38) $\mathbb{Z}_2[i] = \{a+bi : a, b \in \mathbb{Z}_2\}$
$= \{0, 1, i, 1+i\}$

	0	1	i	$1+i$
0	0	0	0	0
1	0	1	i	$1+i$
i	0	i	-1	$-1+i$
$1+i$	0	$1+i$	$-1+i$	0

Since $i, 1+i \in \mathbb{Z}_2[i]$ have no inverses, they are not units.
$\therefore$ By Definition 13.3, $\mathbb{Z}_2[i]$ is not a field.

Since $1+i \neq 0$ but $(1+i)(1+i) = 0$, by Definition 13.2,
$\mathbb{Z}_2[i]$ is not an integral domain. $\square$

41) Let R be a finite commutative ring with no zero-divisors and at least two elements

Assume that R has only two elements, Since R has no zero-divisors, $a,b \in R$, $ab=0 \Rightarrow a=0$ or $b=0$

let $a=0, b\neq 0$, then $0=a=ab$, So

$$ab=(ab)b=abb$$

$$ab-abb=0$$

$$(a-ab)b=0$$

$\therefore a-ab=0 \Rightarrow a=ab$

$\therefore$ b is the unity //

Assume that R with k elements, $1\leq k\leq n$, has a unity.

Assume that R has n elements, then

$$R=\{a_1, a_2, \ldots, a_{n-1}, 0\}$$

Let $a_1 a_i = a_1$, then

$$a_1 a_j = (a_1 a_i) a_j$$

$$a_1 a_j - a_1 a_i a_j = 0$$

$$a_1 (a_j - a_i a_j) = 0$$

$\therefore a_j - a_i a_j = 0 \Rightarrow a_j = a_i a_j$

$\therefore a_i$ is the unity $\square$

45) Let $x, y \in R$, R is commutative, char $R = p$, p is prime.

a) Char $R = p \Rightarrow \forall x \in R,\ px = 0$

By the Binomial Theorem,

$$(x+y)^p = \sum_{r=0}^{p} \binom{p}{r} x^{p-r} y^r$$

$$= \binom{p}{0} x^p y^0 + \binom{p}{1} x^{p-1} y + \binom{p}{2} x^{p-2} y^2 + \cdots + \binom{p}{p} x^0 y^p$$

$$= \frac{p!}{(p-0)!} x^p y^0 + \frac{p!}{(p-1)!} x^{p-1} y + \cdots + \frac{p!}{(p-p)!} x^0 y^p$$

Since $\forall x \in R,\ px = 0$,

$$\frac{p!}{(p-k)!} x^{p-k} y^k = \frac{p(p-1)\cdots(1)}{(p-k)!} x^{p-k} y^k = 0$$

$\therefore (x+y)^p = x^p + y^p$ $\square$

b) WTS $\forall n \in \mathbb{Z}^+$, $(x+y)^{p^n} = x^{p^n} + y^{p^n}$

Let $n=1$, by part (a), $(x+y)^p = x^p + y^p$.

Assume that for $1 \leq k \leq n$, $(x+y)^{p^k} = x^{p^k} + y^{p^k}$

Let $n = k+1$, by the Binomial Theorem,

$$(x+y)^{p^{k+1}} = \binom{p^{k+1}}{0} x^{p^{k+1}} y^0 + \binom{p^{k+1}}{1} x^{p^{k+1}-1} y + \cdots + \binom{p^{k+1}}{p^{k+1}} x^0 y^{p^{k+1}}$$

Since $\binom{p^{k+1}}{r} x^{p^r} y^r = \dfrac{p^{k+1}!}{(p^{k+1}-r)!} x^{p^r} y^r$

$$= \frac{p^{k+1}(p^{k+1}-1)\cdots(1)}{(p^{k+1}-r)!} x^{p^{k+1}} y^r$$

$$= \frac{p\,p^k(p^{k+1}-1)\cdots(1)}{(p^{k+1}-r)!} x^{p^{k+1}} y^r$$

$$= \frac{p^k(p^{k+1}-1)\cdots(1)}{(p^{k+1}-r)!} p\, x^{p^{k+1}} y^r$$

$$= 0$$

$\therefore (x+y)^{p^{k+1}} = x^{p^{k+1}} - y^{p^{k+1}}$

$\therefore$ By induction, $(x+y)^{p^n} = x^{p^n} + y^{p^n}$ $\square$

C) Let char $R = 4$,

Find $x, y \in R$: $(x+y)^4 \neq x^4 + y^4$

Char $R = 4 \Rightarrow \forall x \in R,\ 4x = 0$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$= x + 6x^2y^2 + y^4$

52) $\mathbb{Z}_3[i] = \{a+bi : a, b \in \mathbb{Z}_3\}$

$= \{0, i, 2i,$

$1, 1+i, 1+2i,$

$2, 2+i, 2+2i\}$

$x^2 - x + 2 = 0$

$x^2 - x = -2 = 1$

$\therefore x(x-1) = 1$

$x = 0,\ 0(0-1) = 0$

$x = i,\ i(i-1) = -1-i = -(1+i) = 2+2i$

$x = 2i,\ 2i(2i-1) = 4i^2 - 2i = -1-2i = -(1+2i) = 2+i$

$x = 1,\ 1(1-1) = 0$

$x = 1+i, \quad (1+i)(1+i-1) = (1+i)(i)$

$= i - 1$

$= -1 + i$

$= 2 + i$

$x = 1+2i, \quad (1+2i)(1+2i-1) = (1+2i)(2i)$

$= 2i + 4i^2$

$= 2i - 1$

$= -1 + 2i$

$= 2 + 2i$

$x = 2, \quad 2(2-1) = 2$

$x = 2+i, \quad (2+i)(2+i-1) = (2+i)(1+i)$

$= 2 + 2i + i - 1$

$= 1$

$x = 2+2i, \quad (2+2i)(2+2i-1) = (2+2i)(1+2i)$

$= 2 + 4i + 2i + 4i^2$

$= 2 + i + 2i - 1$

$= 1$

$\therefore$ $x = 2+2i$ and $x = 2+i$ are the solutions for

$x^2 - x + 2 = 0$ over $\mathbb{Z}_3[i]$ $\square$