //

6) WTS $\exists \alpha \in A_8 : |\alpha| = 15$

Since $(123) = (12)(13)$,

$\therefore (123) \in A_8$

Since $(45678) = (48)(47)(46)(45)$,

$\therefore (45678) \in A_8$

$\therefore (123)(45678) \in A_8$,

By Thm 5.3,

$|(123)(45678)| = \text{lcm}(3,5)$
$= 15$ //

7) S_6, let (n) be an n-cycle, then the possibilities of elements of S_6 in distinct cycle form,

(6)

(5)(1)

(4)(2)

(4)(1)(1)

(3)(3)

(3)(2)(1)

(3)(1)(1)(1)

(2)(4)

(2)(3)(1)

(2)(2)(2)

(2)(2)(1)(1)

(2)(1)(1)(1)(1)

(1)(5)

(1)(4)(1)

(1)(3)(2)

$(1)(3)(1)(1)$

$(1)(3)(3)$

$(1)(2)(3)(1)$

$(1)(2)(1)(1)(1)$

$(1)(1)(1)(1)(1)(1)$

By Thm 5.3,

$\operatorname{lcm}(5,1) = 5 = \operatorname{lcm}(1,5)$

$\operatorname{lcm}(4,2) = 4 = \operatorname{lcm}(4,1,1)$

$\operatorname{lcm}(3,3) = 3 = \operatorname{lcm}(3,1,1,1)$

$\operatorname{lcm}(3,2,1) = 6$

$\operatorname{lcm}(2,2,2) = 2 = \operatorname{lcm}(2,2,1,1) = \operatorname{lcm}(2,1,1,1,1)$

$\operatorname{lcm}(1,1,1,1,1,1) = 1$

$\therefore \alpha \in S_6 \Rightarrow |\alpha| \in \{1,2,3,4,5,6\}$

A_6

$(5)(1)$, $(4)(2)$, $(3)(3)$, $(2)(4)$,

$(2)(2)(1)(1)$, $(1)(5)$, $(1)(2)(2)(1)$,

$(1)(1)(1)(1)(1)(1) \in A_6$

$\therefore \alpha \in A_6 \Rightarrow |\alpha| \in \{1, 2, 3, 4, 5\}$

A_7

By Example 5.5,

(7), $(5)(1)(1)$, $(4)(2)(1)$, $(3)(3)(1)$,

$(3)(2)(2)$, $(3)(1)(1)(1)(1)$,

$(2)(2)(1)(1)(1)$, $(1) \in A_7$,

$\therefore \alpha \in A_7 \Rightarrow |\alpha| \in \{1, 2, 3, 4, 5, 6, 7\}$ //

9) d) $(12)(134)(152)$

$(134) = (14)(13)$

$(152) = (12)(15)$

$\therefore (12)(134)(152) = (12)(14)(13)(12)(15)$

Odd

e) $(1243)(3521)$

$(1243) = (13)(14)(12)$

$(3521) = (31)(32)(35)$

$\therefore (1243)(3521) = (13)(14)(12)(31)(32)(35)$

even

13) Done in notes

17) $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$

$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$

a) $\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ //

b) $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$ //

c) $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$ //

23) a) $\alpha_3 = (13)(24)$

$a \in G$, $C(a) = \{x \in G : xa = ax\}$

$\alpha_3\alpha_1 = \alpha_1\alpha_3 = \alpha_3$

$\alpha_3\alpha_2 = \alpha_2\alpha_3 = \alpha_4$

$\alpha_3\alpha_4 = \alpha_4\alpha_3 = \alpha_2$

$\alpha_3\alpha_3 = \alpha_3\alpha_3 = \alpha_1$

$\therefore C(\alpha_3) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ //

b) $\alpha_{12} = (124)$

$\alpha_{12}\alpha_1 = \alpha_1\alpha_{12} = \alpha_{12}$

$\alpha_{12}\alpha_7 = \alpha_7\alpha_{12} = \alpha_1$

$\alpha_{12}\alpha_{12} = \alpha_{12}\alpha_{12} = \alpha_7$

$\therefore C(\alpha_{12}) = \{\alpha_1, \alpha_7, \alpha_{12}\}$ //

30) let $\alpha = (a_1 a_2 \dots a_n)$

$\alpha(a_1) = a_2$, $\alpha^{-1}(a_2) = a_1$

$\alpha(a_2) = a_3$, $\alpha^{-1}(a_3) = a_2$

$\vdots$

$\alpha(a_i) = a_{i+1}$, $\alpha^{-1}(a_{i+1}) = a_i$, $2 \le i < n$

$\alpha(a_n) = a_1$, $\alpha^{-1}(a_1) = a_n$

$\therefore \alpha^{-1} = (a_1 a_n a_{n-1} \dots a_2)$ //

31) G is a group of permutation on set X

$a \in X$, $stab(a) = \{\alpha \in G : \alpha(a) = a\}$

Let $\alpha, \beta \in Stab(a)$,

$\therefore \alpha(a) = a, \beta(a) = a$

$\beta(a) = a \Rightarrow \beta^{-1}(a) = a$

$\therefore \beta^{-1} \in Stab(a)$

$\alpha\beta^{-1}(a) = \alpha(a) = a$

$\therefore \alpha\beta^{-1} \in stab(a)$

By Thm 3.1,

$\alpha, \beta \in Stab(a)$, $\alpha\beta^{-1} \in Stab(a) \Rightarrow Stab(a) \leq G$

41) WTS S_n is non-Abelian for $n \geq 3$.

$(123)(12) \in S_n$, $n \geq 3$

$(12)(123) \in S_n$, $n \geq 3$

$\because (123)(12) \neq (12)(123)$

$\therefore S_n$ is non-Abelian for $n \geq 3$ //

47) $A_n, n \geq 3$

$\alpha \in A_n \Rightarrow \alpha = \beta_1 \beta_2 \dots \beta_r$

$\beta_i = 2\text{-cycle}, r = \text{even}$

$(12)(13) = (123)$

$(12)(23) = (123)$

$(13)(23) = (132)$

$(12)(34) = (324)(132)$

$(13)(24) = (241)(134)$

$(14)(23) = (213)(142)$

5 b) $\beta, \gamma \in S_4$, $\beta\gamma = (1432)$, $\gamma\beta = (1243)$

$\beta(1) = 4$

$\therefore \gamma(1) = 1$, $\gamma(4) = 2$

$\beta\gamma(1) = 4$ $\quad$ $\gamma\beta(1) = 2$

$\beta\gamma(4) = 3$ $\quad$ $\gamma\beta(2) = 4$

$\beta\gamma(3) = 2$ $\quad$ $\gamma\beta(4) = 3$

$\beta\gamma(2) = 1$ $\quad$ $\gamma\beta(3) = 1$

$\beta\gamma(4) = \beta(2) = 3$ $\quad$ $\therefore \beta = (1423)$

$\gamma\beta(2) = \gamma(3) = 4$ $\quad$ $\gamma = (234)$ //

$\beta\gamma(3) = \beta(4) = 2$

$\gamma\beta(4) = \gamma(2) = 3$

$\gamma\beta(1) = \gamma(4) = 2$

$\beta\gamma(2) = \beta(3) = 1$