1) Let $\phi : \mathbb{Z} \to G$,

$\phi(n) = 2n, \quad n \in \mathbb{Z}$

Let $n = m$, $\phi(n) = 2n = 2m = \phi(m)$

$\therefore \phi$ is one-to-one //

Let $x \in G$, WTS $\exists\, n \in \mathbb{Z} : \phi(n) = x$

$x = \text{even} \Rightarrow \frac{x}{2} \in \mathbb{Z}$

Let $n = \frac{x}{2} \in \mathbb{Z}$,

$\phi(n) = 2n = 2\left(\frac{x}{2}\right) = x$

$\therefore \phi$ is onto //

$\phi(n+m) = 2(n+m) = 2n + 2m = \phi(n) + \phi(m)$

$\therefore \phi$ preserves binary operation //

$\therefore \phi : \mathbb{Z} \to G$ is an isomorphism $\square$

6) Let $G \cong H$, $H \cong K$

WTS $G \cong K$

Let $\phi : G \to H$, $\varphi : H \to K$

$\therefore \phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$

$\forall h \in H$, $\exists g \in G : \phi(g) = h$

$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$

$\varphi(h_1) = \varphi(h_2) \Rightarrow h_1 = h_2$

$\therefore \varphi(\phi(g_1)) = \varphi(\phi(g_2)) \Rightarrow \phi(g_1) = \phi(g_2)$

$\Rightarrow g_1 = g_2$, $\therefore \varphi\phi$ is one-to-one

$\forall k \in K, \exists h \in H : \varphi(h) = k$

$\Rightarrow \varphi(\phi(g)) = k$

$\therefore \exists g \in G : \varphi\phi(g) = k$

$\therefore \varphi\phi$ is onto

$\varphi(h_1 h_2) = \varphi(h_1)\varphi(h_2)$

$\therefore \varphi(\phi(g_1 g_2)) = \varphi(\phi(g_1)\phi(g_2))$

$= \varphi\phi(g_1)\,\varphi\phi(g_2)$

$\therefore \varphi\phi$ preserves binary operation on G

$\therefore \varphi\phi : G \to K$ is an isomorphism $\square$

7) WTS $S_4 \not\cong D_{12}$

$|S_4| = 4! = 24$

$R_{30} \in D_{12}$

$(R_{30})^{12} = R_{360} = R_0$

$\therefore |R_{30}| = 12$

By Thm 5.3, $\alpha \in S_4$, α in disjoint cycle form

$\Rightarrow |\alpha| = \text{lcm}(\text{cycle lengths})$

$\forall \alpha \in S_4$, $|\alpha| \in \{\text{lcm}(1,4), \text{lcm}(1,3), \text{lcm}(2,2)\}$
$= \{4, 3, 2\}$

By Thm 6.2 (v),

$\forall a \in G$, $|a| = |\phi(a)|$

Let $\alpha \in S_4$: $\phi(\alpha) = R_{30}$

$\therefore |R_{30}| = |\phi(\alpha)| = 12$

But $\alpha \in S_4 \Rightarrow |\alpha| \neq 12$

$\therefore |\phi(\alpha)| \neq |\alpha|$ $(\Rightarrow\Leftarrow)$ $\square$

4) WTS $U(8) \not\cong U(10)$

By Thm 6.2 (iv),

$\phi: G \to \overline{G}$ is an isomorphism $\Rightarrow G = \langle a \rangle \Leftrightarrow \overline{G} = \langle \phi(a) \rangle$

$U(8) = \{1, 3, 5, 7\}$

$U(10) = \{1, 3, 7, 9\}$

$3 \bmod 10 = 3$

$3^2 \bmod 10 = 9$

$3^3 \bmod 10 = 7$

$3^4 \bmod 10 = 1$

$\therefore U(10) = \langle 3 \rangle$

$3 \bmod 8 = 3$ $\quad 5 \bmod 8 = 5$ $\quad 7 \bmod 8 = 7$

$3^2 \bmod 8 = 1$ $\quad 5^2 \bmod 8 = 1$ $\quad 7^2 \bmod 8 = 1$

$3^3 \bmod 8 = 3$ $\quad 5^3 \bmod 8 = 5$ $\quad 7^3 \bmod 8 = 7$

$\therefore U(8)$ is not cyclic

$\therefore U(8) \not\cong U(10)$ $\square$

10) G is a group

WTS $\forall g \in G$, $\alpha(g) = g^{-1}$ is an automorphism

$\Leftrightarrow$ G is Abelian

($\Rightarrow$) Let $\alpha(g) = g^{-1}$ be an automorphism,

$\therefore \forall g, h \in G$, $\alpha(g) = \alpha(h) \Rightarrow g^{-1} = h^{-1} \Rightarrow g = h$

$\forall h \in G, \exists g \in G : \alpha(g) = g^{-1} = h$

$\alpha(gh) = \alpha(g)\alpha(h) = g^{-1}h^{-1}$

$\therefore (gh)^{-1} = g^{-1}h^{-1}$

$h^{-1}g^{-1} = g^{-1}h^{-1}$

$h^{-1}g^{-1}h = g^{-1}$

$g^{-1}h = hg^{-1}$

$g^{-1}hg = h$

$hg = gh$

$\therefore$ G is Abelian //

($\Leftarrow$) Let G be Abelian,

$\therefore \forall g, h \in G$, $gh = hg$

Let $\alpha(g) = \alpha(h)$

$$g^{-1} = h^{-1}$$

$$g^{-1}h = e$$

$$h = g$$

$\therefore \alpha$ is one-to-one

Let $g \in G$, $\exists g^{-1} \in G$: $\alpha(g^{-1}) = (g^{-1})^{-1} = g$

$\therefore \alpha$ is onto

$$\alpha(gh) = (gh)^{-1}$$

$$= h^{-1}g^{-1}$$

$$= g^{-1}h^{-1}$$

$$= \alpha(g)\alpha(h)$$

$\therefore \alpha$ preserves the binary operation in G

$\therefore \alpha : G \to G$ is an Automorphism

$\therefore \alpha$ is an automorphism $\Leftrightarrow$ G is Abelian $\square$

16) $\phi : U(16) \to U(16)$

$\phi(x) = x^3$

WTS ϕ is an automorphism

$U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$

Let $a, b \in U(16)$, $\phi(a) = \phi(b)$,

$\therefore \phi(a) = \phi(b)$

$a^3 = b^3$

$a = b$

$\therefore \phi$ is one-to-one

Let $a \in U(16)$,

$\exists\, b \in U(16) : \phi(b) = b^3 = a$

$\therefore \phi$ is onto

Let $\phi(ab) = (ab)^3$

$= a^3 b^3$

$= \phi(a)\phi(b)$

$\therefore \phi$ is OP.

$\therefore \phi$ is an automorphism //

$\phi : x \to x^n$, n is even, $n > 1$

is an automorphism of $U(16)$

Since if n is even,

$$\phi(a) = \phi(b)$$

$$a^n = b^n$$

$$a \neq b$$

Example, $a^2 = b^2 \Rightarrow a = 3, b = 5$

$a^4 = b^4 \Rightarrow a = 1, b = 3$

17) $r \in U(n)$

WTS $\alpha : Z_n \to Z_n$, $\alpha \in Aut(Z_n)$

$\alpha(s) = sr \bmod n$, $s \in Z_n$

Let $a, b \in \mathbb{Z}_n$, $\alpha(a) = \alpha(b)$

$\alpha(a) = \alpha(b)$

$ar \bmod n = br \bmod n \iff n \mid (a-b)r$

$r \in U(n) \Rightarrow \gcd(n, r) = 1$

$\therefore n \mid (a-b)$

$\therefore a - b = nt$

$a, b \in \mathbb{Z}_n \Rightarrow a, b < n$

$\therefore a - b = nt \Rightarrow t = 0$

$\therefore a - b = 0 \Rightarrow a = b$

$\therefore \alpha$ is one-to-one

$$\begin{aligned}
\alpha(a+b) &= (a+b)r \bmod n \\
&= (ar + br) \bmod n \\
&= ar \bmod n + br \bmod n \\
&= \alpha(a) + \alpha(b)
\end{aligned}$$

α is O.P.

Let $a \in \mathbb{Z}_n$, WTS $\exists b \in \mathbb{Z}_n : \alpha(b) = a$

$r \in U(n) \Rightarrow \gcd(r, n) = 1$

$\therefore xr \equiv 1 \pmod n$ has a solution

Let $b = yx$,

$\alpha(b) = br = (yx)r = y(xr) = y \pmod n$

$\therefore \alpha$ is onto

$\therefore \alpha$ is an automorphism //

24) $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$U(24) = \{1, 5, 7, 11, 13, 15, 17, 19, 23\}$

$U(20) \not\cong U(24)$ since $|U(20)| \neq |U(20)|$ //

26) $G = \{a + b\sqrt{2} : a, b \text{ are rational}\}$

$H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \text{ are rational} \right\}$

Let $\phi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

let $a+b\sqrt{2},\ c+d\sqrt{2} \in G$,

$$\phi(a+b\sqrt{2}) = \phi(c+d\sqrt{2})$$

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$$

$$\therefore a = c,\ b = d$$

$$\therefore a+b\sqrt{2} = c+d\sqrt{2}$$

let $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in H$,

WTS $\exists\, a+b\sqrt{2} \in G : \phi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

a, b are rational $\Rightarrow \forall \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in H$,

$\exists\, a+b\sqrt{2} \in G : \phi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

$$\phi\left[(a+b\sqrt{2}) + (c+d\sqrt{2})\right]$$

$$= \phi(a+b\sqrt{2}+c+d\sqrt{2})$$

$$= \phi\left[(a+c)+(b+d)\sqrt{2}\right]$$

$$= \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$$

$$= \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$$

$$= \phi(a+b\sqrt{2}) + \phi(c+d\sqrt{2})$$

$\therefore$ ϕ is OP

$\therefore$ $G \approx H$ under addition

$$\phi\left[(a+b\sqrt{2})(c+d\sqrt{2})\right]$$

$$= \phi(ac + ad\sqrt{2} + bc\sqrt{2} + bd2)$$

$$= \phi\left[(ac+bd2) + (ad+bc)\sqrt{2}\right]$$

$$= \begin{pmatrix} ac+2bd & 2(ad+bc) \end{pmatrix}$$

$+ \quad c \quad ac+2$

$$= \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$$

Yes //

30) $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}\}$

$\phi : (a_1, \dots, a_n) \to (-a_1, \dots, -a_n)$

Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}^n$,

$\phi(a_1, \dots, a_n) = \phi(b_1, \dots, b_n)$

$(-a_1, \dots, -a_n) = (-b_1, \dots, -b_n)$

$-a_i = -b_i \Rightarrow a_i = b_i$

$\therefore (a_1, \dots, a_n) = (b_1, \dots, b_n)$

$\therefore \phi$ is one-to-one

let $(a_1, \dots, a_n) \in \mathbb{R}^n$,

WTS $\exists (b_1, \dots, b_n) \in \mathbb{R}^n$ s.t.

$\phi(b_1, \dots, b_n) = (a_1, \dots, a_n)$

$\phi(b_1, \dots, b_n) = (-b_1, \dots, -b_n)$

$a_i \in \mathbb{R} \Rightarrow -a_i \in \mathbb{R}$

let $b_i = -a_i$,

$$\begin{aligned}\phi(b_1, \dots, b_n) &= (-b_1, \dots, -b_n) \\ &= (-(-a_1), \dots, -(-a_n)) \\ &= (a_1, \dots, a_n)\end{aligned}$$

$\therefore \phi$ is onto //

$\phi[(a_1, \dots, a_n) + (b_1, \dots, b_n)]$

$= \phi(a_1+b_1, \dots, a_n+b_n)$

$= (-(a_1+b_1), \dots, -(a_n+b_n))$

$= (-a_1-b_1, \dots, -a_n-b_n)$

$= (-a_1, \dots, -a_n) + (-b_1, \dots, -b_n)$

$= \phi(a_1, \dots, a_n) + \phi(b_1, \dots, b_n)$

$\therefore \phi$ is op //

35) $g, h \in G$, $\forall x \in G$,

$\phi_g(x) = gxg^{-1}$

$\phi_h(x) = hxh^{-1}$

$\phi_g(x) = \phi_h(x)$

$gxg^{-1} = hxh^{-1}$

WTS $h^{-1}g \in Z(G)$

$Z(G) = \{a \in G : \forall x \in G, ax = xa\}$

$gxg^{-1} = hxh^{-1}$

$h^{-1}gxg^{-1} = xh^{-1}$

$h^{-1}gx = xh^{-1}g$

$\therefore h^{-1}g \in Z(G)$ //

39) $\phi \in Aut(D_4)$

$\phi(R_{90}) = R_{270}$

$\phi(V) = V$

$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

$D = R_{90}V$

$H = R_{90}D$

$$\phi(D) = \phi(R_{90} V)$$
$$= \phi(R_{90})\phi(V)$$
$$= R_{270} V$$
$$= D'$$

$$\phi(H) = \phi(R_{90} D)$$
$$= \phi(R_{90})\phi(D)$$
$$= R_{270} D'$$
$$= H$$

42) $\phi \in Aut(\mathbb{Q})$

WTS $\phi(x) = x\phi(1)$

let $\phi \in Aut(\mathbb{Q})$,

$$\phi(x) = \phi(\underbrace{1+1+\cdots+1}_{x})$$

$$= \underbrace{\phi(1) + \cdots + \phi(1)}_{x}$$

$$= x\phi(1)$$

//

40) $\alpha_i \in Aut(Z_9)$

$\alpha_i(1) = i, \quad gcd(i, 9) = 1$

$Z_9 = \{0, 1, \ldots, 8\}$

$\alpha_5(1) = 5$

$\alpha_5(2) = \alpha_5(1+1)$

$= \alpha_5(1) \alpha_5(1)$

$= 5 + 5$

$= 1$

$\alpha_5(3) = \alpha_5(1) + \alpha_5(2)$

$= 5 + 1$

$= 6$

$\alpha_5(4) = \alpha_5(1) + \alpha_5(3)$

$$= 5+6$$
$$= 2$$

$$\alpha_5(5) = \alpha_5(1) + \alpha_5(4)$$
$$= 5+2$$
$$= 7$$

$$\alpha_5(6) = \alpha_5(1) + \alpha_5(5)$$
$$= 5+7$$
$$= 3$$

$$\alpha_5(7) = 5+3 = 8$$

$$\alpha_5(8) = 5+8 = 4$$

$$\alpha_5(0) = 0\alpha_5(1) = 0$$

$$\alpha_5 = (157842)(36)(0)$$

$$\alpha_8(1) = 8$$

$$\alpha_8(2) = 2\alpha_8(1)$$
$$= 2(8)$$

$$= 7$$

$$\alpha_8(3) = 8 + 7$$

$$= 6$$

$$\alpha_8(4) = 8 + 6$$

$$= 5$$

$$\alpha_8(5) = 8 + 5$$

$$= 4$$

$$\alpha_8(6) = 8 + 4$$

$$= 3$$

$$\alpha_8(7) = 8 + 3$$

$$= 2$$

$$\alpha_8(8) = 8 + 2$$

$$= 1$$

$$\alpha_8(0) = 0$$

$$\alpha_8 = (1\,8)(2\,7)(3\,6)(4\,5)(0)$$