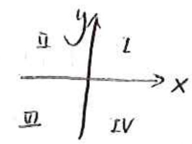
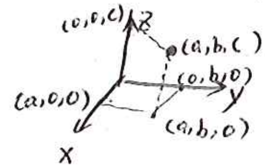
12.1

two-dimensional space (plane) $\{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$



three-dimensional space



2D

$y = x^2$ $\{(x, x^2) : x \in \mathbb{R}\}$

$y - x^2 = 0$

$f(x,y) = y - x^2$

$= \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$

3D

$\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}$

$x^2 + y^2 + z^2 - 1 = 0$

$3x + 4y^2 - z^2 = 0$

...

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ hyperbola

$y = x^2$ parabola

Ex. 1 in 3D, 1) $x = 0$ $\{(0, y, z) : y \in \mathbb{R}, z \in \mathbb{R}\}$ plane

2) $y = -2$ plane

3) $x - z = 0$ plane

4) $y = x^2$ $\{(x, x^2, z) : x \in \mathbb{R}, z \in \mathbb{R}\}$ surface cylinder

5) $x^2 + y^2 = 4$. $\{(x,y,z) : x^2 + y^2 = 4, z \in \mathbb{R}\}$ circular cylinder

Ex 2 1) $y \geq 2$ half-space $\{(x,y,z) : y \geq 2, x, z \in \mathbb{R}\}$

2) $1 \leq x \leq 4$

3) $x = 1$ and $y = 4$ $\{(1, 4, z) : z \in \mathbb{R}\}$ line! (parallel to z-axis passing thru $(1,4,0)$)

4) $x^2 + y^2 < 4$ interior of circular cylinder

5) $x^2 + y^2 = 4$ and $z = 2$ circle

6) $x^2 + y^2 + z^2 = 4$ sphere

Distance Formula

$P_1(x_1, y_1, z_1)$

$P_2(x_2, y_2, z_2)$

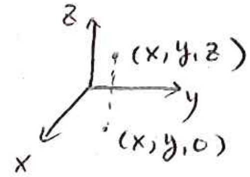
$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Sphere all points whose distance to $\overset{\text{center}}{(x_1, y_1, z_1)}$ is r

$\{(x,y,z) : \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = r\}$

$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$

Ex 3. $P=(-2,3,-5)$



1) distance to x, y plane

$$d = \sqrt{(-2-x)^2+(3-y^2)+(-5-0)^2} = 5$$

2) to x-axis $(x,0,0)$

$$d = \sqrt{9+25} = \sqrt{34}$$

Ex 4. 1) Intersection of $(x-1)^2+(y-2)^2+(z-3)^2=4$ w/ ~~y-z~~ plane

$(0,y,z)$ $\quad 1+(y-2)^2+(z-3)^2=4$

$$\begin{cases}(y-2)^2+(z-3)^2=(\sqrt{3})^2 \\ x=0\end{cases}$$

2) $\{(x,y,z): 1 \le x^2+y^2+z^2 \le 9\}$.

3) Find center & radius $\quad x^2+y^2+z^2=4x-2y+10$

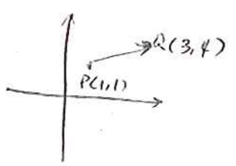
$$x^2-4x+4+y^2+2y+1+z^2=10+5$$

$$(x-2)^2+(y+1)^2+z^2=(\sqrt{15})^2$$

$(2,-1,0)$ w/ $r=\sqrt{15}$

12.2

Vector



$\vec{PQ} = \langle 2,3\rangle = -\vec{QP}$

$\vec{QP} = \langle -2,-3\rangle$

addition, subtraction

$\langle a,b,c\rangle + \langle d,e,f\rangle = \langle a+d, b+e, c+f\rangle$

$\vec{AC} = \vec{AB}+\vec{BC}$

$\sim \vec{u}-\vec{v}$

Unit vector $\quad \langle 2,-3,4\rangle = \vec{u}$

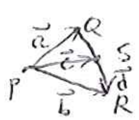
length $= \sqrt{4+9+16} = \sqrt{29} = |\vec{u}|$

Vector = length × direction (unit vector)

Unit vector w/ same direction $\langle \frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\rangle = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{|\vec{u}|}\cdot\vec{u}$

$\hat{i}=\langle 1,0,0\rangle \quad \hat{j}=\langle 0,1,0\rangle \quad \hat{k}=\langle 0,0,1\rangle \qquad \langle a,b,c\rangle = a\hat{i}+b\hat{j}+c\hat{k}$

Ex 5. $a=\vec{PQ}$, $b=\vec{PR}$, S is midpt of QR, $c=\vec{PS}$, $d=\vec{SR}$. Express c, d in terms of a, b



$\vec{b}=\vec{c}+\vec{d}$

$\vec{RQ}=\vec{a}-\vec{b}$

$\vec{d}=\frac{1}{2}(\vec{b}-\vec{a})$

$\vec{c}=\vec{b}-\vec{d} = \vec{b}-\frac{1}{2}\vec{b}+\frac{1}{2}\vec{a} = \frac{1}{2}\vec{b}+\frac{1}{2}\vec{a}$

12.3 Dot product

$\vec{u} = \langle a_1, b_1, c_1 \rangle$

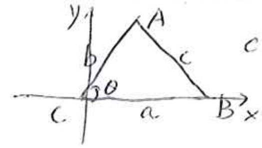
$\vec{v} = \langle a_2, b_2, c_2 \rangle$

$\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2 + c_1 c_2$

$= |\vec{u}||\vec{v}|\cos\theta$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

Law of Cosine

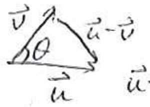


$c^2 = a^2 + b^2 - 2ab\cos\theta$

$C(0,0), B(a,0), A(b\cos\theta, b\sin\theta)$

$c = |\vec{AB}| = \sqrt{a^2 + b^2\cos^2\theta - 2ab\cos\theta + b^2\sin^2\theta}$

$= \sqrt{a^2 - 2ab\cos\theta + b^2}$ proved.



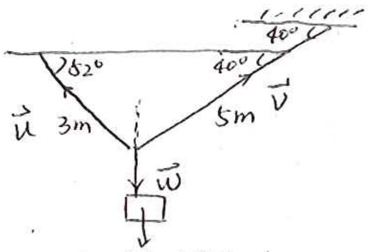
$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

$\vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

$-2\vec{u} \cdot \vec{v} = -2|\vec{u}||\vec{v}|\cos\theta$

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ proved.

Ex.



$G = 5kg \times 9.8 N/kg$

$\vec{u} + \vec{v} + \vec{w} = \vec{0}$

$\vec{w} = \langle 0, 5 \times 9.8 \rangle$

$\vec{u} = \langle -|\vec{u}|\cos 52°, |\vec{u}|\sin 52° \rangle$

$\vec{v} = \langle |\vec{u}|\cos 40°, |\vec{v}|\sin 40° \rangle$

Directional Angles α, β, γ

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow \cos\alpha = \dfrac{\vec{u} \cdot \hat{i}}{|\vec{u}||\hat{i}|} \quad \cos\beta = \dfrac{\vec{u} \cdot \hat{j}}{|\vec{u}||\hat{j}|} \quad \cos\gamma = \dfrac{\vec{u} \cdot \hat{k}}{|\vec{u}||\hat{k}|}$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$= \langle |\vec{v}|\cos\alpha, |\vec{v}|\cos\beta, |\vec{v}|\cos\gamma \rangle$

Ex. angle bet. $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 4, 0, -1 \rangle$

$\cos\theta = \dfrac{\langle 1,2,3 \rangle \cdot \langle 4,0,-1 \rangle}{\sqrt{14} \cdot \sqrt{17}}$

$= \dfrac{4+0-3}{\sqrt{14} \cdot \sqrt{17}} = \dfrac{1}{\sqrt{14}\sqrt{17}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{14} \cdot \sqrt{17}}\right)$

Projection

Scalar projection of $\vec{u}$ to $\vec{v}$

$|\vec{u}|\cos\theta$

$= |\vec{u}| \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

Vector projection of $\vec{u}$ to $\vec{v}$

$\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \text{Proj}_{\vec{v}} \vec{u}$

Ex. Find $\vec{v} = \langle -1, -2, 2 \rangle$ onto $\vec{w} = \langle 3, 3, 4 \rangle$

$\text{Proj}_{\vec{w}} \vec{v} = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w} = \dfrac{-1}{34} \langle 3, 3, 4 \rangle = \langle -\frac{3}{34}, -\frac{3}{34}, -\frac{4}{34} \rangle$

$|\text{Proj}_{\vec{w}} \vec{v}| = \frac{1}{34}\sqrt{34} = \frac{1}{\sqrt{34}}$

The Work

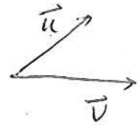
same in physics

Determinant

$\vec{u} = \langle u_1, u_2 \rangle$

$\vec{v} = \langle u_2, -u_1 \rangle$

$\vec{v} \cdot \vec{u} = 0 \quad \vec{u} \perp \vec{v}$



$\left.\begin{array}{l} \vec{u} = \langle u_1, u_2, u_3 \rangle \\ \vec{v} = \langle v_1, v_2, v_3 \rangle \end{array}\right\}$ given

solve $\vec{w} \perp \vec{u}$, $\vec{w} \perp \vec{v}$. $\vec{w} = \langle w_1, w_2, w_3 \rangle$

$\vec{w} \cdot \vec{u} = \vec{w} \cdot \vec{v} = 0$

$$\begin{cases} u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \\ v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \end{cases}$$

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -u_3 w_3 \\ -v_3 w_3 \end{pmatrix}$$

12.4 Cross-Product

$$\vec{u} \times \vec{v} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \qquad \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

properties

- $\vec{u} \times \vec{v}$ is a vector $\perp$ to both vectors (right-hand rule in physics)
- $\vec{u} \times \vec{u} = \vec{0}$
- $\vec{u} \times \vec{v} = \vec{0}$. $\vec{u} \parallel \vec{v}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

Geometry

$\vec{u} = \langle a, b, c \rangle$

$\vec{v} = \langle d, e, f \rangle$

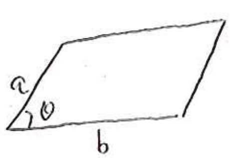
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{i}(bf - ec) - \hat{j}(af - cd) + \hat{k}(ae - bd)$$

$$|\vec{u} \times \vec{v}|^2 = (bf-ec)^2 + (af-cd)^2 + (ae-bd)^2 = (a^2+b^2+c^2)(d^2+e^2+f^2) - (ad+be+cf)^2$$

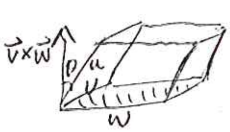
$$= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u} \cdot \vec{v}|^2$$

$$= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2\theta) = |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta$$

$\Rightarrow |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$



$A = ab\sin\theta = |\vec{a} \times \vec{b}|$



$h = |\vec{u}| \cos\theta$

$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

triple product

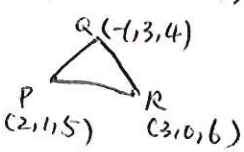
$$u\cdot(v\times w) = (u\times v)\cdot w = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ex. 1. $\vec{u} = \langle 5,1,4\rangle \quad \vec{v} = \langle -1,0,2\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 1 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \hat{i}(2-0) - \hat{j}(10+4) + \hat{k}(+1)$$
$$= 2\hat{i} - 14\hat{j} + \hat{k}$$

$$(\vec{u}\times\vec{v})\cdot\vec{u} = \langle 2,-14,1\rangle\langle 5,1,4\rangle$$
$$= 10-14+4 = 0$$

3. $\overrightarrow{PQ} = \langle -3,2,-1\rangle \quad \overrightarrow{PR} = \langle 1,-1,1\rangle$

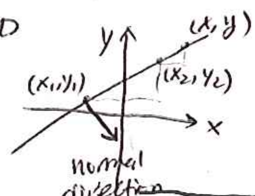


$$A = \frac{1}{2}\overrightarrow{PQ}\times\overrightarrow{PR}$$
$$= \frac{1}{2}\langle -3,2,-1\rangle\times\langle 1,-1,1\rangle$$
$$= \langle 1,2,1\rangle = \sqrt{1^2+2^2+1^2}$$

12.5

Equation of a line

2D



$$k = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-y_2}{x-x_2} = \frac{y_2-y_1}{x_2-x_1}$$

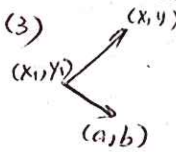
(1) $\boxed{\text{Point-slope}}$ $y = y_2 + \frac{y_2-y_1}{x_2-x_1}(x-x_2)$

or $\boxed{y-y_0 = k(x-x_0)}$

(2) $\begin{cases} x = x_1 + t \\ y = y_1 + kt \end{cases}$

$\langle x,y\rangle = \langle x_1,y_1\rangle + t\langle A,B\rangle$ parametric

(3)



$\boxed{\langle a,b\rangle\cdot\langle x-x_0,\ y-y_0\rangle = 0}$ vector

Ex 1.
$P = (1,2)$
$Q = (-3,7)$

① $k = \frac{7-2}{-3-1} = \frac{5}{-4} = -\frac{5}{4}$

$y-2 = -\frac{5}{4}(x-1)$

(pt-slope)

② direction $= \overrightarrow{PQ} = (-4,5)$

$\langle x,y\rangle = \langle 1,2\rangle + t\langle -4,5\rangle$ (parametric)

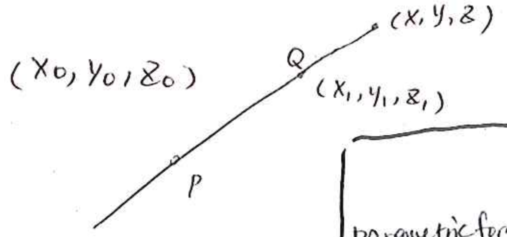
$t\in\mathbb{R}$.

③ normal vector $\langle a,b\rangle$

$$\langle a,b\rangle\cdot\langle -4,5\rangle = 0$$
$$-4a+5b=0$$
$$a=5,\ b=4$$

$$\langle 5,4\rangle\cdot\langle -4,5\rangle = 0$$

$$\langle 5,4\rangle\langle x-1,y-2\rangle = 0$$
$$5(x-1)+4(y-2) = 0$$
$$5x+4y = 13$$



Line in 3-D

$\overrightarrow{PQ} = \langle A, B, C \rangle$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \cdot \overrightarrow{PQ}$$

parametric form
$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct \end{cases}$$

vector $\vec{r} = \vec{r_0} + t\vec{v}$

derived from param.

Sym. form $t = \dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}$

Ex 3.(1) $P = (1, 2, 3)$
$Q = (-4, 7, 0)$

$\overrightarrow{PQ} = \langle -5, 5, -3 \rangle$

$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + t \langle -5, 5, -3 \rangle$

or $\begin{cases} x = 1 - 5t \\ y = 2 + 5t \\ z = 3 - 3t \end{cases}$ $\qquad \dfrac{x-1}{-5} = \dfrac{y-2}{5} = \dfrac{z-3}{-3}$

(2) z-axis

$P = (0, 0, 0)$
$Q = (0, 0, 1)$

$\overrightarrow{PQ} = \langle 0, 0, 1 \rangle$

$\vec{v} = \langle 0, 0, 0 \rangle + t \langle 0, 0, 1 \rangle$

(3) $\begin{cases} x + y + z = 0 \\ 2x - 3y + 7z = 9 \end{cases}$

$5y - 5z = -9$

$\begin{cases} x = -2z + \frac{9}{5} \\ y = z - \frac{9}{5} \\ z = z \end{cases}$

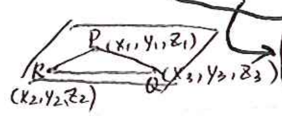
or find one pt
$\overrightarrow{\text{line}} = \langle 1, 1, 1 \rangle \times \langle 2, -3, 7 \rangle$

Plane in 3-D

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

Vector: $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

general: $ax + by + cz = d$
$\langle a, b, c \rangle$ = normal vector



$\left.\begin{matrix} \overrightarrow{PR} \\ \overrightarrow{PQ} \end{matrix}\right\} \to \overrightarrow{PQ} \times \overrightarrow{PR}$ = normal vector

Ex 2. (1) $P(1, 2, 3)$
$Q(2, -4, 9)$
$R(0, 5, -4)$

$\overrightarrow{PQ} = \langle 1, -6, 6 \rangle$
$\overrightarrow{PR} = \langle -1, 3, -7 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -6 & 6 \\ -1 & 3 & -7 \end{vmatrix} = \hat{i}(42 - 18) - \hat{j}(-7 + 6) + \hat{k}(3 - 6)$$
$$= 24\hat{i} + \hat{j} - 3\hat{k}$$
$$= \langle 24, 1, -3 \rangle$$

$\langle 24, 1, -3 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$
$24(x - 1) + (y - 2) + (z - 3)(-3) = 0$
$24x - 24 + y - 2 - 3z + 9 = 0$
$24x + y - 3z = 17$

Ex 2 (2) angle betw plane $x - y + z = 4$
and $3x + 4y - 5z = 0$

$\vec{n_1} = \langle 1, -1, 1 \rangle$
$\vec{n_2} = \langle 3, 4, -5 \rangle$

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

(3) Plane // to plane $2x + 4y + 7z = 18$ & going thru $(1, 2, 3)$

$\langle 2, 4, 7 \rangle \langle x - 1, y - 2, z - 3 \rangle = 0$

$\begin{cases} ax + by + cz = d_1 \\ ax + by + cz = d_2 \end{cases}$

Summary

Equation of plane
line

line and plane relationship

2-D Lines: parallel or intersecting

$y_1 = m_1x + b_1$ $\quad m_1 = m_2$ $\qquad m_1 \neq m_2$

$y_2 = m_2x + b_1$ $\qquad\qquad m_1 m_2 = -1\ (\perp)$

3-D planes: parallel or intersect

$P_1 = a_1x + b_1y + c_1z + d_1 = 0$ $\qquad \langle a_1, b_1, c_1\rangle \neq k\langle a_2, b_2, c_2\rangle$

$P_2 = a_2x + b_2y + c_2z + d_2 = 0$ $\qquad \Rightarrow P_1 \cap P_2 \neq \phi$

$\langle a_1, b_1, c_1\rangle \,//\, \langle a_2, b_2, c_2\rangle$

$\Rightarrow \langle a_1; b_1, c_1\rangle = k\langle a_2, b_2, c_2\rangle$

$\Rightarrow P_1 // P_2$

Ex4 (1) $\begin{cases} x+2y+3z=10 \\ x+2y+3z=0 \end{cases}$ no solution $\Rightarrow P_1 // P_2$

3-D Lines $\quad l_1 = \langle x, y, z\rangle = \langle x_1, y_1, z_1\rangle + \langle a_1, b_1, c_1\rangle t$

$l_2 = \langle x, y, z\rangle = \langle x_2, y_2, z_2\rangle + \langle a_2, b_2, c_2\rangle t$

① parallel: $\langle a_1, b_1, c_1\rangle = k\langle a_2, b_2, c_2\rangle$

Ex4 (2) $l_1 \begin{cases} x = 2t+3 \\ y = -3t-4 \\ z = 5t+7 \end{cases}$ $\quad l_2 \begin{cases} x = 4s-9 \\ y = 8s-3 \\ z = -3s+5 \end{cases}$

Suppose $(x, y, z) \in l_1 \cap l_2$

$\begin{cases} 2t+3 = 4s-9 & ① \\ -3t-4 = 8s-3 & ② \\ 5t+7 = -3s+5 & ③ \end{cases}$ $\quad t = -\frac{17}{7},\ s = \frac{25}{14}$ from ①, ②

③: $-\frac{85}{7} + 7 = -\frac{75}{14} + 5$

$-\frac{36}{7} = \frac{-5}{14}$

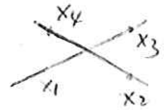
$\Rightarrow$ no s, t satisfy all 3 eqs.

② $\Rightarrow$ skew.

(3) $L_1 \begin{cases} x = 2t+3 \\ y = -3t-4 \\ z = 5t+7 \end{cases}$ $\quad l_2 \begin{cases} x = 4s+3 \\ y = -6s-4 \\ z = 10s+7 \end{cases}$

$\langle 2, -3, 5\rangle \qquad \langle 4, -6, 10\rangle \Rightarrow$ parallel.

③ intersect.



$\vec{x} = \vec{x_1} + (\vec{x_3} - \vec{x_1})t$

$\vec{x} = \vec{x_2} + (\vec{x_4} - \vec{x_2})t$

Distance

$P_1(x_1, y_1, z_1)$

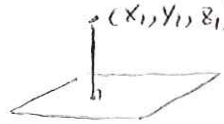
$P_2(x_2, y_2, z_2)$

2D.

$|P_1P_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

from point P

(x_1, y_1, z_1)



$p = ax + by + cz + d = 0$

① find line $L \perp P$ thru (x_1, y_1, z_1)

$\langle a, b, c\rangle$

$L = \langle x, y, z\rangle = \langle x_1, y_1, z_1\rangle + \langle a, b, c\rangle t$

② find $(x_2, y_2, z_2) \in L \cap P$

$a(x_1 + at) + b(y_1 + bt) + c(z_1 + ct) + d = 0$

$t_* = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

$(x_2, y_2, z_2) = (x_1, y_1, z_1) + t_*(a, b, c)$

$(x_2 - x_1, y_2 - y_1, z_2 - z_1) = t_*(a, b, c)$

③ $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Ex 5 (1) $5x-7y+z=22$ and $(1,2,3)$

$$d=\frac{|5-14+3-22|}{\sqrt{5^2+7^2+1^2}}$$

Distance between parallel planes

find any pt on plane 1 $\xrightarrow{\text{distance}}$ plane 2.

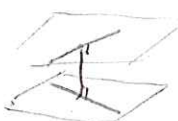
Ex 5 (2) $P_1: 5x-7y+z=22$

$P_2: 5x-7y+z=1$

$(0,0,1)$ on P_2.

$$d=\frac{|1-22|}{\sqrt{5^2+7^2+1^2}}$$

Distance between lines and lines (skew)



① finding planes

or ② optimization

$(x_1,y_1,z_1)=(2t+3,-3t+4,5t+7)$

$(x_2,y_2,z_2)=(4s-9,8s-3,-3s+5)$

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

find $(s_1,t_1)\in R\times R$ s.t.

$d(s_1,t_2)=\min d(s,t)$

Cylinder

parabolic cylinder $z=x^2$

$x^2+y^2=1$

Chapter 13.

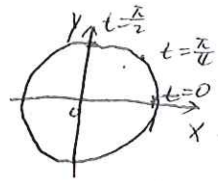
13.1-13.2 Vector Valued function

$x = f(t)$ $\quad$ $x = f(t)$ $\quad$ $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

[3D] $y = g(t)$ $\quad$ $y = g(t)$ $\quad$ $x^2+y^2=1.$

$z = h(t)$ $\quad$ [2D]

(parametric equation)



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

Limit $\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$

Derv. $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Intg. $\int_a^b \vec{r}(t)dt = \langle$

Special Space Surves

Lines: $\vec{r}(t) = \langle x_0+at, y_0+bt, z_0+ct \rangle$

Helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Deriv. and physics

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $\qquad$ $\vec{r}(t_0)$: tangent vector at $t=t_0$, velocity at $t=t_0$

tangent line at $t=t_0$ $\quad$ $\vec{u}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$

Unit tangent vector $\quad$ $T(t_0) = \dfrac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$

Examples.

1. $\vec{r}(t) = (\ln(4-t^2), \sqrt{t+1}, \cos 2t)$

domain of func.?

$\begin{cases} 4-t^2 > 0 \\ t+1 \geq 0 \end{cases} \Rightarrow \begin{cases} -2<t<2 \\ t \geq -1 \end{cases}$

$-1 \leq t < 2$

2. $P(-2,4,0)$ $\quad$ equation of line segment

$Q(6,-1,2)$

Vector:

$\vec{PQ} = (8,-5,2)$. $\quad \vec{r}(t) = \langle -2,4,0 \rangle + t\langle 8,-5,2 \rangle$

param. $\begin{cases} x = -2+8t \\ y = 4-5t \\ z = 0+2t \end{cases}$ $\qquad t \in [0,1]$

3. Intersection

$\begin{cases} z = 4x^2+y^2 \\ y = x^2 \end{cases} \Rightarrow \begin{cases} z = 4x^2+x^4 \\ x = x \\ y = x^2 \end{cases}$ or change $x \to t$.

4. $\vec{r}(t) = \langle \ln t, 2\sqrt{t+1}, t^2 \rangle$ at $P_0(0, 2\sqrt{2}, 1)$

$\vec{r}'(t) = \langle \frac{1}{t}, \frac{1}{\sqrt{t+1}}, 2t \rangle$. $\qquad \ln t = 0$

$\vec{r}'(t_0) = \langle 1, \frac{\sqrt{2}}{2}, 2 \rangle$ $\qquad t_0 = 1$

$\vec{u}(t) = \langle 0, 2\sqrt{2}, 1 \rangle + t\langle 1, \frac{\sqrt{2}}{2}, 2 \rangle$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'g-g'f}{g^2}$

5. $\vec{r}(t) = \langle \ln(4-t^2), \sqrt{t+1}, \cos 2t \rangle$

$\vec{v}(t) = \vec{r}'(t) = \langle \frac{1}{4-t^2}(-2t), \frac{1}{2}\cdot\frac{1}{\sqrt{t+1}}, -\sin 2t \cdot 2 \rangle$

$= \langle \frac{2t}{t^2-4}, \frac{1}{2\sqrt{t+1}}, -2\sin 2t \rangle$

$\vec{a}(t) = \vec{r}''(t) = \langle \frac{(2t^2-8)-4t^2}{(t^2-4)^2}, \frac{-1}{4}(t+1)^{-\frac{3}{2}}, -2\cos 2t \cdot 2 \rangle$

$= \langle \frac{-8-2t^2}{(t^2-4)^2}, -\frac{1}{4(t+1)^{\frac{3}{2}}}, -4\cos 2t \rangle$

13.3 Arc Length

2D

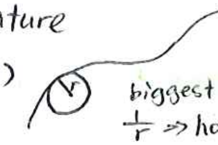
$y = f(x)$ or $(x(t), y(t))$ $a \le t \le b$

$a \le x \le b$

$\int_a^b \sqrt{1+f'(x)^2}\,dx$ $\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

Curvature

(not included)



biggest ball

$\frac{1}{r} \Rightarrow$ how curve the function is

3D $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $a \le t \le b$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b |\vec{r}'(t)|\,dt$$

Example

a. $\vec{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$ $-10 \le t \le 10$

$\vec{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle$ $L = \int_{-10}^{10} \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2}\,dt = \sqrt{29} \cdot 20$

b. Intersect. $\begin{cases} \text{cylinder } 4x^2 + y^2 = 4 \\ \text{plane } x + y + z = 2 \end{cases} \longrightarrow \begin{cases} x = \cos\theta \\ y = 2\sin\theta \\ z = 2 - x - y = 2 - \cos\theta - 2\sin\theta \end{cases}$ $0 \le \theta \le 2\pi$

$\begin{cases} x' = -\sin\theta \\ y' = 2\cos\theta \\ z' = \sin\theta - 2\cos\theta \end{cases}$

$$L = \int_0^{2\pi} \sqrt{(-\sin\theta)^2 + 4\cos^2\theta + (\sin\theta - 2\cos\theta)^2}\,d\theta$$

13.4 physics

$\vec{r}(t)$ = location

$\vec{v}(t) = \vec{r}'(t)$ velo.

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$

$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(s)\,ds$ (Fundamental Theorem of Calc). $\int_a^b f'(x)\,dx = f(b) - f(a)$)

$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(s)\,ds$

Ex. (6) $\vec{a}(t) = \langle t, t^2, \cos 2t \rangle$

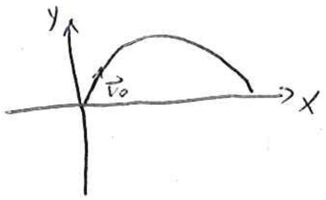
$\vec{v}(t) = \vec{v}(0) + \int_0^t \langle s, s^2, \cos 2s \rangle\,ds$

$= \langle 1, 0, 1 \rangle + \langle \frac{1}{2}s^2\Big|_0^t, \frac{1}{3}s^3\Big|_0^t, \frac{1}{2}\sin 2s\Big|_0^t \rangle$

$= \langle 1, 0, 1 \rangle + \langle \frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{2}\sin 2t \rangle$

$= \langle 1 + \frac{1}{2}t^2, \frac{1}{3}t^3, 1 + \frac{1}{2}\sin 2t \rangle$

Projectile



$$\vec{a}(t) = \langle 0, -g \rangle$$
$$\vec{v}(t) = \vec{v}(0) + \int_0^t \langle 0, -g \rangle \, ds$$
$$= \vec{v}(0) + \langle 0, -gt \rangle = \vec{v}(0) + \vec{a}t$$
$$\vec{r}(t) = \vec{r}(0) + \int_0^t (\vec{v}(0) + \vec{a}s)\, ds$$
$$= \vec{r}(0) + \vec{v}(0)t + \tfrac{1}{2}\vec{a}t^2 \qquad \vec{r}(0) = \langle x_0, y_0 \rangle$$
$$= \langle x_0, y_0 \rangle + \langle |\vec{v_0}|\cos\theta, |\vec{v_0}|\sin\theta \rangle t + \tfrac{1}{2}\langle 0, -g \rangle t^2$$

$$\begin{cases} x(t) = x_0 + |\vec{v_0}|\cos\theta \cdot t \\ y(t) = y_0 + |\vec{v_0}|\sin\theta \cdot t - \frac{1}{2}gt^2 \end{cases}$$

$$\begin{cases} V_x(t) = |\vec{v_0}|\cos\theta \\ V_y(t) = |\vec{v_0}|\sin\theta - gt \end{cases}$$

highest point $t = t_1$, $\quad V_y(t_1) = |\vec{v_0}|\sin\theta_1 - gt_1 = 0$, $\quad t_1 = \dfrac{|\vec{v_0}|\sin\theta}{g}$

$$y(t_1) = y_0 + |\vec{v_0}|\sin\theta \cdot \frac{|\vec{v_0}|\sin\theta}{g} - \frac{1}{2}g\frac{|\vec{v_0}|^2\sin^2\theta}{g^2}$$
$$= y_0 + \frac{|\vec{v_0}|^2\sin^2\theta}{2g} \quad \text{Highest distance}$$

farest point $t = t_2$. $\quad y(t_2) = 0 \quad (t_2 = 2t_1)$

$$x(t_2) = x_0 + |\vec{v_0}|\cos\theta \cdot \frac{2|\vec{v_0}|\sin\theta}{g}$$
$$= x_0 + \frac{2|\vec{v_0}|^2\sin\theta\cos\theta}{g} \quad \text{Furthest distance}$$
$$= x_0 + \frac{|\vec{v_0}|^2\sin 2\theta}{g}$$

$\theta = 45^\circ \to$ gives you furthest distance

Example. $|V_0| = 200$, $\theta = 60^\circ$

range $= \dfrac{|\vec{v_0}|^2}{g}\sin(2\theta)$ $\qquad$ Height $= \dfrac{|\vec{v_0}|^2\sin^2\theta}{2g}$.

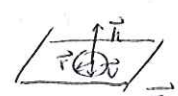
Speed at impact $= \langle 100, -100\sqrt{3} \rangle$

Kapler's Law, Prove.

$\vec{r}(t) =$ $\qquad \vec{r}'' = -\dfrac{GM}{r^3}\vec{r}$

$\vec{F}(t) = m\vec{a}$ $\qquad \vec{a} \parallel \vec{r} \quad \vec{r} \times \vec{a} = 0$ $\qquad \vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{F} = -\dfrac{GMm}{|r|^3}\vec{r}$ $\qquad \dfrac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r}' \times \vec{v} + \vec{v}' \times \vec{r}$



$\vec{r} \times \vec{v} = \vec{h}$

$|\vec{F}| = \dfrac{GMm}{r^2}$ $\qquad = \vec{v} \times \vec{v} + \vec{a} \times \vec{r} = 0$

$\vec{a} \times (\vec{r} \times \vec{v}) = \vec{a} \times \vec{h}$

$= -\dfrac{GM}{|\vec{r}|^3}\vec{r} \times (\vec{r} \times \vec{v}')$

$\vec{r} \times \vec{v} = \vec{h}$

$\vec{r} \times \vec{h} = 0 \qquad \vec{r}, \vec{v}$ both $\perp \vec{h}$ $\qquad \vec{r} = |\vec{r}| \cdot \dfrac{\vec{r}}{|\vec{r}|} = R \cdot \vec{u}$

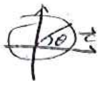
$\vec{r}' = R'\vec{u} + R\vec{u}'$

$$\vec{a} \times \vec{h} = -\frac{GM}{R^3} \cdot (R\vec{u}) \times (R\vec{u} \times (R'\vec{u} + R\vec{u}')) = -GM\,\vec{u} \times (\vec{u} \times \vec{u}')$$
$$= -GM((\vec{u} \cdot \vec{u}') \times \vec{u} - (\vec{u} \cdot \vec{u})\vec{u}') = GM\vec{u}'$$
$$\vec{r} \cdot (\vec{v} \times \vec{h}) = (GM\vec{u} + \vec{c}) \cdot \vec{r}$$

$(GM\vec{u} + \vec{c})R\vec{u} = h^2$ $\qquad R = \dfrac{h^2}{GM + c\cos\theta}$

$R(GM + |\vec{c}|\cos\theta) = h^2$

$R(t) = \dfrac{h^2}{GM + |\vec{c}|\cos\theta(t)}$



Chapter 14 - partial Derivatives

14.1

vector-valued function $\vec{r}(t) = \langle x(t), y(t) \rangle$ from ch. 13 — domain $t \rightarrow$ range (x, y)

function of two variables $z = f(x, y)$

domain $(x, y) \longrightarrow$ range z

e.g. $V = \pi r^2 h$

$V(r, h) = \pi r^2 h$

$P(L, K) = bL^{\alpha}K^{1-\alpha}$ $\alpha \in (0, 1)$

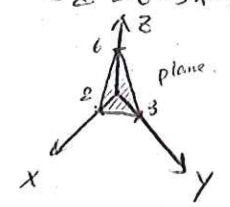
prod. labor capital

4 ways of expression $\{(x, y) \in R^2 : \text{restriction}\}$

- Word description
- formula (math)
- spread sheet w/ #s
- graph

eg. graph: $z = -3x - 2y + 6$

$(z = 6 - 3x - 2y)$



ex. - graph $g(x, y) = \sqrt{9 - x^2 - y^2}$ domain $9 - x^2 - y^2 \geq 0$

$z^2 = 9 - x^2 - y^2$ $\{(x, y) \in R^2 : x^2 + y^2 \leq 9\}$

$x^2 + y^2 + z^2 = 9$ $(z \geq 0)$

half of sphere

level curves

$z = f(x, y)$

$L_c = \{(x, y) \in R^2 : f(x, y) = c\}$

$z = 3x^2 + 4y^2$

$L_{12} = \{(x, y) \in R^2 : 3x^2 + 4y^2 = 12\}$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Contour Map : collections of bunch of level curves

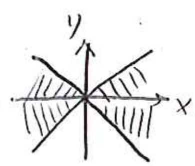
e.g. geographic map

Ex. 1. $f(x, y) = \sqrt{x^2 - y^2}$

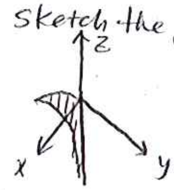
(a) $f(2, 1) = \sqrt{3}$

(b) domain = $\{(x, y) \in R^2 : x^2 \geq y^2\}$

range $[0, \infty)$



(c) sketch the graph



14.2

Continuous (review)

$f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$ $f(x) = x^n, e^x, \ln x, \sin x, \cos x$

not continuous

$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

lim does not exist

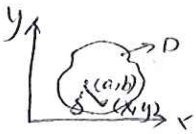
$x = 0$ undef.

$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$

if (a, b) is in the domain

Combination of cont. func is cont. $\pm, \cdot, \div, f \circ g = f(g(x))$

$z = f(x,y)$ domain $D = \{(x,y) \in R^2 : \cdots\}$



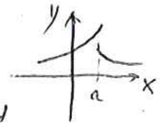
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$\lim_{x\to a} f(x) = L$

(1) $\lim_{x\to a^-} f(x)$ exists ($\pm\infty$ not included)

(2) $\lim_{x\to a^+} f(x)$ exists ($\neq \pm\infty$)

(3) $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$



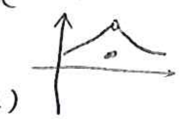
$\lim_{x\to a} f(x)$ DNE

$f(x)$ is continuous at $x=a$

(1) $\lim_{x\to a} f(x)$ exisits

(2) $\lim_{x\to a} f(x) = f(a)$

($f(a)$ must exists)



$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = L}$ (1) For any path approaching to (a,b) the limit of $f(x,y)$ along that path exists

(2) Any limit along the path is the same

$f(x,y)$ is cont. at $(x,y)=(a,b)$

(1) $\lim_{(x,y)\to(a,b)} f(x,y)$ exist

(2) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Example 3. (a) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

Domain $\{(x,y)\in R^2 ; (x,y) \neq (0,0)\}$

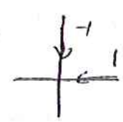
$f(x,y)$ cont. for any (x,y) as long as $x\neq 0$ or $y \neq 0$

$\lim_{(x,y)\to(0,0)} f(x,y)$

path $(x,0)$ $\dfrac{x^2-0^2}{x^2+0^2} = 1$

path $(0,y)$ $\dfrac{-y^2}{y^2} = -1$ $\Rightarrow$ limit DNE

path $y=kx$ $\dfrac{x^2-k^2x^2}{x^2+k^2x^2} = \dfrac{1-k^2}{1+k^2}$

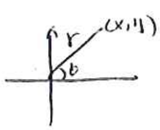


(b) $f(x,y) = \dfrac{xy^3}{x^2+y^2}$

cont., not $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} = 0$$

. $y=kx$ $\dfrac{x(k^3x^3)}{x^2+k^2x^2} = \dfrac{k^3\cancel{x^4}x^2}{(1+k^2)\cancel{x^2}} \xrightarrow{x\to 0} 0$



. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $= \lim_{r\to 0} \dfrac{r\cos\theta\, r^3\sin^3\theta}{r^2}$

$r = \sqrt{x^2+y^2}$ $= \lim_{r\to 0} r^2\cos\theta\sin^3\theta$

$(x,y)\to(0,0)$ $= 0$

$r \to 0$

$-r^2 \le r^2\cos\theta\sin^3\theta \le r^2$

$\downarrow$ 0 $\quad\quad \downarrow$ 0

$h(x) \le f(x) \le g(x)$ $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$

$\Rightarrow \lim_{x\to a} f(x) = L$

(c) $f(x,y) = \frac{xy^3}{x^2+y^6}$ $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ DNE

path $(x,0)$ $\lim_{(x,y)\to(0,0)} = 0$

$(0,y)$ $= 0$

path $y=kx$ $\lim_{(x,y)\to(0,0)} f(x) = \lim_{(x,y)\to(0,0)} \frac{x k^3 x^3}{x^2+k^6x^6}$

$= \lim_{(x,y)\to(0,0)} \frac{k^3x^2}{1+k^6x^4}$

$= 0$

path. $x=y^3$ $f(x) = \frac{y^6}{2y^6} = \frac{1}{2}$ DNE

$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$ not defined at $(0,0)$

$\lim_{(x,y)\to(0,0)} x^3y = 0$

$\frac{0}{0}$ type

$\lim_{(x,y)\to(0,0)} x^4+y^4 = 0$

L'Hopital's Rule in 1D

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

but no such rule in 2D

- limit exist = ① prove it w/ ε-δ
  ② using squeezing theorem + polar coordinate

- limit DNE = ① if path $(x, y_1(x))$ limit on that path $= L_1$
  $(x, y_2(x))$ $= L_2$ $L_1 \neq L_2$
  ② limit along any path DNE
  ($\frac{0}{0}$ form)

Prob. $\lim_{(x,y)\to(0,0)} \frac{3x-2y}{x^2+y^2}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

polar coord.

$= \frac{3r\cos\theta - 2r\sin\theta}{r^2}$

$= \frac{3\cos\theta - 2\sin\theta}{r}$

path $y=kx$ $\frac{3x-2kx}{x^2+k^2x^2} = \frac{3-2k}{x+k^2x}$ DNE

$\lim_{r\to0} 3\cos\theta - 2\sin\theta$ DNE

$\lim_{r\to0} r = 0$ $\frac{\text{DNE}}{0} = \text{DNE}$

$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$ power 4 / power 4 $= \frac{r^3\cos^3\theta\, r\sin\theta}{r^4(\cos^4\theta+\sin^4\theta)} = \frac{\cos^3\theta\sin\theta}{\cos^4\theta+\sin^4\theta}$ DNE

polar coord.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$y=kx$ limit $= \frac{k}{1+k^4}$

$$\lim_{(x,y)\to(0,0)} \frac{3x^3-2y^3}{x^2+y^2} = \frac{3r^3\cos^3\theta - 2r^3\sin^3\theta}{r^2} = 3r\cos^3\theta - 2r\sin^3\theta$$

$$\lim_{r\to 0}(3r\cos^3\theta - 2r\sin^3\theta)$$

$$-5r \le \lim_{r\to 0}(r(3\cos^3\theta - 2\sin^3\theta)) \le r(5)$$

$$\lim_{r\to 0} -5r = \lim_{r\to 0} 5r = 0 \Rightarrow \lim_{r\to 0} r(3\cos^3\theta - 2\sin^3\theta) = 0$$ squeeze Theorem

14.3 Partial Derivative

Derivative in 1D

$y=f(x)$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists

$= f'(x)$ derivative of $f(x)$

$\frac{df}{dx}$

In 2D

$z=f(x,y)$

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(x,y) = f_x(x,y) = D_x f(x,y)$$

partial derivative of $f(x,y)$ with respect to x

Fix $y=y_0$ $z=f(x,y_0)$ slope

prob

1. $f(x,y) = x^5+3x^3y^2+3xy^4$ (treat y as constant)

(1) find f_x, f_y

$f_x = 5x^4+9y^2x^2+3y^4$

$f_y = 0+(3x^3)(2y)+(3x)(4y^3)$

$= 6x^3y+12xy^3$

(2) $f_x(1,2) = 5+36+48 = 89$

$x=1 \to 1+h$

$f(1+h,2) \approx f(1,2)+89h$ (meaning)

2. $f(x,y) = x^y$

$f_x = (y)x^{y-1}$

$f_y = x^y \ln x$

$(a^x)' = a^x \ln a$

$(x^k)' = kx^{k-1}$

Second derivative.

$y=f(x)$

$f'(x)$

$f''(x) = \frac{df'(x)}{dx}$

$f_{xx} = (y)(y-1)x^{y-2}$

$f_{yx} = (x^y\ln x)_x$

$= (yx^{y-1})(\ln x) + (\frac{1}{x})x^y$

$= y\ln x \cdot x^{y-1} + x^{y-1}$

$f_{yy} = (x^y \ln x)_y$

$= (x^y\ln x)\ln x + 0 = x^y \ln^2 x$

$f_{xy} = (yx^{y-1})_y = (x^{y-1})_y + (x^{y-1})y_y$

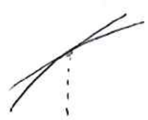
$= x^{x-1} + yx^{y-1}\ln x(y-1)_y$

$= x^{y-1} + y\ln x \cdot x^{y-1}$

$\boxed{f_{yx} = f_{xy}}$ if func. continuous

Clairaut's Theorem

$$\begin{cases} z = f(x,y) \\ y = y_0 \end{cases}$$

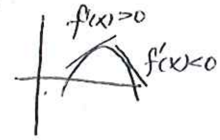


$\frac{\partial f}{\partial x}(x_0, y_0)$

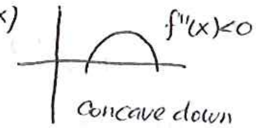
$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$

$y = f(x)$

$f'(x)$



$f''(x)$



Concave down

$f''(x) > 0$

Concave up (convex)

Implicit Differentation

① $x^2+y^2+z^2=9$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ $\quad z = z(x,y)$

$$x^2+y^2+(z(x,y))^2=9$$

$$\frac{\partial}{\partial x}\left(x^2+y^2+(z(x,y))^2\right)=0$$

$$2x+0+2z(x,y)\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial}{\partial y}(x^2+y^2+z^2) = \frac{\partial}{\partial y}(9)$$

$$0+2y+2z\frac{dz}{dy}=0$$

$$\frac{dz}{dy} = -\frac{y}{z}$$

$z = f(x,y)$ $\quad F(x,y,z)=0$

$$\underline{\frac{\partial}{\partial x}\left(F(x,y,z(x,y))=0\right)}$$

② $z = x^y$

("ln" technique)

$$\ln z = \ln(x^y)$$

$$\ln z = y\ln x$$

$$\frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(y\ln x)$$

$$\frac{1}{z}\cdot\frac{dz}{dx} = y\cdot\frac{1}{x}$$

$$\frac{dz}{dx} = \frac{yz}{x} = \frac{yx^y}{x}$$

$$= yx^{y-1}$$

③ $x - z = \tan^{-1}(yz)$

$$\frac{\partial}{\partial x}(x-z) = \frac{\partial}{\partial x}(\tan^{-1}(yz))$$

$$1-\frac{\partial z}{\partial x} = \frac{1}{1+(yz)^2}\frac{\partial}{\partial x}(yz)$$

$$1-\frac{\partial z}{\partial x} = \frac{1}{1+y^2z^2}(y)\frac{\partial z}{\partial x}$$

$$1 = \frac{y+1+y^2z^2}{1+y^2z^2}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{1+y^2z^2}{1+y^2z^2+y}$$

$$\frac{\partial}{\partial y}: \quad 0-\frac{\partial z}{\partial y} = \frac{1}{1+y^2z^2}\frac{\partial}{\partial y}(yz)$$

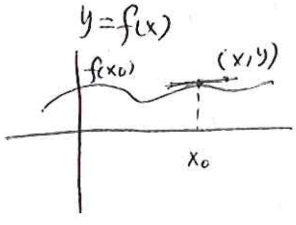
$$= \frac{1}{1+y^2z^2}\left(\frac{\partial z}{\partial y}y+z\right)$$

$$0 = \frac{1+y^2z^2+y}{1+y^2z^2}\frac{\partial z}{\partial y}+\frac{z}{1+y^2z^2}$$

$$\frac{\partial z}{\partial y} = -\left(\frac{z}{\cancel{1+y^2z^2}}\right)\left(\frac{\cancel{1+y^2z^2}}{1+y^2z^2+y}\right) = -\frac{z}{1+y+y^2z^2}$$

14.4 Linear Approximation

$y = f(x)$



$$\frac{y-f(x_0)}{x-x_0} = f'(x_0)$$

$$y-f(x_0) = f'(x_0)(x-x_0)$$

tangent vector $\langle 1, f'(x_0)\rangle$

$z = f(x,y)$ $\quad (x_0, y_0)$

$(x_0, y_0, f(x_0,y_0))$

$$\begin{cases} z = f(x,y) \\ y = y_0 \end{cases} \quad z = f(x, y_0)$$

$$T_1 = \left\langle 1, 0, \frac{\partial f}{\partial x}(x_0,y_0)\right\rangle$$

$$\begin{cases} z = f(x,y) \\ x = x_0 \end{cases} \rightarrow z = f(x_0, y)$$

$$T_2 = \left\langle 0, 1, \frac{\partial f}{\partial y}(x_0,y_0)\right\rangle$$

Tangent plane

$$N = \vec{T_1} \times \vec{T_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

$$\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle -f_x, -f_y, 1 \rangle = 0$$

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

linearization of $z=f(x,y)$ at (x_0, y_0)

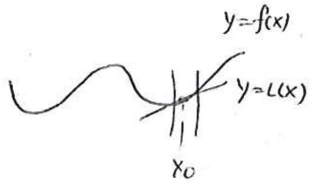
Differential

$$z = f(x_0,y_0) + \underbrace{\boxed{f_x(x_0,y_0)(x-x_0)}}_{x\text{-differential}} + \underbrace{\boxed{f_y(x_0,y_0)(y-y_0)}}_{y\text{-differential}}$$

Δx

$\Delta x \to 0$ in func.

total diff.



(x,y) close to (x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+ \tfrac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$
$$+ \tfrac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$$

$(\tfrac{1}{2}f_{xy} + \tfrac{1}{2}f_{yx})$

Qudratic approx.

1D Tyler Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 + \dots + \tfrac{1}{n!} f^{(n)}(x_0)(x-x_0)^n$$

eg sell tickets

$R(p,n) = pn$

R = revenue

P = price

n = # sold

if increase price by \$2, and # of ticket ↓ by 10.

total differential $dR = \frac{\partial R}{\partial p} \cdot \Delta p + \frac{\partial R}{\partial n} \Delta n$

$$= \frac{\partial R}{\partial p}(2) + \frac{\partial R}{\partial n}(-10)$$

$$= 2n - 10p$$

$$dR = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$$

$\parallel$ differential

$$\Delta R = f(x_0+dx, y_0+dy) - f(x_0,y_0)$$

$\parallel$ difference

$\Delta R \underset{\text{by}}{\overset{\text{approx'd}}{\approx}} dR$

prob eg 3

$z = y \ln x$ tangent plane at $(1,4,0)$

$$z - z_0 = f_x(x-x_0) + f_y(y-y_0)$$

$z_x = y \cdot \frac{1}{x}$ $z_x(1,4) = 4$

$z_y = \ln x$ $z_y(1,4) = \ln 1 = 0$

$f(x,y) = \sqrt{x^2+y^2}$ $(x_0,y_0) = (3,4)$

$$L(x,y) = \underbrace{f(x_0,y_0)}_{\sqrt{3^2+4^2}} + f_x(x-x_0) + f_y(y-y_0)$$

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \tfrac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$f_y(3,4) = \frac{4}{5}$

$f_x(3,4) = \frac{3}{5}$

$$\Rightarrow L(x,y) = 5 + \tfrac{3}{5}(x-3) + \tfrac{4}{5}(y-4)$$

$$\sqrt{(3.01)^2+(3.98)^2} = f(3.01, 3.98) \approx L(3.01, 3.98)$$

$$= 5 + \tfrac{3}{5}(0.01) + \tfrac{4}{5}(-0.02)$$

$$= 5 + 0.006 + (-0.016)$$

$$= 4.99$$

$z = f(x,y)$ $\qquad$ $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$ difference

$\frac{\partial f}{\partial x}(x,y)$ rate of change in x direction $\qquad$ $dz = \frac{\partial f}{\partial x}(x,y)\Delta x + \frac{\partial f}{\partial y}(x,y)\Delta y$ differential

$\Delta z \approx dz \Rightarrow f(x,y)$ is differentiable

1D $\quad$ $y = f(x)$ $\quad$ $f'(x)$

$f(x)$ is continuous $\qquad$ $f(x)$ is differentiable

$\lim_{x\to a}(f(x)-f(a)) = 0$ $\qquad$ $\lim_{x\to a}\left|\frac{f(x)-f(a)-f'(x)(x-a)}{x-a}\right| = 0 = \lim_{x\to a}\frac{|\Delta y - dy|}{|\Delta x|} = 0$

Theorem: if $f(x)$ continuous, $f'(x)$ exists

$f'(x)$ continuous, then $f(x)$ is differentiable

(smooth) 

 not smooth

2D

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\left(f(x,y)-f(x_0,y_0)\right) - \left(\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

If $f(x,y)$ continuous, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and continuous

$\Rightarrow f(x,y)$ is differentiable at (x_0,y_0)

Eg. 3(3) $z = x^2+y^2$

$\Delta x = dx$ $\quad$ $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$\Delta y = dy$

$\Delta z \approx dz$ $\quad$ $\frac{\partial z}{\partial x} = 2x$ $\quad$ $\frac{\partial z}{\partial y} = 2y$

if f is differentiable

$dz = 2x\,dx + 2y\,dy$

$(x,y) = (2,5)$

$dz = 4dx + 10dy$

$dz = (4)(0.1) + 10(0.1)$

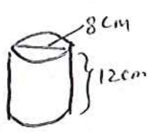
$= 1.4$

$\Delta z = z(2.1, 5.1) - z(2,5)$

$= (2.1)^2 + (5.1)^2 - 2^2 - 5^2$

$= 1.42$

$dz \approx \Delta z$

3(4) 

thick = 0.04 cm

What is ΔV or dV if thick = 0.04 cm

$V = \pi r^2 h$

$V(r,h) = \pi r^2 h$

$dr = 0.04$

$dh = 0.08$

$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$

$= (2\pi rh)dr + (\pi r^2)dh$

$= (2\pi \cdot 4 \cdot 12)(0.04) + (\pi \cdot 4^2)(0.08)$

$= 3.84\pi + 1.28\pi = 5.12\pi$ cm^3

$\Delta V = V(4+0.04, 12+0.08) - V(4,12)$

V2: Error b/c measurement $|dr| \le 0.04$, $|dh| \le 0.08$

percentage $\frac{|dr|}{r} \le \frac{0.04}{4} = 1\%$

$\frac{|dh|}{h} \le \frac{0.08}{12} \approx 0.67\%$

What's percentage error for V

$\frac{|dV|}{V} \le \frac{5.12\pi}{\pi \cdot 4^2 \cdot 12} = \frac{5.12}{192}\%$

14.5 Chain Rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right) \cdot \left(\underbrace{\frac{g(x+h) - g(x)}{h}}_{\to g'(x)}\right)$$

$$= \lim_{M\to 0} \frac{f(g(x)+M) - f(g(x))}{M}$$

$M = g(x+h) - g(x)$

$$\downarrow$$

$$f'(g(x))$$

$$F(t) = f(x(t), y(t))$$

$$F'(t) = ?$$

$$\lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}$$

$$= \frac{\left[\lim_{h\to 0} f(x(t+h), y(t+h)) - f(x(t+h), y(t))\right]^{①} + ②\ f(x(t+h), y(t)) - f(x(t), y(t))}{h}$$

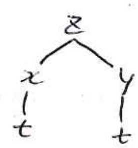
$$① = \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t+h), y(t))}{y(t+h) - y(t)} \cdot \frac{y(t+h) - y(t)}{h}$$

$$= \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$② = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t)$$

$$\frac{\partial f(x(t), y(t))}{\partial t} = f_t = f_y \cdot y_t + f_x \cdot x_t$$

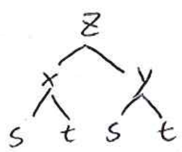
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$



$z = f(x, y)$

$x = g(s, t)$

$y = h(s, t)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\begin{pmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial x}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix}$$

Problems

1. $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{2t}$, $\frac{\partial z}{\partial t} = ?$, $\frac{\partial z}{\partial t}(1)$

w/o chain rule $\quad z(t) = \sqrt{x(t)^2 + y(t)^2} = \sqrt{2e^{4t}}$

$x(1) = e^2 = y(1)$ $\quad \frac{\partial z}{\partial t}(1)$

$$= \frac{2e(2e^2)}{\sqrt{e^4+e^4}} = 2\sqrt{2}e^2$$

w/ chain rule $\quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

$$\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dx}{dt} = 2e^{2t} = \frac{dy}{dt}$$

$$\frac{\partial z}{\partial t} = \frac{x}{\sqrt{x^2+y^2}} \cdot 2e^{2t} + \frac{y}{\sqrt{x^2+y^2}} \cdot 2e^{2t}$$

$$= \frac{2e^{2t}(x+y)}{\sqrt{x^2+y^2}}$$

2. $z = e^{xy}\tan y,\ x = s+2t,\ y = \frac{s}{t},$ $\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \frac{1}{t} & -\frac{s}{t^2} \end{pmatrix}$

$$\frac{\partial z}{\partial x} = (\tan y)(e^{xy})(y)$$

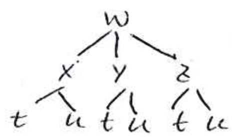
$$\frac{\partial z}{\partial y} = (\cancel{\tfrac{1}{1+y^2}}\ \sec^2 y)(e^{xy}) + (e^{xy})'\tan y$$

$$= (\sec^2 y)(e^{xy}) + (e^{xy})(x)\tan y$$

$$\frac{\partial z}{\partial s} = (\tan y)(y)(e^{xy}) + (e^{xy})(x)(\tan y)\cdot(\tfrac{1}{t})$$

$$\frac{\partial z}{\partial t} = (\tan y)(y)(e^{xy})(2) + (e^{xy})(x)(\tan y)(-\tfrac{s}{t^2})$$

3. $W = f(x,y,z), x=x(t,u), y=y(t,u), z=z(t,u)$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

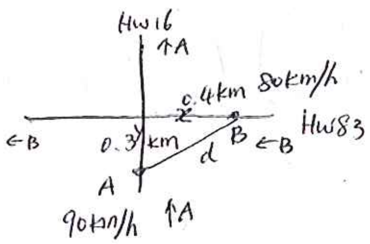
4. $z_x, z_y = ?$ if $xyz = \cos(x+y+z)$

$xyz(x,y) = \cos(x+y+z(x,y))$ or $F(x,y,z) = xyz - \cos(x+y+z) = 0$

$$\frac{\partial}{\partial x}(F_x\cdot x_x + F_y\cdot y_x + F_z\cdot z_x) = 0$$

$F_x + F_z z_x = 0$ $\quad z_x = -\frac{F_x}{F_z}$

5.



$d = \sqrt{x^2+y^2}$

$x(t) = 0.4 - 80t$

$y(t) = -0.3 + 90t$

$$\frac{\partial d}{\partial t} = \frac{\partial d}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial d}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{x}{\sqrt{x^2+y^2}}(-80) + \frac{y}{\sqrt{x^2+y^2}}(90)$$

$$= \frac{0.4}{0.5}(-80) + \frac{-0.3}{0.5}(90)$$

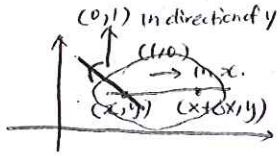
14.6 Directional Derivative

$f(x,y)$

$\frac{\partial f}{\partial x}(x,y)$ = rate of change of $f(x,y)$ in x

$x \to x+\Delta x$

$f(x,y) \to f(x+\Delta x, y)$



= rate of change of $f(x,y)$ in direction $(1,0)$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$x(t) = x_0 + at$

$y(t) = y_0 + bt$

$\langle a,b\rangle$ direction — unit vector!

$\underline{a^2+b^2=1}$

$$\frac{\partial f(x(t),y(t))}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{\partial f}{\partial x}(x_0,y_0)\cdot a + \frac{\partial f}{\partial y}(x_0,y_0)\cdot b$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \langle a,b\rangle$$

2D $\quad = \langle f_x, f_y \rangle \langle a, b \rangle = \nabla f(x_0, y_0) \cdot \vec{u}$

$D_u f(x,y) = f_x \cdot a + f_y \cdot b \quad \vec{u} = \langle a, b \rangle$

3D $\quad D_u f(x,y,z) = f_x \cdot a + f_y \cdot b + f_z \cdot c \quad \vec{u} = \langle a, b, c \rangle$

$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$

Gradient Vector of $f(x,y,z)$

$\underline{\nabla f = \langle f_x, f_y \rangle \text{ or } \langle f_x, f_y, f_z \rangle}$

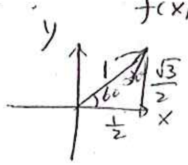
Prob 1. $\quad f(x,y) = 5xy^2 - 4x^3y$ at $(x,y) = (1,2)$, $\vec{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

$\nabla f = \langle \underbrace{5y^2 - 12x^2y}_{f_x}, \underbrace{10xy - 4x^3}_{f_y} \rangle \qquad \nabla f(1,2) = \langle -4, 16 \rangle$

$D_u f(1,2) = \langle -4, 16 \rangle \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle = \frac{172}{13}$

$f(1 + \frac{5}{13}\Delta z, 2 + \frac{12}{13}\Delta z) - f(1,2) \approx D_u f(1,2) \cdot \Delta z = \frac{172}{13}\Delta z$

2 $\quad f(x,y) = x\sin(xy)$ at $(x,y) = (2,0)$ in angle $\theta = \frac{\pi}{3}$ w/ $+x$ axis.



direction $= \langle \cos\theta, \sin\theta \rangle = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\nabla f(2,0) \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle 0, 4 \rangle \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = 2\sqrt{3}$

$\nabla f = \langle \sin(xy) + \cos(xy)xy, \ x^2\cos(xy) \rangle$

$\nabla f(2,0) = \langle 0, 4 \rangle$

3 $\quad f(x,y,z) = x^2 + y^2 + z^2$, $P(1,2,3)$ in direction of Origin.

$\vec{u} = \frac{(0,0,0) - (1,2,3)}{|\vec{u}|} = \frac{\langle -1, -2, -3 \rangle}{\sqrt{14}} = \langle -\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \rangle$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$D_u f(1,2,3) = \langle 2, 4, 6 \rangle \cdot \langle -\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \rangle = -\frac{28}{\sqrt{14}} = -2\sqrt{14}$

Geometry

$D_u f(x_0, y_0) = \langle f_x, f_y \rangle \langle a, b \rangle$

$= |\nabla f| \cdot |\vec{u}| \cos\theta = |\nabla f| \cos\theta$

θ = angle between ∇f and $\vec{u}$

$\xrightarrow{\nabla f}$ $\xrightarrow{\vec{u}}$ $\quad \theta = 0$, $\cos\theta = 1$ $\quad D_u f = |\nabla f|$ max direction derivative

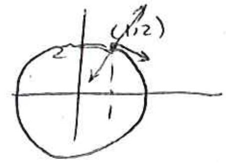
$\xrightarrow{\nabla f}$ $\xleftarrow{\vec{u}}$ $\quad \theta = \pi$, $\cos\pi = -1$ $\quad D_u f = -|\nabla f|$ min direction deriv.

$\xrightarrow{\nabla f}$ $\downarrow \vec{u}$ $\quad \theta = \frac{\pi}{2}$, $\cos\theta = 0$ $\quad D_u f = 0$

direction where f increases fastest is $\vec{u} = \frac{\nabla f}{|\nabla f|}$

$f(x,y) = x^2+y^2$ $\quad \nabla f = \langle 2x, 2y\rangle$ $\quad \dfrac{\nabla f}{|\nabla f|} = \dfrac{\langle 2,4\rangle}{\sqrt{20}} = \dfrac{\langle 1,2\rangle}{\sqrt{5}}$

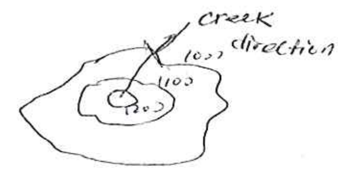
$(x,y) = (1,2)$ $\quad \nabla f(1,2) = \langle 2,4\rangle$



direction $\perp$ level curve (2D) = gradient

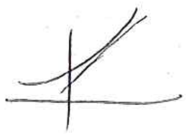
surface (3D)

$f(x,y)=5$



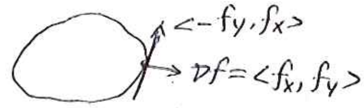
$y = f(x)$ $\quad y - y_0 = f'(x_0)(x-x_0)$

(x_0, y_0)



$f(x,y) = c$

(x_0, y_0)



tangent line to level curve

$x = x_0 + (-f_y(x_0,y_0))t$

$y = y_0 + (-f_x(x_0,y_0))t$

$\dfrac{x-x_0}{-f_y} = \dfrac{y-y_0}{f_x}$

$f_x(x-x_0) = -f_y(y-y_0)$

or $\langle f_x, f_y\rangle \cdot \langle x-x_0, y-y_0\rangle = 0$

$f_x(x-x_0) + f_y(y-y_0) = 0$

$z = f(x,y)$

tangent plane $\quad z - z_0 = f_x(x-x_0) + f_y(y-y_0)$

$f(x,y,z) = c$

$\langle f_x, f_y, f_z\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

Prob. 1 $\quad f(x,y) = x^3y$ at $(2,3)$ what direction increase the fastest? rate?

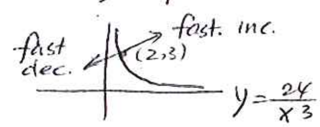
$\vec{u} = \dfrac{\nabla f(2,3)}{|\nabla f|} = \dfrac{\langle 3x^2y, x^3\rangle}{|\nabla f|} = \dfrac{\langle 36, 8\rangle}{\sqrt{36^2+8^2}} = \dfrac{\langle 9,2\rangle}{\sqrt{85}}$

decrease fastest $\dfrac{\langle -9,-2\rangle}{\sqrt{85}}$

max rate $= |\nabla f| = \sqrt{36^2+8^2} = 4\sqrt{85}$

min rate $-4\sqrt{85}$

$x^3y = 24$



Prob 2. $\quad$ find tangent plane and normal line of surface $x^2-2y^2+z^2+yz=2$ at $(1,0,1)$

$\langle f_x, f_y, f_z\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

$\langle 2x, -4y+z, 2z+y\rangle$

tang plane $\langle 2,1,2\rangle \langle x-1, y, z-1\rangle = 0$

normal line $\quad \vec{r} = \vec{r_0} + \vec{v}t$

$\langle x, y, z\rangle = \langle 1,0,1\rangle + \langle 2,1,2\rangle t$

$\begin{cases} x = 1+2t \\ y = t \\ z = 1+2t \end{cases}$

14.7 Optimization

$f(x,y)$

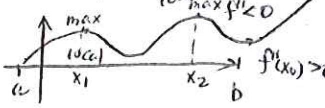
find $\max f(x,y)$

$\min f(x,y)$

$y=f(x)$

critical point $x \to f'(x)=0$ global max

local max $f''<0$



x_1, x_2, a, b

1st derv. test local max/min $\Rightarrow$ critical point

reverse? $y=x^3$, not true $\Leftarrow$

2nd derv test

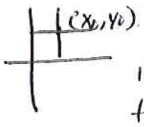
$f'(x_0)=0$ $\quad f''>0 \Rightarrow$ local min

$f''<0 \Rightarrow$ local max

2-D. $z=f(x,y)$

critical pt $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = 0$

$D_u f(x,y) = a f_x + b f_y$



$g(x) = f(x, y_0)$

if $f(x_0,y_0) = \max f(x,y)$

then $g(x) = \max f(x, y_0)$

Theorm A $\Rightarrow g'(x_0)=0$

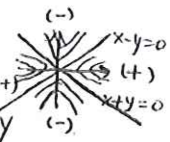
$\frac{\partial f}{\partial x}(x_0,y_0)$

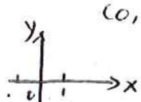
saddle point

$f(x,y) = x^2-y^2$

$f_x = 2x \quad f_y=-2y$

$(0,0)$ = critical pt





$f(x,0)=x^2$ $(0,0)$ local min along $y=0$

$f(0,y)=-y^2$ $(0,0)$ local max along $x=0$

$\Rightarrow (0,0)$ saddle point

Ex. $f(x) = x^3-3x^2+2x-7$

$f'(x) = 3x^2-6x+2=0 \quad x = \frac{6 \pm 2\sqrt{3}}{6}$

$f''(x) = 6x-6$

$f''\left(\frac{3+\sqrt{3}}{3}\right)>0 \quad f''\left(\frac{3-\sqrt{3}}{3}\right)<0$

local min $\qquad$ local max

in $[0,7]$, evaluate $x = \frac{3\pm\sqrt{3}}{3}, 0, 7$

find abs. max and min

2nd derv test

$f(x,y) = x^2-y^2$

$f_{xx}=2 \quad f_{yy}=-2$

$f_{xy}=f_{yx}=0$

symmetric

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Determinant$(H) = f_{xx}f_{yy} - (f_{xy})^2$

$T = \text{trace}(H) = f_{xx}+f_{yy}$

- local min if $f_{xx}>0, f_{yy}>0, D>0$
- local max if $f_{xx}<0, f_{yy}<0, D>0$
- $D<0$, then (x_0,y_0) saddle point
- $D, f_{xx}, f_{yy} = 0$ (any), undetermined or other type

OR

- T.

$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} = 0$

$(a-\lambda)(c-\lambda)-b^2=0 \quad \lambda^2-(a+c)+ac-b^2=0$

$D = ac-b^2<0$

$= \lambda_1\lambda_2$

$\Rightarrow \lambda_1>0, \lambda_2<0$ (saddle)

$D>0 \quad \Delta = (a+c)^2-4(ac-b^2) = (a-c)^2+4b^2>0$

if $D>0$

- $T=a+c<0$ (local max)

$\Rightarrow \lambda_1<0, \lambda_2<0$

- $T=a+c>0$

$\Rightarrow \lambda_1>0, \lambda_2>0$

(local min)

Ex1 $\quad f(x,y) = x^3y+12x^2-8y$

$\begin{cases} f_x = 3x^2y+24x=0 \\ f_y = x^3-8=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=-4 \end{cases}$

Critical point $(2,-4)$

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6xy+24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix}$

$H(2,-4) = \begin{pmatrix} -24 & 12 \\ 12 & 0 \end{pmatrix}$

$\text{Det}(H) = -144 \quad \Rightarrow (2,-4)$ saddle point

Ex 2 $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

$f_x = 6x^2 + y^2 + 10x = 0$

$f_y = 0 + 2xy + 0 + 2y = 2xy + 2y = 0 \Rightarrow 2y(x+1) = 0 \Rightarrow y = 0 \text{ or } x = -1$

$\begin{cases} y = 0 \\ x = 0 \text{ or } -\frac{5}{3} \end{cases}, \begin{cases} x = -1 \\ y = \pm 2 \end{cases}$ critical points $(-1,-2)$ $(-1,2)$ $(0,0)$ $(0,-\frac{5}{3})$

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{pmatrix}$

$H(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$ $f_{xx} > 0$ $f_{yy} > 0$ $D > 0$ $(0,0)$ local min

$H(-\frac{5}{3},0) = \begin{pmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix}$ $f_{xx} < 0$ $f_{yy} < 0$ $D > 0$ $(-\frac{5}{3},0)$ local max

$H(-1,2) = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$ $D < 0$ Saddle point

$H(-1,-2) = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix}$ $D = -16$ Saddle

Ex 3 shortest distance from $(0,1,1)$ to plane $x - 2y + 3z = 6$

$d = \sqrt{(x-0)^2 + (y-1)^2 + (z-1)^2}$ ← $x = 6 + 2y - 3z$

$= \sqrt{(6+2y-3z)^2 + (y-1)^2 + (z-1)^2}$ min ? $y, z \in \mathbb{R}^2$

$f(y,z) = (6+2y-3z)^2 + (y-1)^2 + (z-1)^2$

$f_y = \cancel{2}(6+2y-3z) \cdot 2 + \cancel{2}(y-1) = 0$

$f_z = \cancel{2}(6+2y-3z) \cdot (-3) + \cancel{2}(z-1) = 0$

$\begin{cases} 12 + 4y - 6z + y - 1 = 11 + 5y - 6z = 0 \\ -18 - 6y + 9z + z - 1 = -19 - 6y + 10z = 0 \end{cases}$

$\Rightarrow \begin{cases} y = \frac{2}{7} \\ z = \frac{29}{14} \end{cases}$

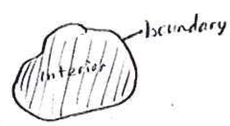
$H = \begin{pmatrix} 10 & -12 \\ -12 & 20 \end{pmatrix}$ $f_{yy} = 10 > 0$ $f_{zz} = 20 > 0$ $D > 0 \Rightarrow$ min

in 12.5 $D = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

$D = \frac{5}{\sqrt{14}}$

Abs Max, Min

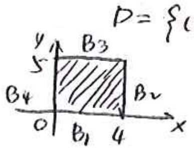
on closed subset/boundary



- ① find critical pt $f_x = f_y = 0$ (inside)
- ② find boundary pts + 1-D derivatives (critical pts on boundary)
- ③ evaluate each pts.

Ex 2 $f(x,y) = 4x + 6y - x^2 - y^2$

$D = \{(x,y) \mid 0 \le x \le 4, 0 \le y \le 5\}$



Step 1. $\nabla f = \langle 4 - 2x, 6 - 2y \rangle$

$= \langle 0, 0 \rangle$

$\begin{cases} x = 2 \\ y = 3 \end{cases}$ $(2,3)$ critical pt and inside region

step 2. max/min on boundary

B_1: $y = 0$, $0 \le x \le 4$

$f(x,y) = f(x,0) = 4x - x^2 = f_1(x)$

$f_1'(x) = 0 \Rightarrow x = 2$ CP$(2,0)$

Boundary pt $(0,0)$, $(4,0)$

B_2: $x = 4$, $0 \le y \le 5$

$f(x,y) = f(4,y) = 6y - y^2 = f_2(y)$

$f_2'(y) = 6 - 2y = 0$ $y = 3$

CP $(4,3)$

BP $(4,0)$, $(4,5)$

B_3: $y = 5$, $0 \le x \le 4$

$f(x,y) = f(x,5) = 4x - x^2 + 5 = f_3(x)$

$f_3'(x) = 0 \Rightarrow x = 2$

CP$(2,5)$ BP$(4,5)$, $(0,5)$

B_4: $x = 0$, $0 \le y \le 5$

$f(x,y) = 6y - y^2 = f_4(y)$

$f_4'(y) = 0 \Rightarrow y = 3$

CP$(0,3)$ BP$(0,5)$ $(0,0)$

Step 3

(x,y)	$f(x,y)$	
$(2,3)$	17	$\to$ max (abs.)
$(2,0)$	4	
$(0,0)$	0	$\to$ min, abs.
$(4,0)$	0	$\to$ min, abs.
$(4,5)$	5	
$(4,3)$	9	
$(2,5)$	9	
$(0,5)$	5	
$(0,3)$	9	

Ex5. $\langle 1+t, 1+6t, 2t\rangle$ closest to $(1,2,3)$

$d^2 = (1+t-1)^2 + (1+6t-2)^2 + (2t-3)^2$

Ex6 $L_1: \langle 1+t, 1+6t, 2t\rangle$

$L_2: (1+2s, 5+15s, -2+6s)$

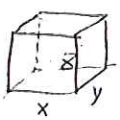
$d^2 = (1+t-1-2s)^2 + (1+6t-5-15s)^2 + (2t+2-6s)^2$

$= (t-2s)^2 + (6t-15s-4)^2 + (2t+2-6s)^2$

$= f(s,t)$

$f_s = f_t = 0$

Ex7.



$V = xyz$

$C = \underset{\text{cost is 5}}{5xy} + 2yz + \underset{\text{cost is 1.}}{2xz}$

min $C(x,y,z)$ given $xyz = V$.

$z = \frac{V}{xy} \Rightarrow C = 5xy + \frac{2V}{x} + \frac{2V}{y} = C(x,y)$

~~$= 5xy$~~ $x > 0,\ x < \infty$

$\lim\limits_{\substack{x\to 0\\ \text{or } y\to 0}} C = \infty$ $\quad \lim\limits_{\substack{x\to\infty\\ y\to\infty}} C(x,y) = \infty$

$\frac{\partial C}{\partial x} = 5y - \frac{2V}{x^2} = 0$

$\frac{\partial C}{\partial y} = 5x - \frac{2V}{y^2} = 0$

$\Rightarrow \begin{cases} x = \sqrt[3]{\frac{2V}{5}} \\ y = \sqrt[3]{\frac{2V}{5}} \\ z = \frac{5}{2}\cdot\sqrt[3]{\frac{2V}{5}} \end{cases}$ local min

$H = \begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix} = 75 > 0$

$C_{xx} > 0$

$C_{yy} > 0$

global min in this case.

Ex8. $x + y + z = 12$ $\quad z = 12 - x - y$

$x^2 + y^2 + z^2$ smallest.

$f(x,y,z) = x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2$

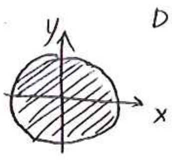
$12 > x > 0,\ 12 > y > 0$

Ans: $x = y = z = 4$

Ex9. find abs max, min

$f(x,y) = 2x^2 + 3y^2 - 4x + 6y - 5$

$D = \{(x,y) \mid x^2 + y^2 \le 16\}$.



Crit. Point $\begin{cases} f_x = 4x - 6 = 0 \\ f_y = 6y + 6 = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$ $(1,-1)$

min/max $f(x,y)$ given $x^2 + y^2 = 16$?

$x = 4\cos\theta \quad y = 4\sin\theta$

On $x^2 + y^2 = 16$

$f(x,y) = 2(4\cos\theta)^2 + 3(4\sin\theta)^2 - 4(4\cos\theta) + 6(4\sin\theta) - 5$

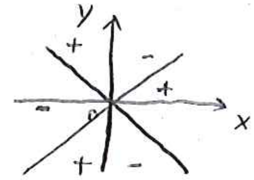
$g(\theta) = 16\sin^2\theta - 16\cos\theta + 24\sin\theta + 27$

$0 \le \theta \le 2\pi$

$g'(\theta) = \ldots$

See HW 14.7 35, 39, 43

Saddle point



$f(x,y) = x^3 - 3xy^2 = x(x-\sqrt{3}y)(x+\sqrt{3}y)$

$f_x = 3x^2 - 3y^2 = 0$

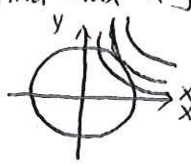
$f_y = -6xy = 0 \quad \Rightarrow (0,0)$

$f_{xx} = 6x \quad f_{xy} = -6y \quad f_{yy} = -6x$ degerate critical point

$H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

14.8 Constrained Optimization

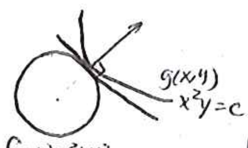
Find max x^2y, given $x^2+y^2=1$ (constraint)



$x^2y = c$

$y = \frac{c}{x^2}$

max should occur when $x^2y=c$ tangent to $x^2+y^2=1$

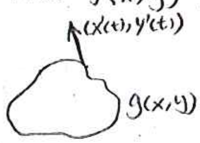


$\nabla g = \langle 2xy, x^2 \rangle$

$\nabla f = \langle 2x, 2y \rangle$

$\boxed{\nabla f = \lambda \nabla g}$

Prove: find max $f(x,y)$ given $g(x,y)=0$



$x = x(t)$

$y = y(t)$

$g(x(t), y(t)) = 0 \Rightarrow g_x x'(t) + g_y y'(t) = 0 \quad \langle g_x, g_y \rangle \cdot \langle x'(t), y'(t) \rangle = 0$

find max $f(x(t), y(t)) \Rightarrow f_x x'(t) + f_y y'(t) = 0 \quad \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle = 0$

$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$

$\nabla f = \lambda \nabla g \qquad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases}$ solve x, y, λ

- solve $\nabla f = \lambda \nabla g$ and $g(x,y) = k$

Ex1 Find abs. min, max $f(x,y) = 3x+y$, given $x^2+y^2=10$

① $\nabla f = \langle 3, 1 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

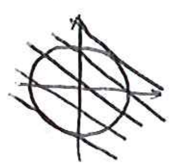
$\langle 3, 1 \rangle = \lambda \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\Rightarrow \begin{cases} 3 = \lambda 2x \Rightarrow x = \frac{3}{2\lambda} \\ 1 = \lambda 2y \Rightarrow y = \frac{1}{2\lambda} \\ x^2+y^2=10 \end{cases}$

$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10$

$\frac{10}{4\lambda^2} = 10 \qquad \lambda = \pm\frac{1}{2}$



case 1, $\lambda = \frac{1}{2}$

$x = 3, y = 1$

$(x, y, \lambda) = (3, 1, \frac{1}{2})$

$f(3,1) = 10$

abs. max

case 2, $\lambda = -\frac{1}{2}$

$x = -3, y = -1$

$(x, y, \lambda) = (-3, -1, -\frac{1}{2})$

$f(-3,-1) = -10$

abs. min

Ex2 $f(x,y) = x^2+y^2$, $g(x) = 3x+y = 10$

Ex 3 abs. min/max $f(x,y)=2x^2+3y^2-4x+6y-5$ on $D=\{(x,y)\mid x^2+y^2\le 16\}$

given $x^2+y^2=16$

$\nabla f=\langle 4x-4,\ 6y+6\rangle$

$\nabla g=\langle 2x, 2y\rangle$

$$\begin{cases} 4x-4=\lambda 2x \\ 4y+6=\lambda 2y \\ x^2+y^2=16 \end{cases} \qquad x=\frac{2x-2}{\lambda} \Rightarrow x=\frac{2}{2-\lambda} \qquad y=\frac{-3}{3-\lambda}$$

$\lambda_1 \approx 1.43078$

$\lambda_2 \approx 3.78141$

Ex 5 points on $x^2+y^2+z^2=1$ closest to $(1,2,3)$

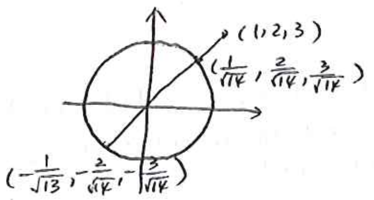
$g(x,y,z)=x^2+y^2+z^2=1$

$f(x,y,z)=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$

$f_1(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$

$\nabla f=\langle 2(x-1), 2(y-2), 2(z-3)\rangle$

$\nabla g=\langle 2x, 2y, 2z\rangle$



$$\begin{cases} 2x-2=\lambda 2x \\ 2y-4=\lambda 2y \\ 2z-6=\lambda 2z \\ x^2+y^2+z^2=1 \end{cases} \qquad x=\frac{1}{1-\lambda} \quad y=\frac{2}{1-\lambda} \quad z=\frac{3}{1-\lambda}$$

$(\frac{1}{1-\lambda})^2+(\frac{2}{1-\lambda})^2+(\frac{3}{1-\lambda})^2=1$

$(1-\lambda)^2=14 \qquad \lambda=1\pm\sqrt{14}$

$$\begin{cases} \lambda=1-\sqrt{14} \\ x=-\frac{1}{\sqrt{14}} \\ y=-\frac{2}{\sqrt{14}} \\ z=-\frac{3}{\sqrt{14}} \end{cases} \qquad \begin{cases} \lambda=1+\sqrt{14} \\ x=\frac{1}{\sqrt{14}} \\ y=\frac{2}{\sqrt{14}} \\ z=\frac{3}{\sqrt{14}} \end{cases}$$

See Hw 14.8 31, 33, 37, 43. (2-const. Prob)

Find min $f(x,y)$ (or max), given $g(x,y)=c$

Define $F(x,y,\lambda)=f(x,y)-\lambda(g(x,y)-c)$

max, min $F(x,y,\lambda) \sim f(x,y)$

Hamiltonian Function

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= f_x-\lambda g_x=0 \\ \frac{\partial F}{\partial y} &= f_y-\lambda g_y=0 \end{aligned}\right\} \Rightarrow \nabla f=\lambda\nabla g$$

$\frac{\partial F}{\partial \lambda}=-(g(x,y)-c)=0 \Rightarrow g(x,y)=c$

$F(x,y,\lambda,c)$

$\lambda=\frac{\partial F}{\partial c}$.

Example in Econ

$u(x,y)=x^\alpha y^{1-\alpha}$ utility func. $(0<\alpha<1)$

x = labor, y = capital (Cobb-Douglas)

$u(kx,ky)=(kx)^\alpha(ky)^{1-\alpha}=k\,u(x,y)$ homogeneous

max $x^\alpha y^{1-\alpha}$, budget $= W=P_x x+P_y y$ (P_x, P_y price)

Method 1: $x=\frac{W-P_y y}{P_x}$

max $\left(\frac{W-P_y y}{P_x}\right)^\alpha y^{1-\alpha}=f(y)$

Method 2: multiplier

$F(\lambda,x,y)=x^\alpha y^{1-\alpha}-\lambda(P_x x+P_y y-W)$

$\frac{\partial F}{\partial \lambda}=P_x x+P_y y-W=0$ ①

$\frac{\partial F}{\partial x}=\alpha x^{\alpha-1}y^{1-\alpha}-\lambda P_x=0$ ②

$\frac{\partial F}{\partial y}=(1-\alpha)x^\alpha y^{-\alpha}-\lambda P_y=0$ ③

$$\left.\begin{aligned} ② &\Rightarrow \alpha x^\alpha y^{1-\alpha}=\lambda P_x x \\ ③ &\Rightarrow (1-\alpha)x^\alpha y^{1-\alpha}=\lambda P_y y \end{aligned}\right\} \Rightarrow x^\alpha y^{1-\alpha}=\lambda(P_x x+P_y y)=\lambda W$$

$\lambda=\frac{x^\alpha y^{1-\alpha}}{W}$ if (x,y,w) is the optimal value

$=$ optimal $\frac{\text{utility}}{\text{budget}}$

$x=\frac{\alpha W}{P_x}$

$y=\frac{(1-\alpha)W}{P_y}$

Theorem 4 (first deriv. test for constrain prob) If f has a local min/max on $S = \{(x,y) \mid g(x,y)=c\}$

then $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$

If S is closed and bounded subset of $\mathbb{R}^2$, then abs max/min alway exists

Two Constraints Problem

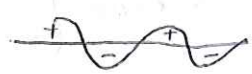
find max, min $f(x,y,z)$

$g(x,y,z) = c_1$

$h(x,y,z) = c_2$

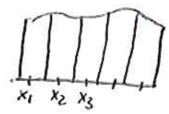
$F(\lambda, \mu, x, y, z) = f - \mu(h - c_2) - \lambda(g - c_1)$

15.1 Integrals



can be counted neg.

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$



$= \int_a^b f(x)\,dx = F(b) - F(a)$

$F(x) \to F'(x) \xrightarrow{\int_a^b} F(b) - F(a)$

Foundament Theorem of Calc

$F'(x) = f(x)$

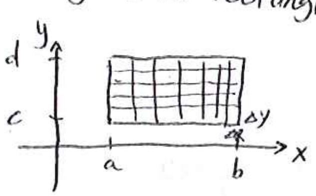
Indefinite Integrals $F(x) = \int_0^x f(s)\,ds$ or $F(x) = \int f(x)\,dx$

techniques subst. $\int \sin(2x)\,dx$ $u = 2x$

$= \int \sin u \cdot \frac{1}{2}\,du$ $du = 2dx$

$= -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(2x) + C$

Double Integrals on rectangle



$R = [a,b]\times[c,d] = \{(x,y) \in \mathbb{R}^2 : a<x<b,\ c<y<d\}$

$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A$ height $(\Delta x \cdot \Delta y)$ in the base square

find the volume under the function, above the square

$\lim_{m,n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\,\Delta A = \iint_R f(x,y)\,dA$

$\Delta A = \frac{b-a}{n} \cdot \frac{d-c}{m}$

15.2 Iterated Integral

$g(y) = \int_a^b f(x,y)\,dx$ for fixed y

$\int_c^d g(y)\,dy$

$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$

Fubini's Theorem

if $f(x,y)$ continuous on $R = [a,b]\times[c,d]$

$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$

Ex1 $\iint_R \frac{xy^2}{x^2+1}\,dA$ $dA = dx \cdot dy$ $\boxed{dA}\,dy$ dx

$R = [0,1]\times[-3,3]$

$= \int_0^1 \left(\int_{-3}^{3} \frac{xy^2}{x^2+1}\,dy \right) dx$

$= \int_0^1 \left(\frac{x}{x^2+1} \int_{-3}^{3} y^2\,dy \right) dx$

$= \int_0^1 \left(\frac{x}{x^2+1} \left. \frac{y^3}{3} \right|_{y=-3}^{3} \right) dx$

$= 18\int_0^1 \frac{x}{x^2+1}\,dx = 18 \cdot \frac{1}{2} \ln(x^2+1) \Big|_{x=0}^{x=1} = 9 \ln 2$

$u = x^2+1$

$du = 2x\,dx$

$\int \frac{x}{x^2+1}\,dx$

$= \int \frac{\frac{1}{2}}{u}\,du = \frac{1}{2}\ln u$

$= \frac{1}{2}\ln(x^2+1)$

$\boxed{\begin{aligned} &\int_a^b \left(\int_c^d f(x)\cdot g(y)\,dy \right) dx \\ &= \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy \\ &\text{true only when } F(x,y) = f(x)\cdot g(y) \end{aligned}}$

Ex 2 $\iint_R \frac{x}{x^2+y^2}\, dA,\ R=[1,2]\times[0,1]$

$= \int_1^2 \left(\int_0^1 \frac{x}{x^2+y^2} dy\right) dx$

$= \int_1^2 \left(x \int_0^1 \frac{1}{x^2+y^2} dy\right) dx$ $\qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$

$= \int_1^2 x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)\Big|_{y=0}^{y=1} dx$

$= \int_1^2 \left(\tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}(0)\right) dx.$

$= \int_1^2 \tan^{-1}\left(\frac{1}{x}\right) dx$ $\qquad$ IbyP $\qquad u=\tan^{-1}$

$\int \tan^{-1}\left(\frac{1}{x}\right) dx = x\tan^{-1}\left(\frac{1}{x}\right) - \int x\cdot\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)dx$

$= x\tan^{-1}\left(\frac{1}{x}\right) + \int \frac{x}{x^2+1}\, dx$

$= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1)$

$= \left(x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1)\right)\Big|_{x=1}^{x=2}$

or $\int_0^1 \left(\int_1^2 \frac{x}{x^2+y^2} dx\right) dy$ $\qquad u=x^2+y^2,\ du=2x\,dx,$

$= \int_0^1 \left(\int_1^2 \frac{\frac{1}{2}du}{u}\right) dy$

$= \int_0^1 \frac{1}{2}\ln|u|\Big|_{x=1}^{x=2} dy$

$= \int_0^1 \frac{1}{2}\ln(x^2+y^2)\Big|_{x=1}^{x=2} dy$

$= \frac{1}{2}\int_0^1 \left(\ln(y^2+4) - \ln(y^2+1)\right) dy$

Ex3 $\quad z = 1-\frac{x^2}{4}-\frac{y^2}{9}$

$\iint_R \left(1-\frac{x^2}{4}-\frac{y^2}{9}\right) dA$

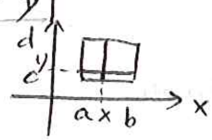
Avg Value of 1D func. $f(x)$ on (a,b)

$\frac{1}{b-a}\int_a^b f(x)\,dx$

Avg val. of $f(x,y)$ on $R=[a,b]\times[c,d]$

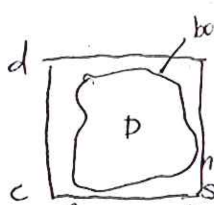
$\frac{1}{(b-a)(d-c)}\iint_R f(x,y)\,dA$

- $\boxed{\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA}$



$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

15.3 Double Integrals over General Regions



not in rectangular shape

$\iint_D f(x,y)\,dA$

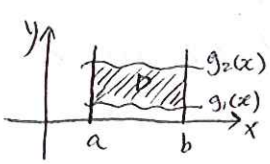
$g(x,y) = \begin{cases} f(x,y) & (x,y)\in D \\ 0 & (x,y)\notin D \end{cases}$

$\iint_D f(x,y)\,dA = \iint_R f(x,y)\,dA$ $\quad (D\subset R)$

$= \int_c^d\int_a^b g(x,y)\,dx\,dy$

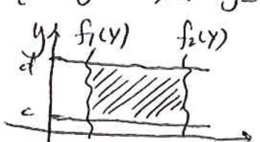
Type I.

$D=\{(x,y)\in\mathbb{R}^2;\ a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

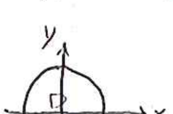


Type II

$D=\{(x,y)\in\mathbb{R}^2;\ d\le y\le c,\ f_1(y)\le x\le f_2(y)\}$



both Type I & II



region bounded by circle $r=3$, center $(2,3)$

$(x-2)^2+(y-3)^2=3^2$

$D=\{(x,y): (x-2)^2+(y-3)^2\le 3^2\}$

$=\{(x,y): -1\le x\le 5,\ 3-\sqrt{9-(x-2)^2}\le y\le 3+\sqrt{9-(x-2)^2}\}$

Type I: $\quad -1\le x\le 5$

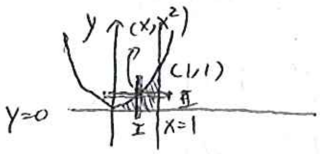
$(y-3)^2 = 3^2-(x-2)^2$

$y = 3\pm\sqrt{9-(x-2)^2}$

$\iint_D f(x,y)\,dA$

$= \int_{-1}^{5}\int_{3+\sqrt{9-(x-2)^2}}^{3-\sqrt{9-(x-2)^2}} f(x,y)\,dy\,dx$

Ex1 D bounded by $y=0$, $y=x^2$, and $x=1$



$D=\{(x,y): 0\le x\le 1,\ 0\le y\le x^2\}$ (type 1)

$= \{(x,y): 0\le y\le 1,\ \sqrt{y}\le x\le 1\}$ (type 2)

$\iint_D f(x,y)\,dA$

$= \int_0^1\left(\int_0^{x^2} f(x,y)\,dy\right)dx$

$= \int_0^1\left(\int_{\sqrt{y}}^1 f(x,y)\,dx\right)dy$

$$\iint_D xy\,dA = \int_0^1\int_0^{x^2} xy\,dy\,dx$$

$$= \int_0^1 x\frac{y^2}{2}\Big|_{y=0}^{y=x^2} dx$$

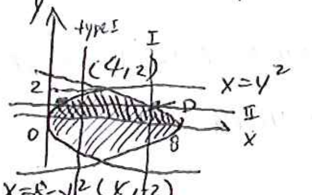
$$= \int_0^1 \frac{x^5}{2}\,dx = \frac{x^6}{12}\Big|_{x=0}^{x=1} = \frac{1}{12}$$

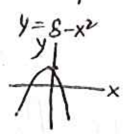
$$\text{or } = \int_0^1\int_{\sqrt{y}}^1 xy\,dx\,dy$$

$$= \int_0^1 \frac{x^2}{2}y\Big|_{x=\sqrt{y}}^{x=1} dy$$

$$= \int_0^1\left(\frac{y}{2}-\frac{y^2}{2}\right)dy = \frac{y^2}{4}-\frac{y^3}{6}\Big|_{y=0}^{y=1} = \frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

Ex2 D bounded by $x=y^2$, $x=8-y^2$, first quadrant, $\iint_D y\,dA$





$\begin{cases} x=8-y^2 \\ x=y^2 \end{cases}$ find intersection pts

$$\iint_D y\,dA = \int_0^2\left(\int_{y^2}^{8-y^2} y\,dx\right)dy$$

$$= \int_0^2 yx\Big|_{x=y^2}^{x=8-y^2} dy$$

$$= \int_0^2 (8y-y^3-y^3)\,dy = \int_0^2 (8y-2y^3)\,dy$$

$$= 4y^2-\frac{y^4}{2}\Big|_{y=0}^{y=2} = (16-8) = 8.$$

Type II $D=\{(x,y),\ 0\le y\le 2,\ y^2\le x\le 8-y^2\}$

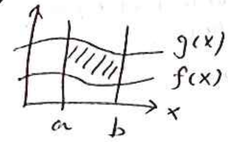
Type I $D=\{(x,y),\ 0\le x\le 4,\ 0\le y\le\sqrt{x}\}\cup\{(x,y);\ 4\le x\le 8,\ 0\le y\le\sqrt{8-x}\}$

$$\int_0^4\int_0^{\sqrt{x}}(\ )\,dy\,dx + \int_4^8\int_0^{\sqrt{8-x}}(\ )\,dy\,dx$$

Volume

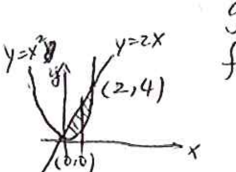
bounded by surface $z=f(x,y)$, $z=g(x,y)$ over D.

$$V=\iint_D[g(x,y)-f(x,y)]\,dA$$



Area $=\int_a^b (g(x)-f(x))\,dx$

Ex3 Volume under $z=x^2+y^2$ and above region D in xy-plane bounded by $y=2x$, $y=x^2$



$g(x,y)=x^2+y^2$

$f(x,y)=0$

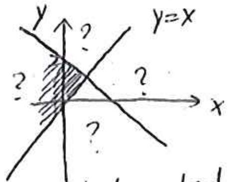
$D=\{(x,y): 0\le x\le 2,\ x^2\le y\le 2x\}$

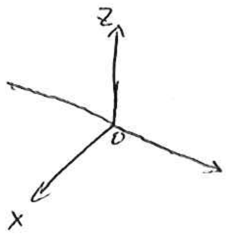
$$V=\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$$

$\begin{cases} y=x^2 \\ y=2x \end{cases}$ $x^2-2x=0$

(0,0), (2,4)

Ex 4 Volume bounded by $z=x$, $y=x$, $x+y=2$, and $z=0$





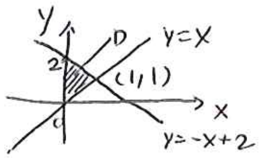
$g(x,y) = x$

$f(x,y) = 0$

$$V = \iint_D x\, dA = \int_0^1 \int_x^{2-x} x\, dy\,dx$$

$$= \int_0^1 xy\Big|_{y=x}^{y=2-x} dx$$

$$= \int_0^1 \left((2-x)x - x^2\right) dx$$



$D = \{(x,y) : 0 \le x \le 1,\ x \le y \le -x+2\}$

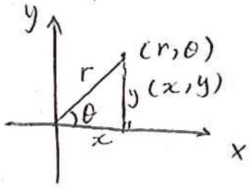
$z = x = 0$
$\Rightarrow x = 0$

15.4

polar coordinate

$-\infty < x < \infty$, $-\infty < y < \infty$

$0 \le r < \infty$, $0 \le \theta < 2\pi$



$(r,\theta) = (r, \theta + 2k\pi)$

$\sin\theta = \frac{y}{r}$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $(r,\theta) \to (x,y)$

$\cos\theta = \frac{x}{r}$

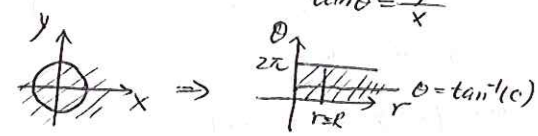
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $r = \sqrt{x^2+y^2}$ $\quad (x,y) \to (r,\theta)$

$\tan\theta = \frac{y}{x}$

$x^2+y^2 = R^2$ $\quad r^2 = R^2$ $\quad r = R$

$x = (R)\cos\theta$ $\quad r = \sqrt{x^2+y^2}$

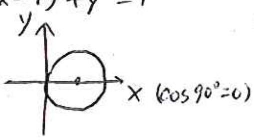
$y = (R)\sin\theta$ $\quad \tan\theta = \frac{y}{x}$



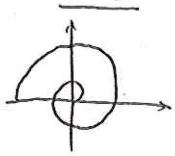
$r = 2\cos\theta = 2 \cdot \frac{x}{r}$

$x^2 + y^2 = 2x$

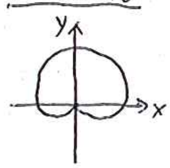
$(x-1)^2 + y^2 = 1$



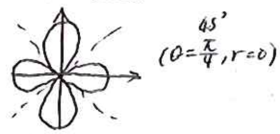
$r = \theta$



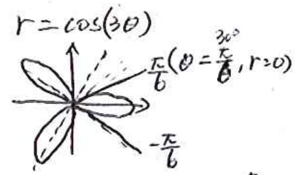
$r = 1 + \sin\theta$



$r = \cos(2\theta)$

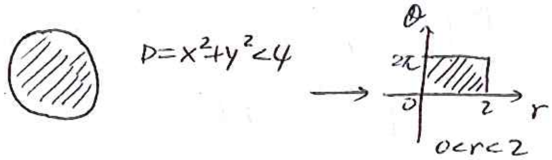


$r = \cos(3\theta)$



Integrals in polar coordinates

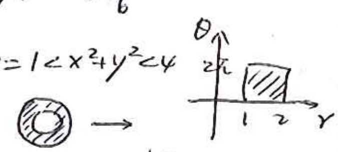
$D = x^2+y^2 < 4$



$0 < r < 2$

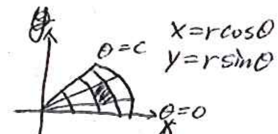
$0 < \theta < 2\pi$

$D = 1 < x^2+y^2 < 4$



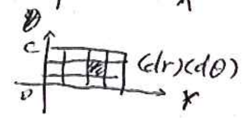
$1 < r < 2$

$0 < \theta < 2\pi$



$x = r\cos\theta$

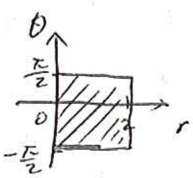
$y = r\sin\theta$



$D = \{x^2+y^2 < 4,\ x > 0\}$

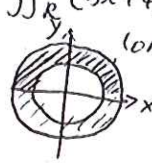
$0 < r < 2$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



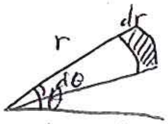
Ex $\iint_R (3x+4y)\, dA$, R bounded by $x^2+y^2=1$ and $x^2+y^2=4$

(only $y>0$.)



$D = \{1 < r < 2,\ 0 < \theta < \pi\}$

$$= \int_1^2 \int_0^{\pi} (3r\cos\theta + 4r\sin\theta)\, \underline{r\,d\theta\,dr}$$



$\Rightarrow \underline{dA = r\,dr\,d\theta}$

$$\left(\pi(r+dr)^2 - \pi r^2\right)\frac{d\theta}{2\pi} = \frac{(2\pi r \cdot dr + \cancel{\pi(dr)^2})d\theta}{2\pi}$$

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta,\ r\sin\theta)\,\underline{r}\,dr\,d\theta$$

Ex con't

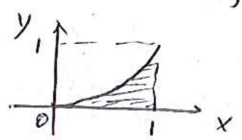
$$\int_0^2\int_0^\pi (3r\cos\theta+4r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^2\int_0^\pi 3r^2\cos\theta\,d\theta\,dr + \int_0^2\int_0^\pi 4r^2\sin\theta\,d\theta\,dr$$

$$= \left(\int_0^2 3r^2\,dr\right)\left(\int_0^\pi \cos\theta\,d\theta\right) + \left(\int_0^1 4r^2\,dr\right)\left(\int_0^\pi \sin\theta\,dr\right)$$

$$= \frac{56}{3}$$

15.3 Ex 5 $\quad \int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$ transformation between Type I and Type II



$x=\sqrt{y}$

$x^2=y$

$$= \int_0^1\left(\int_0^{x^2}\sqrt{x^3+1}\,dy\right)dx$$

$$= \int_0^1 \sqrt{x^3+1}\,y\Big|_{y=0}^{y=x^2}\,dx$$

$$= \int_0^1 \sqrt{x^3+1}\,(x^2)\,dx$$

$$= \frac{2}{9}(x^3+1)^{\frac{3}{2}}\Big|_{x=0}^{x=1}$$

type II $\rightarrow$ type I

$D=\{(x,y)\mid 0\le y\le 1,\ \sqrt{y}\le x\le 1\}$

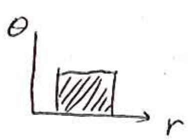
$\Downarrow$

$D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x^2\}$

See HW 15.3 17, 37, 43 (graph)

$x=r\cos\theta \quad r=\sqrt{x^2+y^2}$

$y=r\sin\theta \quad \theta=\tan^{-1}\left(\frac{y}{x}\right) \rightarrow$



Prob 1 $\quad r=4+3\cos\theta$ change to x, y coordinate

$$\sqrt{x^2+y^2} = 4+3\cdot\frac{x}{r} = 4+3\cdot\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 4\sqrt{x^2+y^2}+3x$$

$$(x^2+y^2-3x)^2 = 16(x^2+y^2)$$

$$4-3\le r\le 4+3$$

Area $= \iint_D 1\,dA$

$$= \int_0^{2\pi}\left(\int_{r=0}^{r=4+3\cos\theta} 1\cdot r\,dr\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}r^2\Big|_{r=0}^{r=4+3\cos\theta}\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}(4+3\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(16+24\cos\theta+9\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\left(16\theta+24\sin\theta+\frac{9}{2}\theta+\frac{9}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= 16\pi+\frac{9}{2}\pi = \frac{41}{2}\pi$$

$$\int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\theta+\frac{1}{4}\sin 2\theta$$

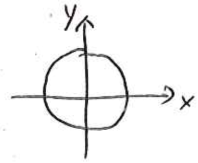
Prob 2

$z = 3x^2 + 3y^2 \quad z = 4 - x^2 - y^2$

$3x^2 + 3y^2 = 4 - x^2 - y^2$

$4x^2 + 4y^2 = 4$

$x^2 + y^2 = 1$



$(0,0) \quad z_1 = 3(0+0) = 0$ bottom

$z_2 = 4 > 0$ top func.

$$V = \iint_D (4 - x^2 - y^2 - 3x^2 - 3y^2)\, dA$$

$$= \int_0^1 \int_0^{2\pi} (4 - 4r^2)\, r\, d\theta\, dr$$

Prob 3 $\quad e^{-x^2}$

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy = erf(x)$$

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \qquad I = \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-y^2} dx\, dy$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr$$

$$= \int_0^{\infty} e^{-r^2}\, r\, dr \cdot \int_0^{2\pi} 1\, d\theta$$

$$= \frac{2\pi}{2} \int_0^{\infty} e^{-u}\, du = \pi$$

$I^2 = \pi$

$I = \sqrt{\pi}$

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$0 \le x < \infty$

15.7 Triple Integral

Domain: Box $= B = [a,b] \times [c,d] \times [r,s]$

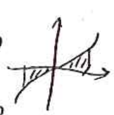
$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz \quad \text{Rectangular box}$$

$$\iiint_E f(x,y,z)\, dV = \int_r^s \int_{g_1(z)}^{g_2(z)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\, dx\, dy\, dz$$

Prob 1. $\iiint_B (z^3 + \sin y + 3)\, dV \quad$ B is unit Ball $x^2 + y^2 + z^2 \le 1$

odd function

$f(-x) = f(x)$



$\int_{-L}^{L} f(x)\, dx = 0$

even function

$\int_{-L}^{L} f(x)\, dx = 2\int_0^L f(x)\, dx$

$\iiint_B z^3 = 0 \quad z^3$ is odd in z

$(-z)^3 = -(z^3)$

$\sin(-y) = -\sin y$

$$\iiint_B 3\, dV = 3 \text{ Volume (Ball)}$$

$$= 3 \cdot \frac{4}{3}\pi \cdot 1^3 = 4\pi$$

2D	3D
$x \to y$	$x \to y \to z$
$y \to x$	$x \to z \to y$
	$y \to x \to z$
	$y \to z \to x$
	$z \to x \to y$
	$z \to y \to x$

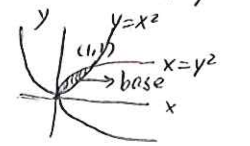
base $D = \{(x,y):\quad\}$

$$B = \{(x,y,z):\ (x,y) \in D,\ \underset{\text{bottom}}{f(x,y)} \le z \le \underset{\text{upper func.}}{g(x,y)}\}$$

$$\iiint_B f(x,y,z)\,dV = \iint_D \left(\int_{f(x,y)}^{g(x,y)} dz\right) dA$$

Prob 1 bounded by $y = x^2$ and $z = 0 \to$ lower

$x = y^2$ $\qquad z = x + y > 0 \to$ upper $\qquad \iiint_E xy\,dV = ?$



$D = \{(x,y): 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

$$\iint_D \left(\int_0^{x+y} xy\,dz\right) dA = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(\int_0^{x+y} xy\,dz\right) dy\,dx.$$

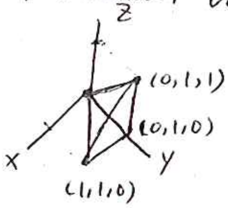
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xyz \Big|_{z=0}^{z=x+y} dy\,dx$$

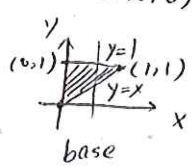
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2y + xy^2\,dy\,dx$$

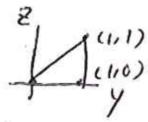
$$= \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3} \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{x^3}{2} + \frac{x^2\sqrt{x}}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx = \frac{3}{28}$$

Prob 2 tetrahedron vertices $(0,0,0), (0,1,0), (1,1,0)\ (0,1,1)$







$$\iiint_E xz\,dV = \iint_D \left(\int_0^{y-x} xz\,dz\right) dA$$

$$= \int_0^1 \int_x^1 \int_0^{y-x} xz\,dz\,dy\,dx.$$

top: plane w/ $(0,0,0), (0,1,1), (1,1,0)$

bottom $z = 0$

$ax + by + cz + d = 0$

$d = 0$

$b + c = 0$

$a + b = 0$

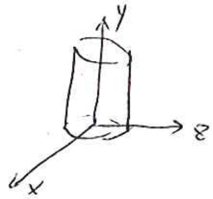
$a = 1$, $b = -1$, $c = 1$, $d = 0$

$x - y + z = 0$

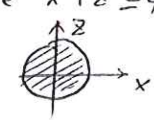
$z = y - x$ (top)

$D = \{(x,y): 0 \le x \le 1,\ x \le y \le 1\}$

Prob 3 $x^2 + z^2 = 4,\ y = -1,\ y + z = 4$



base $x^2 + z^2 \le 4$



$(x,z) = (0,0)$

$y = -1$ (bottom)

$y = 4 - z = 4$ (top)

$x = r\cos\theta$

$z = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$V = \iiint_E 1\,dV = \iint_D \int_{-1}^{4-z} 1\,dy\,dA$$

$$= \int_0^2 \int_0^{2\pi} \left(\int_{-1}^{4-z} 1\,dy\right) r\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} (4 - z + 1)\,r\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} (5 - z)\,r\,d\theta\,dr$$

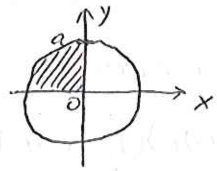
$$= \int_0^2 \int_0^{2\pi} 5r - r^2\sin\theta\,d\theta\,dr$$

$$= \int_0^2 5r\theta + r^2\cos\theta \Big|_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_0^2 10\pi r\,dr = 5\pi r^2 \Big|_0^2 = 20\pi.$$

Integral Conversion

$\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y \, dx\, dy$ to r, θ.



$D = \{ 0 \le y \le a, -\sqrt{a^2-y^2} \le x \le 0 \}$

$a^2 - y^2 = x^2 \quad (x<0)$

$a^2 = x^2 + y^2$

$D = \{ 0 < r < a, \frac{\pi}{2} < \theta < \pi \}$. $\quad x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow \int_0^a \int_{\frac{\pi}{2}}^{\pi} r^4 \cos^2\theta \sin\theta \, d\theta\, dr$

$= \left(\int_0^a r^4 dr\right) \int_{\frac{\pi}{2}}^{\pi} \cos^2\theta \sin\theta \, d\theta$.

$= \left(\frac{1}{5} r^5\right)\Big|_0^a \left(-\frac{1}{3}\cos^3\theta\right)\Big|_{\frac{\pi}{2}}^{\pi} = \frac{1}{15} a^5$

$\int \cos^2\theta \sin\theta \, d\theta \qquad u = \cos\theta$

$= \int u^2 (-du) \qquad du = -\sin\theta \, d\theta$

$= -\frac{u^3}{3} + c$.

15.8 Cylindrical Coordinate

$(x, y, z) \to (r, \theta, z)$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$(r, \theta, z) \to (x, y, z)$

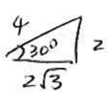
$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases}$

$E = \{(x,y,z) : (x,y) \in D, f(x,y) \le z \le g(x,y)\}$

$= \{(r,\theta,z) : (r,\theta) \in B, f(r\cos\theta, r\sin\theta) \le z \le g(r\cos\theta, r\sin\theta)\}$

$B = \{ r_1 < r < r_2, \theta_1 < \theta < \theta_2 \}$

Prob 1 $\quad (x, y, z) = (2\sqrt{3}, 2, -1) \to (r, \theta, z)$

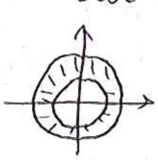


$r = \sqrt{x^2+y^2} = 4$

$\theta = \frac{\pi}{6}$

$z = -1$

Prob 2 $\quad \iiint_E x \, dV$. base $x^2+y^2=4$, $x^2+y^2=9$ $\quad z=0$, $z = x+y+5$



$2 < r < 3$

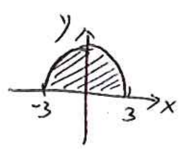
$0 < \theta < 2\pi$

$0 < z < x+y+5$

$\iiint_E x \, dV = \iint_D \int_0^{x+y+5} x \, dz\, dA = \int_2^3 \int_0^{2\pi} \left(\int_0^{r\cos\theta + r\sin\theta + 5} r\cos\theta \, dz \, r \right) d\theta\, dr$

Prob 4 $\quad$ change to cylindrical coord.

$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} \, dz\, dy\, dx$



$D = \{(x,y,z) : -3 \le x \le 3, 0 \le y \le \sqrt{9-x^2}, 0 \le z \le 9-x^2-y^2\} = \{(r,\theta,z) : 0 \le r \le 3, 0 \le \theta \le \pi, 0 \le z \le 9 - r^2\}$

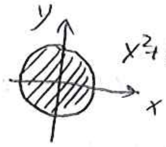
$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

$I = \int_0^3 \int_0^{\pi} \left(\int_0^{9-r^2} r^2 \, dz \right) d\theta\, dr$

cannot switch order in first step!

Prob 3 Lies within $x^2+y^2=1$, $x^2+y^2+z^2=4$



$x^2+y^2<1$

$E: 0<r<1$
$0<\theta<2\pi$

$z^2=4-x^2-y^2$
$z=\pm\sqrt{4-x^2-y^2}$
bottom $z=-\sqrt{4-x^2-y^2}$
top $z=\sqrt{4-x^2-y^2}$

$\int 2\sqrt{4-r^2}\,r\,dr$
$u=4-r^2$
$du=-2r\,dr$
$=\int\sqrt{u}\,(-\,du)$
$=-\frac{2}{3}u^{\frac{3}{2}}$

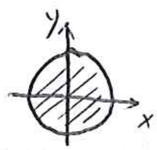
$V=\iiint_E 1\,dV$
$=\int_0^1\int_0^{2\pi}\left(\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\,dz\right) r\,d\theta\,dr$
$=\int_0^1\int_0^{2\pi} 2\sqrt{4-r^2}\,r\,d\theta\,dr$
$=\left(\int_0^1 2\sqrt{4-r^2}\,r\,dr\right)\left(\int_0^{2\pi} 1\,d\theta\right)$
$=-\frac{2}{3}(4-r^2)^{\frac{3}{2}}\Big|_0^1\cdot 2\pi$

15.10 Change of variable

$$\int_{t=t_1}^{t=t_2} f(u(t))\,u'(t)\,dt=\int_{u(t_1)}^{u(t_2)} f(u)\,du$$
$du=u'(t)\,dt$

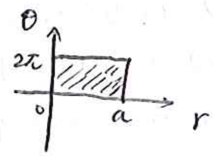
$\int_0^2 ye^{-y^2}dy=\int_0^4 e^{-u}\cdot\frac{1}{2}du=\frac{1}{2}\int_0^4 e^{-u}du$
$u=y^2$ $y=0\Rightarrow u=0$
$du=2y\,dy$ $y=2\Rightarrow u=4$
original $y\in[0,2]\rightarrow u\in[0,4]$

Prob 1



$D=\{x^2+y^2<a^2\}$

$x=r\cos\theta$
$y=r\sin\theta$



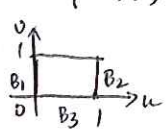
$D_1=\{(r,\theta): 0<r<a,\ 0<\theta<2\pi\}$

$x=g(u,v)$
$y=h(u,v)$
$(x,y)=F(u,v)$

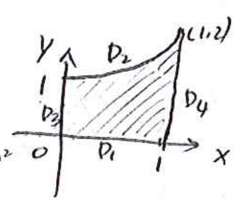
$F: D_2\rightarrow D$
$D_2=\{(u,v)\}$
$D=\{(x,y)\}$

F is injective and surjective
if $F(u_1,v_2)=F(u_2,v_2)$
then $u_1=u_2$, $v_1=v_2$

Prob 2 $S_2=\{(u,v): 0\le u\le 1,\ 0\le v\le 1\}$ $\begin{cases}x=v\\ y=u(1+v^2)\end{cases}$



boundary ① $u=0$, $0\le v\le 1$ (B_1)
② $u=1$, $0\le v\le 1$ (B_2) $\begin{cases}x=v\\ y=1+v^2\end{cases}$
③ $0\le u\le 1$, $v=0$ (B_3) $\begin{cases}x=0\\ y=u\end{cases}$
④ $0\le u\le 1$, $v=1$ (B_4) $x=1$, $y=2u$



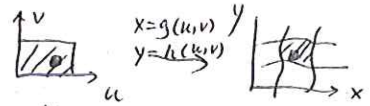
① $0\le x\le 1$, $y=0$ D_1
② $y=1+x^2$ D_2
③ $x=0$, $0\le y\le 1$ D_3
④ $x=1$, $y=2u$ D_4

$$x = g(u,v) \qquad y = h(u,v)$$

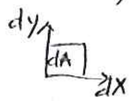
$$\begin{pmatrix} x_u = \frac{\partial g}{\partial u} & x_v = \frac{\partial g}{\partial v} \\ y_u = \frac{\partial h}{\partial u} & y_v = \frac{\partial h}{\partial v} \end{pmatrix} \quad \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - x_v y_u = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

Jacobian Matrix.

prove



dA
$= dx \times dy$

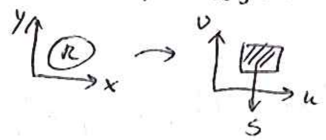


$$dx = g_u du + g_v dv$$
$$dy = h_u du + h_v dv$$
$$dx \times dy = (g_u du + g_v dv) \times (h_u du + h_v dv)$$
$$= (g_u h_v - g_v h_u)\, du \times dv$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

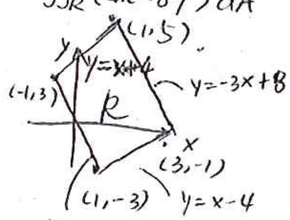
$$J = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad \det J = r\cos^2\theta + r\sin^2\theta = r.$$

In 2D $$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

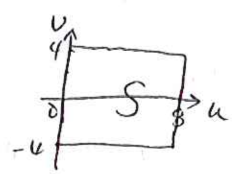


Prob 3

$\iint_R (4x+8y)\,dA$ $\quad R$: parallelogram w/ $(-1,3), (1,-3), (3,-1), (1,5)$
$x = (u+v)/4 \quad y = (v-3u)/4.$



$$R = \{(x,y) : 0 \le y+3x \le 8,\ -4 \le y-x \le 4\}$$
$$\Downarrow$$
$$S = \{(u,v) : 0 \le u \le 8,\ -4 \le v \le 4\}$$



$\begin{cases} u = y+3x \\ v = y-x \end{cases}$ (Information given)

$$\begin{cases} x = \frac{u+v}{4} \\ y = \frac{v-3u}{4} \end{cases} \quad J = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix} \quad \det J = \frac{1}{8} + \frac{3}{8} = \frac{1}{4}.$$

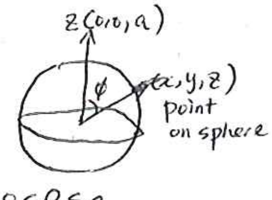
$$\iint_S (u+v+2v-6u) \cdot \frac{1}{4}\, dA$$
$$= \frac{1}{4}\int_0^8 \int_{-4}^4 (u+v+2v-6u)\, du\, dv.$$

See HW 15.10/13

In 3-D $\quad x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ $\quad$ Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

cylindrical $\quad x = r\cos\theta$, $y = r\sin\theta$, $z = z$ $\qquad \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = r$

15.9 $\quad$ Spherical Coordinate : orginal domain 3-D ball $\quad x^2+y^2+z^2 \le a^2$

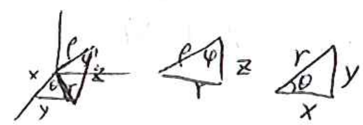


$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$(x,y,z) \to (\rho,\theta,\varphi)$
$\rho = \sqrt{x^2+y^2+z^2}$
θ in x-y plane
φ

$r = \rho\sin\phi \quad r = \sqrt{x^2+y^2}$
$x = r\cos\theta$
$y = r\sin\theta$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\Big\}$ indep. of z

$0 \le \rho \le a$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$ $\qquad \phi = 0 \Rightarrow$ north pole
$\phi = \pi \Rightarrow$ South pole

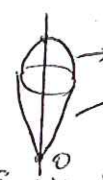


Jacobian for spherical, $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \boxed{\rho^2\sin\varphi}$

Prob 4 $\quad$ Volume of ball $D = \{(x,y,z) : x^2+y^2+z^2 \le a^2\}$

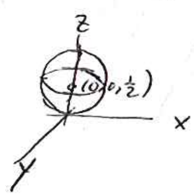
$$V = \iiint_D 1\,dV = \int_0^a\int_0^{2\pi}\int_0^{\pi} 1\,\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho = \left(\int_0^a \rho^2 d\rho\right)\left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^{\pi}\sin\varphi\,d\varphi\right)$$
$$= \frac{\rho^3}{3}\Big|_0^a \cdot \theta\Big|_0^{2\pi} \cdot (-\cos\varphi)\Big|_0^{\pi} = \left(\tfrac{1}{3}a^3\right)(2\pi)(2) = \frac{4\pi}{3}a^3$$

Prob 5



Find its Volume

$x^2+y^2+z^2 - z + \frac{1}{4} = \frac{1}{4}$
$x^2+y^2+(z-\frac{1}{2})^2 = (\frac{1}{2})^2$
Sphere w/ center $(0,0,\frac{1}{2})$
$r = \frac{1}{2}$



$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

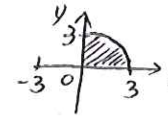
$x^2+y^2+z^2 = z$
$\rho^2 = \rho\cos\varphi$
$\rho = \cos\varphi$

$z = \sqrt{x^2+y^2} \Rightarrow \varphi = \frac{\pi}{4}$
$\rho\cos\varphi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} = \rho\sin\phi$
$\rho\cos\varphi = \rho\sin\varphi \quad \tan\varphi = 1 \quad \varphi = \frac{\pi}{4}$

$E = \{(x,y,z) : \sqrt{x^2+y^2} \le z \le \sqrt{\tfrac{1}{4} - x^2 - y^2} + \tfrac{1}{2}\}$
$= \{(\rho,\theta,\varphi) : 0 < \rho < \cos\varphi,\ 0 < \theta < 2\pi,\ 0 < \varphi < \frac{\pi}{4}\}$

$$V = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\left(\int_0^{\cos\varphi}\rho^2\sin\varphi\,d\rho\right)d\varphi\,d\theta.$$
$$= 2\pi\int_0^{\frac{\pi}{4}} \frac{\rho^3}{3}\sin\varphi\Big|_{\rho=0}^{\rho=\cos\varphi} d\varphi$$
$$= 2\pi\int_0^{\frac{\pi}{4}} \frac{1}{3}\cos^3\varphi\sin\varphi\,d\varphi$$
$$= \left(\frac{2\pi}{3}\right)\left(-\frac{1}{4}\right)\cos^4\varphi\Big|_0^{\frac{\pi}{4}} = \frac{2\pi}{3}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{\pi}{8}$$

prob6 $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\,dx\,dy$

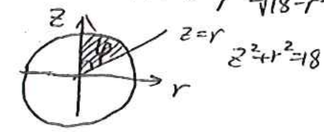


$x^2 = 9-y^2$

$x^2+y^2=3^2$

bottom $z=\sqrt{x^2+y^2}=r$

top $z=\sqrt{18-x^2-y^2}=\sqrt{18-r^2}$



Sphere tical

$D=\{(x,y,z) - 0\le y\le 3,\ 0\le x\le \sqrt{9-y^2},\ \sqrt{x^2+y^2}\le z\le \sqrt{18-x^2-y^2}\}$

$=\{(\rho,\theta,\varphi): 0\le\varphi\le\frac{\pi}{4},\ 0\le\rho\le 3\sqrt{2},\ 0\le\theta\le\frac{\pi}{2}\}$

$I=\int_0^{\frac{\pi}{4}}\int_0^{3\sqrt{2}}\int_0^{\frac{\pi}{2}} \rho^2\ \underline{\rho^2\sin\varphi}\ d\theta\,d\rho\,d\varphi$

$=\left(\int_0^{3\sqrt{2}}\rho^4\,d\rho\right)\left(\int_0^{\frac{\pi}{2}}d\theta\right)\left(\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi\right)$

$z=\sqrt{x^2+y^2} \Rightarrow$ $\varphi=\frac{\pi}{4}$

$\rho\cos\varphi=\rho\sin\varphi$

$z^2+x^2+y^2=18 \Rightarrow \rho^2=18$ $\rho=\sqrt{18}=3\sqrt{2}$

Cylindrical : r,θ,z.

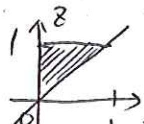
$D=\{(r,\theta,z): r\le z\le\sqrt{18-r^2},\ 0\le r\le 3,\ 0\le\theta\le\frac{\pi}{2}\}$

$I=\int_0^3\int_0^{\frac{\pi}{2}}\int_r^{\sqrt{18-r^2}}(r^2+z^2)\,r\,dz\,d\theta\,dr$

Change of variable, Integral conversion (15.7)

$\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy$, 5 other iterations.

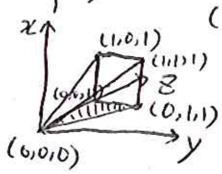
$D=\{(x,y,z)\}: 0\le y\le 1,\ y\le z\le 1,\ 0\le x\le z\}$



$(y,z)=(0,0)$ $0\le x\le 0$

$(y,z)=(0,1)$ $0\le x\le 1$

$(y,z)=(1,1)$ $0\le x\le 1$

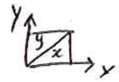


① $=\int_0^1\int_0^y\int_0^z f(x,y,z)\,dx\,dy\,dz$

② $D=\{(x,y,z)\}: 0\le x\le 1,\ 0\le y\le 1,\ \max\{x,y\}\le z\le 1\}$

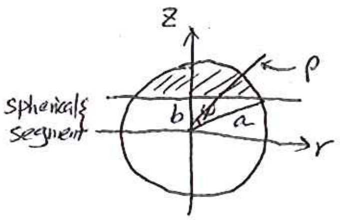
Plane 1 : $(1,0,1)(1,1,1)(0,0,0)$

$ax+by+cz+d=0 \Rightarrow x-z=0$



Plane 2: $z=y$

Spherical cap (15.9 quiz) - how to find (ρ,θ,φ)



$x^2+y^2+z^2\le a^2$

$z\ge b$

$\cos\varphi=\frac{b}{a}$

$$\begin{cases}0\le\theta\le 2\pi\\ 0\le\varphi\le\cos^{-1}\left(\frac{b}{a}\right)\\ \frac{b}{\cos\varphi}\le\rho\le a\end{cases}$$